# Semidefinite programming by Projective-Cutting-Planes

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We focus on the following standard (semidefinite programming) SDP problem, where $\boxed{\mathcal{A}^\top \mathbf{y} = \sum_{i=1}^{k} A_i y_i}$. The implication in the last constraint indicates that the SDP constraint can be expressed using an infinite number of linear cuts.

$$\mathscr{P}\begin{cases} \max & \mathbf{b}^\top \mathbf{y} \\ s.t & X = C - \mathcal{A}^\top \mathbf{y} \\ & X \succeq \mathbf{0} \iff X \bullet \mathbf{s}\mathbf{s}^\top \geq 0 \ \forall \mathbf{s} \in \mathbb{R}^n \end{cases}$$

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We could address the problem by progressively separating infeasible solutions $\mathbf{y}_{\text{out}} \in \mathbb{R}^n$. Such very standard `Cutting-Planes` will likely result in failure.

Three years ago I introduced Projective-Cutting-Planes to upgrade the separation sub-problem to the projection sub-problem: given feasible $\mathbf{y}$ in a polytope $\mathscr{P}$ and an arbitrary direction $\mathbf{d}$, what is the maximum step-length $t^*$ so that $t^*\mathbf{d} \in \mathscr{P}$?

In SDP programming, projecting $\mathbf{y} \to \mathbf{d}$ requires solving $t^* = \max\{t : X + tD \succeq \mathbf{0}\}$ for this $X \succeq \mathbf{0}$ and $D$:

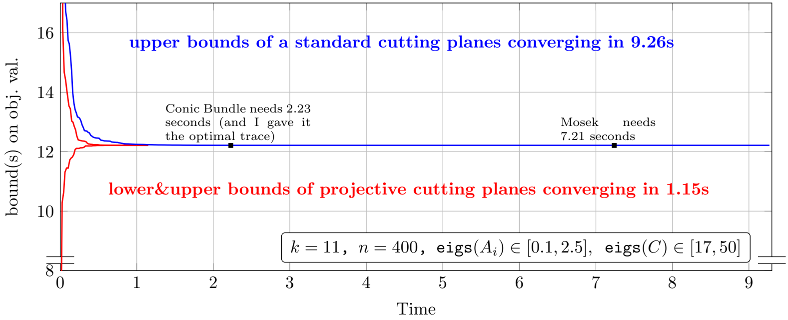
- $X = C - \mathcal{A}^\top \mathbf{y}$ is SDP when $\mathbf{y}$ is feasible
- $D = C - \mathcal{A}^\top \mathbf{d}$ may be SDP or not.

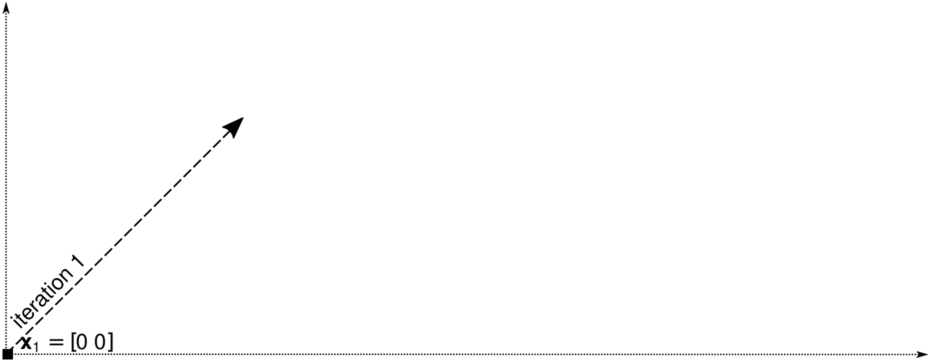
We have to project $X \to D$ in the SDP cone.

Th projection sub-problem is quite simple if $X \succ \mathbf{0}$. In this case, there is a unique Cholesky decomposition $X = KK^\top$ and $K$ is non-singular. We'll see later that the following projections are equivalent, using $D'$ = the unique solution of $D = KD'K^\top$.

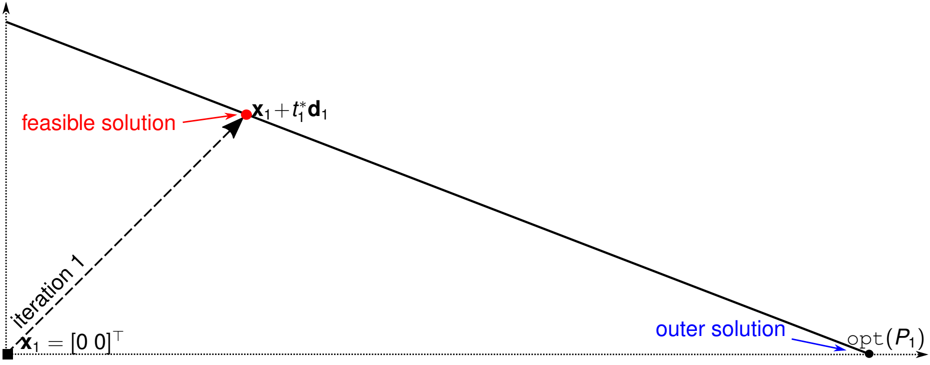
- $X \to D$;
- $I_n \to D' \implies$ finding $\max\{t : I_n + tD' \succeq \mathbf{0}\}$ is easy if you know $\lambda_{\min}(D')$

This projection is more difficult if $X$ is not strictly SDP. Yet, this simplified case enabled us to solve some particular instances very rapidly:

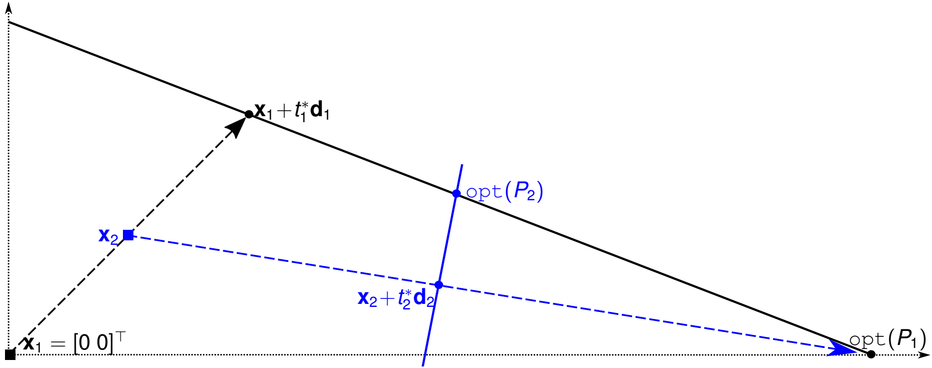




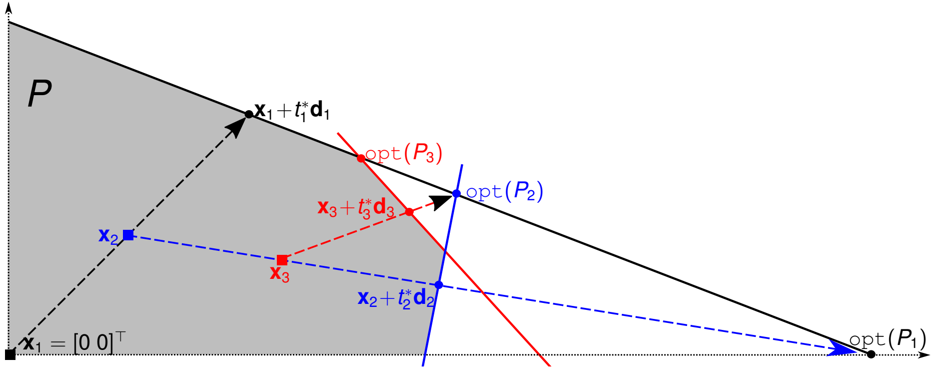

Iteration 1 : uncharted territory, follow objective function, i.e., advance along $\mathbf{x}_1 \to \mathbf{d}_1$



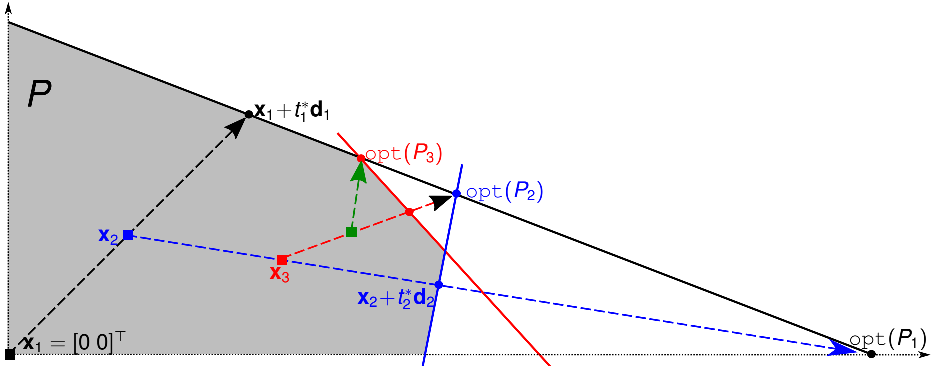

Iteration 1 : found a first outer solution $\texttt{opt}(P_1)$ and a first inner solution (contact point) $\mathbf{x}_1 + t_1^* \mathbf{d}_1$




Iteration 2 : an inner feasible solution (contact point) $\mathbf{x}_2 + t_2^*\mathbf{d}_2$ and a new outer solution. We take $\mathbf{d}_2 = \texttt{opt}(P_1) - \mathbf{x}_2$.

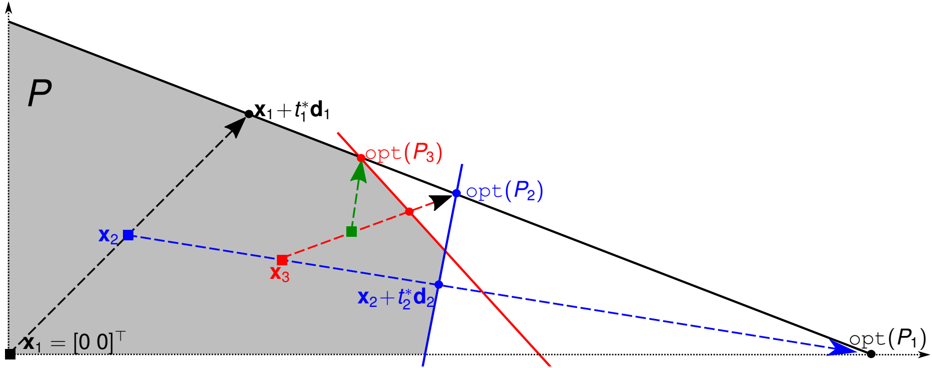



Iteration 3 : the feasible solution $\mathbf{x}_3 + t_3^* \mathbf{d}_3$ is almost optimal




Iteration 4 : optimality of $\texttt{opt}(P_3)$ proved
You can see the proposed method is convergent because it solves a separation problem on $\texttt{opt}(P_k)$ at each iteration $k$

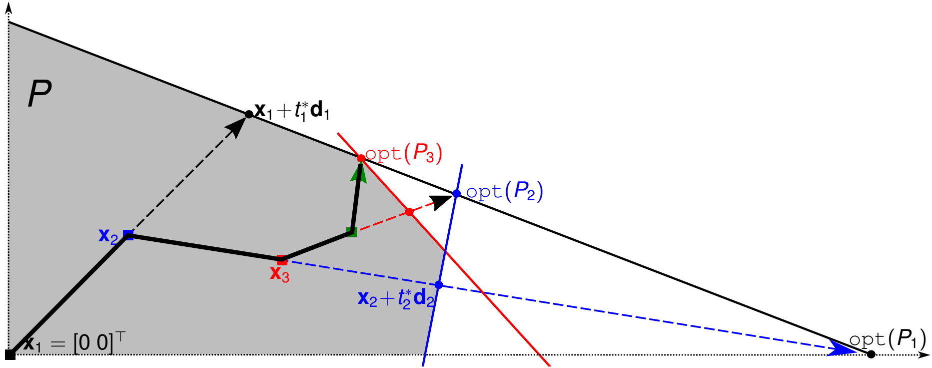
- The convergence proof takes two lines, cool !




Building on existing work [1,2], the new method was deliberately designed to be more general and when possible simpler


[1] Daniel Porumbel. Ray projection for optimizing polytopes with prohibitively many constraints in set-covering column generation. *Mathematical Programming*, 155(1) :147–197, 2016.

[2] Daniel Porumbel. From the separation to the intersection subproblem for optimizing polytopes with prohibitively many constraints in a Benders decomposition context. *Discrete Optimization*, 2018.




Notice the trajectory of the inner points — there is no built-in feature in the `Cutting-Planes` to generate inner points

- each $x_k$ is a point on the last projected segment, i.e., between $\mathbf{x}_{k-1}$ and $\mathbf{x}_{k-1} + t^*_{k-1}\mathbf{d}_{k-1}$
- in this example we choose : $x_k = \mathbf{x}_{k-1} + \frac{1}{2} \cdot t^*_{k-1}\mathbf{d}_{k-1}$

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everything was like a movie until here : let's move to real life

*Property 1.* We will see that the projection $X \to D$ can be calculated more rapidly if $D$ belongs to the image of $X$. This means that each column (and row) of $D$ can be written as a linear combination of the columns (or rows, resp.) of $X$. We can equivalently say that the null space of $X$ is included in the null space of $D$; thus, $X\mathbf{d} = 0 \implies D\mathbf{d} = 0 \ \forall \mathbf{d} \in \mathbb{R}^n$. We will show below in cases A) and B) how it is easier to project when this property holds; if possible, `Projective Cutting-Planes` should thus adapt its own evolution to seek this property.

A) This case is characterized by $X \succ \mathbf{0}$, *i.e.*, $X$ is non-singular; Prop 1 surely holds because the image of a non-singular $X$ is $\mathbb{R}^n$. We apply the Cholesky decomposition to determine the unique non-singular $K$ such that $X = KK^\top$. We then solve $D = KD'K^\top$ in variables $D'$ by back substitution; this may require $O(n^3)$ in theory, but `Matlab` is able to compute it much more rapidly in practice because $K$ is triangular. Let us re-write (3) as:

$$\max \left\{ t : KI_nK^\top + tKD'K^\top \succeq \mathbf{0} \right\}. \tag{4}$$

This is equivalent (by congruence according to Prop 2) to

$$\max \left\{ t : I_n + tD' \succeq \mathbf{0} \right\}. \tag{5}$$

The sought step length is $t^* = -\frac{1}{\lambda_{\min}(D')}$, or $t^* = \infty$ if $\lambda_{\min}(D') \geq 0$.

We still have to find a first-hit cut $\mathbf{v} \in \mathbb{R}^n$; in fact, technically, the first-hit cut will be $\left(A_1 \bullet \mathbf{v}\mathbf{v}^\top\right) y_1 + \left(A_2 \bullet \mathbf{v}\mathbf{v}^\top\right) y_2 + \cdots + \left(A_k \bullet \mathbf{v}\mathbf{v}^\top\right) y_k \leq C \bullet \mathbf{v}\mathbf{v}^\top$.

If $\mathbf{v}$ is an eigenvector of $K(I_n + t^* D')K^\top$ with an eigenvalue of 0, this means $\mathbf{v}^\top K(I_n + t^* D')K^\top \mathbf{v} = 0$. Thus, $\mathbf{u} = K^\top \mathbf{v}$ is eigenvector of $I_n + t^* D'$ with an eigenvalue of 0. This latter eigenvector $\mathbf{u}$ can be computed when determining $\lambda_{\min}(D') < 0$ above, because if the eigenvalue of $\mathbf{u}$ with regards to $D'$ is $\lambda_{\min}(D')$ its eigenvalue with regards to $I_n + t^* D'$ is 0 (since recall $t^* = -\frac{1}{\lambda_{\min}(D')}$). The sought $\mathbf{v}$ solves $K^\top \mathbf{v} = \mathbf{u}$ and it can rapidly be computed by back-substitution. We have $\mathbf{u}^\top D' \mathbf{u} < 0 \implies \mathbf{v}^\top K D' K^\top \mathbf{v} < 0 \implies \mathbf{v}^\top D \mathbf{v} < 0$. We thus have $\mathbf{v}^\top (X + t^* D)\mathbf{v} = 0$ and $\mathbf{v}^\top (X + (t^* + \epsilon) D)\mathbf{v} < 0$ for any $\epsilon > 0$.

B) In this case Prop 1 is still satisfied, but $X$ has rank $c < n$. This means $X$ contains $c$ independent rows (and columns by symmetry), referred to as *core* rows (or columns); the other dependent rows (or columns) are *non-core* positions. Using the LDL decomposition of $X$, we will factorize $X = K_{nc}K_{nc}^\top$, where $K_{nc} \in \mathbb{R}^{n\times c}$. The image of $K_{nc}$ is equal to the image of $X$. Since Prop 1 is satisfied, we will see we can still solve $D = K_{nc}D'K_{nc}^\top$ in variables $D'$. A first intuition is to notice that we can project $X \to D$ only over the core rows and columns, because the non-core positions are dependent on the core ones.

But the most difficult task is to determine these core positions. We first apply the LDL decomposition and write $X = L\mathtt{diag}(\mathbf{p})L^\top$ with $\mathbf{p} \geq \mathbf{0}_n$. The contribution of each $p_i$ in $L\mathtt{diag}(\mathbf{p})L^\top$ is actually $p_i L_i L_i^\top$, where $L_i$ is column $i$ of $L$ ($\forall i \in [1..n]$). If all $n \times n$ elements of $p_i L_i L_i^\top$ are below some precision parameter, we consider $i$ is a non-core position; otherwise, it is a core position. By reducing all non-core positions $p_i$ to zero, we can say that all $n - c$ non core columns of $L$ vanish in the decomposition $X = L\mathtt{diag}(\mathbf{p})L^\top$. After removing these vanished $n - c$ columns from $L$ and the corresponding zeros from $\mathbf{p}$, we can write $X = L\mathtt{diag}(\mathbf{p})L^\top = L\mathtt{diag}(\mathbf{p})^{\frac{1}{2}}\mathtt{diag}(\mathbf{p})^{\frac{1}{2}}L^\top = K_{nc}K_{nc}^\top$ with $K_{nc} \in \mathbb{R}^{n\times c}$.

We next solve $D = K_{nc}D'K_{nc}^\top$ in variables $D'$. For this, we first reduce this system to work on $c \times c$ matrices, *i.e.*, we transform it into $D_{cc} = K_{cc}D'K_{cc}$ where $K_{cc}$ is $K_{nc}$ restricted to the $c$ core rows and $D_{cc}$ is $D$ restricted to the $c \times c$ core rows and columns. To solve this square system, we apply back-substitution twice and this is very fast because $K_{cc}$ is lower triangular. If the resulting solution $D'$ also satisfies $D = K_{nc}D'K_{nc}^\top$, then we are surely in case B). We obtained a reduced-size version of (5) working in the space of $c \times c$ matrices:

| Instance | | | | Projective Cutting-Planes | | | | | | ConicBundle | | Mosek |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | k | Eigs $A_i$'s | Eigs C | Itera-tions | All time | Compute $X$ & $D$ | Proj time | LP time (cplex) | Send data to LP | Trace unknown | Trace provided | |
| 800 | 80 | [-20, 100] | [0,100] | 1108 | 410 | 179 | 44 | 70 | 102 | 1051 | 94 | 320 |
| 600 | 40 | [-20, 100] | [0,100] | 155 | 17 | 4 | 6 | 1 | 3 | 148 | 22 | 72 |
| 400 | 100 | [-20, 100] | [0,100] | 2075 | 572 | 94 | 13 | 384 | 71 | 490 | 42 | 60 |
| Huge instances below have $\mathbf{y} \geq 0$, a random $\mathbf{b}$ and $\frac{n}{5}$ fixed null eigenvectors for all $A_i$'s and $C$ | | | | | | | | | | | | |
| 200 | 2000 | [40, 100] | [10,40] | 31 | 11 | 5 | 0.2 | 0.2 | 5 | timeout | | 717 |
| 200 | 3000 | [40, 100] | [10,40] | 70 | 49 | 27 | 0.4 | 0.7 | 18 | timeout | | 1346 |
| 4000 | 20 | [20,25] | [20,25] | 8 | 76 | 17 | 44 | 0 | 11 | timeout | | timeout |
| 5000 | 20 | [20,25] | [20,25] | 7 | 139 | 27 | 87 | 0 | 18 | timeout | | timeout |

**Table 2.** Seven runs of `Projective Cutting-Planes`, `ConicBundle` and `Mosek` on more varied instances. The last four instances have $\mathbf{y} \geq \mathbf{0}$; such linear constraints on $\mathbf{y}$ simplify the problem for `Projective Cutting-Planes`, but this may be a non-trivial change for `ConicBundle` (or other algorithms that do not embed the SDP problem in a lightweight LP over $\mathbf{y}$).