Non Linear Integer Programming for Exact Solution of Sequence Problems

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Subject

The Low Autocorrelation Binary Sequence problem with a tunable interaction range (or LABS problem for short) is the problem of finding a sequence involving terms equal to only $+1$ and -1 such that an energy function called autocorrelation is minimized. The problem is inspired by an application in physics and has many other applications in cryptography, for synchronization in digital communication systems, and in modulation of radar pulses.

Given the size of the desired sequence n and the range r with $3 \leq r \leq n$, the LABS problem can be formulated as follows:

$$E_{n,r} = \min_{s \in \{-1,1\}^n} \sum_{d=1}^{(r-1)} \sum_{i=1}^{(n-r+1)} \left(\sum_{j=i}^{(i+r-1-d)} s_j s_{j+d} \right)^2$$

For $r \geq 4$ the problem is to minimize a quartic function in -1,1 variables that can be easily transformed into an equivalent function in 0,1 variables by the simple variable change $s_i = 2x_i - 1$ where x_i is a 0,1 variable. So, this problem is a particular case of the polynomial unconstrained binary optimization problem (PUBO). In the case $r = 3$, the problem is of degree 2 and amounts to a QUBO.

The exact solution of this problem with algorithmic or mathematical programming tools is known to be very hard. The MINLPLib library (https://www.minlplib.org) gathers a set of 44 instances with up to $n = r = 60$ and a kind of competition takes place and is reported around different solution methods for these instances. Still, many of them remain unsolved to optimality. These instances are also often used as benchmark for the PUBO problem.

In the most recent studies that solve general PUBOs by integer programming tools and use the LABS as test problem, there are two categories of methods: (i) transform the PUBO into an equivalent MILP and use sophisticated valid inequalities within a branch-and-cut framework. This approach is used, for example, in [2]. In the second category, (ii) transform the PUBO into a quadratic problem thanks to additional variables and then use a tailored quadratic convex reformulation in order to obtain a quadratic problem in which continuous relaxation is a quadratic convex problem. This last problem can be solved by branch-and-bound through standard solvers. This obtained method, based on semidefinite programming to build a good convex reformulation, is called PQCR and is presented in [3]. An alternative solution method is proposed in [1] and is based on dynamic programming. This method is very efficient for instances where the parameter r, also called the reach of a polynomial, is small.

Our recent findings for this problem are twofold. First, we prove that for small values of the range, the problem is easy to solve. Precisely, we prove that $E_{n,3} = n - 2$ and $E_{n,4} = 2(n-3)$ and in both cases, an optimal solution can be built through a simple algorithm. Second, we have a new mathematical programming formulation with some convexity properties that may offer promising improvements in the solution using a branch-and-bound or a branch-and-cut framework.

The **objectives** of the internship include:

- new results on the special cases: for which other values of r is the problem easy?
- implementation of the new formulation, apply machinery of integer programming tools like symmetry breaking, valid inequalities, or efficient branching in order to evaluate this formulation and compare it to existing ones.

Supervision

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Location and dates

UMA, ENSTA, Institut Polytechnique de Paris, 91120 Palaiseau, France

The intership should last 5-6 months between february and september 2026

References

[1] Jens Vinther Clausen et al. “Solving unconstrained binary polynomial programs with limited reach: Application to low autocorrelation binary sequences”. In: *Computers & Operations Research* 165 (2024), p. 106586.

[2] Alberto Del Pia and Matthias Walter. “Simple odd β-cycle inequalities for binary polynomial optimization”. In: *Mathematical programming* 206.1 (2024), pp. 203–238.

[3] Sourour Elloumi, Amélie Lambert, and Arnaud Lazare. “Solving unconstrained 0-1 polynomial programs through quadratic convex reformulation”. In: *Journal of Global Optimization* 80.2 (2021), pp. 231–248.