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A RECURSIVE PROCEDURE TO GENERATE ALL CUTS FOR 0–1 MIXED INTEGER PROGRAMS

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We study several ways of obtaining valid inequalities for mixed integer programs. We show how inequalities obtained from a disjunctive argument can be represented by superadditive functions and we show how the superadditive inequalities relate to Gomory's mixed integer cuts. We also show how all valid inequalities for mixed 0–1 programs can be generated recursively from a simple subclass of the disjunctive inequalities.

Key words: Cutting planes, valid inequalities, disjunctive inequalities, superadditive functions, 0–1 mixed integer programs.

1. Introduction

This paper, which is a substantial revision of a technical report that appeared in 1984 [9], was motivated by Chvátal's [3] description of a simple recursive procedure for generating all valid inequalities for pure integer programs. We examine several ways of obtaining valid inequalities for mixed integer programs. In particular, we show how inequalities based on a disjunctive argument can be represented as superadditive inequalities and we show how these inequalities relate to Gomory's mixed integer cuts. We then show how all valid inequalities for mixed 0–1 programs can be generated recursively from a simple subclass of the disjunctive inequalities.

Cook, Kannan and Schrijver [4] have continued the development of this approach; they have established that the recursive use of valid inequalities obtained by combining a disjunctive argument with a rounding argument based on the size of numbers gives all valid inequalities for any mixed integer program with rational data.

2. Disjunctive and mixed integer rounding (MIR) inequalities

Let

$$P = \{x \in \mathbb{R}^n,\ y \in \mathbb{R}^p\colon Ax + Gy \leqslant b\}$$

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be a rational polyhedron and $T = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$ be a mixed integer set, i.e. the feasible region for a mixed integer program.

We describe three procedures for generating valid inequalities for T using a pair of valid inequalities for P.

(A) *Disjunctive method* [1, 2, 6]. If

$$cx + hy - \alpha(\pi x - \pi_0) \leq c_0 \tag{1a}$$

and

$$cx + hy + \beta(\pi x - \pi_0 - 1) \leq c_0 \tag{1b}$$

with $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\alpha, \beta \geq 0$ are valid inequalities for P, then

$$cx + hy \leq c_0 \tag{1c}$$

is valid for T.

(B) *Split method* [4]. If $cx + hy \leq c_0$ is a valid inequality for $P \cap \{(x, y): \pi x \leq \pi_0\}$ and for $P \cap \{(x, y): \pi x \geq \pi_0 + 1\}$, then $cx + hy \leq c_0$ is valid for T.

(C) *MIR method* [9]. If

$$c^1 x + hy \leq c_0^1 \tag{2a}$$

and

$$c^2 x + hy \leq c_0^2 \tag{2b}$$

are valid inequalities for P, and $\pi = c^2 - c^1 \in \mathbb{Z}^n$, $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$ and $\gamma = c_0^2 - c_0^1 - \pi_0$, then

$$\pi x + (c^1 x + hy - c_0^1)/(1 - \gamma) \leq \pi_0 \tag{2c}$$

is valid for T.

The following result establishes the validity of the MIR method.

Proposition 1. *The disjunctive, split and MIR inequalities are equivalent.*

Proof. The equivalence of the disjunctive and split methods follows immediately from linear programming duality. We now show that every MIR inequality is disjunctive.

Multiplying (2a) by $1/(1-\gamma) > 0$, we obtain that

$$\pi x + (c^1 x + hy - c_0^1)/(1 - \gamma) - \alpha(\pi x - \pi_0) \leq \pi_0$$

where $\alpha = 1$. Multiplying (2b) by $1/(1-\gamma)$ and rewriting the inequality gives

$$\pi x + (c^1 x + hy - c_0^1)/(1 - \gamma) + \beta(\pi x - \pi_0 - 1) \leq \pi_0$$

where $\beta = \gamma/(1-\gamma)$. Hence the disjunctive method gives (2c).

Finally to see that the disjunctive method is MIR, multiply the inequalities (1a) and (1b) by $1/(\alpha + \beta)$. The MIR method with $\gamma = \beta/(\alpha + \beta)$ then gives after simplification

$$(cx + hy - c_0)/\alpha \leq 0. \qquad \square$$

In the next section we examine how to represent the inequalities by superadditive functions.

3. Superadditivity and Gomory's mixed integer inequalities

Here we consider a functional description of inequalities for mixed integer programs. As the theory is easier to describe when the variables are nonnegative, we consider $T^+ = P^+ \cap (\mathbb{Z}^n \times \mathbb{R}^p)$ where $P^+ = P \cap \mathbb{R}_+^{n+p}$. Let a_j for $j \in N = \{1, \ldots, n\}$ be the jth column of A and g_j for $j \in J = \{1, \ldots, p\}$ be the jth column of G.

A function $F: \mathbb{R}^m \to \mathbb{R}$ is called *superadditive* if $F(0) = 0$, and $F(u) + F(v) \leqslant F(u+v)$ for all $u, v \in \mathbb{R}^m$. F is called *nondecreasing* if $u \leqslant v$ implies $F(u) \leqslant F(v)$.

It is known, see [7, 8], that if F is superadditive and nondecreasing and $\bar{F}$ defined by $\bar{F}(d) = \lim_{\lambda \downarrow 0_+} (F(\lambda d)/\lambda)$ exists and is finite for all $d \in \mathbb{R}^m$, then

$$\sum_{j \in N} F(a_j)x_j + \sum_{j \in J} \bar{F}(g_j)y_j \leqslant F(b) \tag{3}$$

is a valid inequality for T^+.

We now develop two superadditive functions that we will use to generate inequalities closely related to those of the previous section, and also to generate Gomory's mixed integer cuts.

Let $x^+ = \max(0, x)$ and $x^- = \min(0, x)$.

Proposition 2. *Let* $F_\gamma : \mathbb{R}^1 \to \mathbb{R}^1$ *for* $0 < \gamma < 1$ *be defined by*

$$F_\gamma(d) = \lfloor d \rfloor + (d - \lfloor d \rfloor - \gamma)^+ / (1 - \gamma).$$

Then

(i) F_γ *is superadditive and nondecreasing*;

(ii) $\bar{F}_\gamma$ *exists, and* $\bar{F}_\gamma(d) = \min\{d/(1-\gamma), 0\}$.

Proof. (i) F_γ is nondecreasing because it is piecewise linear with slope of either 0 or $1/(1-\gamma)$ and has no jumps. To prove superadditivity, let $f_i = d_i - \lfloor d_i \rfloor$ for $i = 1, 2$.

Case 1. $f_1 + f_2 < 1$.

$$F_\gamma(d_1) + F_\gamma(d_2) = \lfloor d_1 \rfloor + \frac{(f_1 - \gamma)^+}{1-\gamma} + \lfloor d_2 \rfloor + \frac{(f_2 - \gamma)^+}{1-\gamma}$$

$$\leqslant \lfloor d_1 + d_2 \rfloor + \frac{(f_1 + f_2 - \gamma)^+}{1-\gamma} = F_\gamma(d_1 + d_2).$$

Case 2. $f_1 + f_2 \geqslant 1$, $f_2 \leqslant \gamma$.

$$F_\gamma(d_1) + F_\gamma(d_2) = \lfloor d_1 \rfloor + \frac{(f_1 - \gamma)^+}{1-\gamma} + \lfloor d_2 \rfloor$$

$$< \lfloor d_1 \rfloor + \lfloor d_2 \rfloor + 1 = \lfloor d_1 + d_2 \rfloor \leqslant F_\gamma(d_1 + d_2).$$

(The same argument applies if $f_1 \leqslant \gamma$.)

Case 3. $f_1+f_2 \geqslant 1, f_1, f_2 > \gamma$.

$$F_\gamma(d_1)+F_\gamma(d_2) = \lfloor d_1 \rfloor + \frac{f_1-\gamma}{1-\gamma} + \lfloor d_2 \rfloor + \frac{f_2-\gamma}{1-\gamma}$$

$$= \lfloor d_1 \rfloor + \lfloor d_2 \rfloor + 1 + \frac{f_1+f_2-1-\gamma}{1-\gamma} \leqslant F_\gamma(d_1+d_2).$$

(ii) If the magnitude of d is sufficiently small then $F_\gamma(d)=0$ for $d>0$ and $F_\gamma(d)=d/(1-\gamma)$ for $d<0$. □

To represent inequalities derived from two inequalities, we consider a two-dimensional function

$$H_\gamma(d_1, d_2) = d_1/(1-\gamma) + F_\gamma(d_2-d_1), \quad 0<\gamma<1. \tag{4}$$

The contours of this function are exhibited in Figure 1 for $\gamma = \frac{1}{3}$.

To show the superadditivity of H_γ, we use the following result on the composition of superadditive functions.

Proposition 3. *Let $L:\mathbb{R}^k \to \mathbb{R}^1$ be superadditive and nondecreasing and let $F_i:\mathbb{R}^m \to \mathbb{R}^1$ be superadditive for $i=1,\ldots,k$. The composite function $L(F_1,\ldots,F_k)$ is superadditive. Moreover if the F_i are also nondecreasing, then $L(F_1,\ldots,F_k)$ is nondecreasing.* □

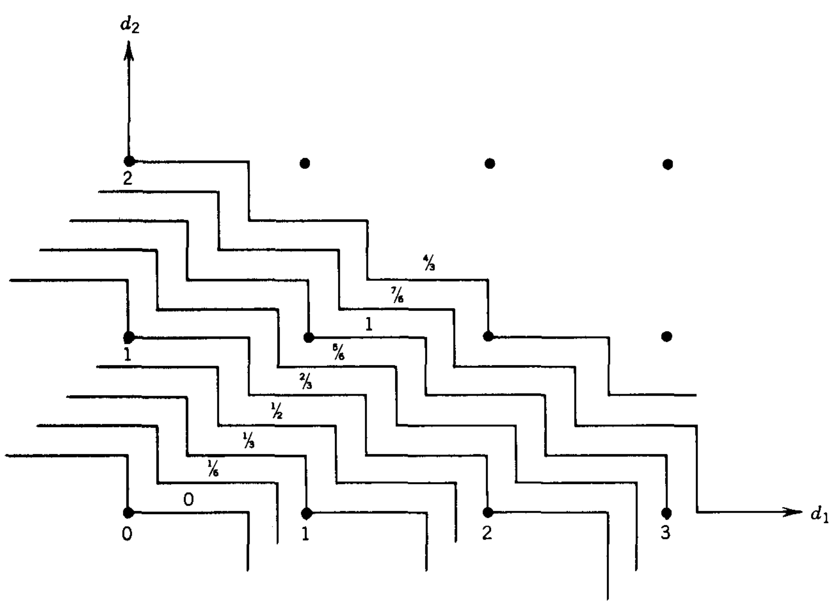

Fig. 1. Contours of $(1-\gamma)H_\gamma(d_1, d_2)$ for $\gamma = \frac{1}{3}$.

Proposition 4. *H_γ given by* (4) *is nondecreasing and superadditive, and $\bar{H}_\gamma(d_1, d_2) = \min(d_1, d_2)/(1-\gamma)$.*

Proof. Since F_γ is nondecreasing, H_γ is nondecreasing in d_2. With respect to d_1, the first term in the definition of H_γ has slope $1/(1-\gamma)$ and the second term is piecewise linear with slope of $-1/(1-\gamma)$ or 0. Hence, H_γ is nondecreasing in d_1.

Since the first term is linear, to prove that H_γ is superadditive it suffices to show that the second term is superadditive. But this follows from Proposition 3 and the superadditivity of F_γ and $d_2 - d_1$.

Now we establish the form of H_γ. For λ positive and sufficiently close to zero, if $d_1 < d_2$ then

$$H_\gamma(\lambda d_1, \lambda d_2) = \frac{\lambda d_1}{1-\gamma} + 0 = \frac{\lambda \min(d_1, d_2)}{1-\gamma},$$

and if $d_1 \geq d_2$ then

$$H_\gamma(\lambda d_1, \lambda d_2) = \frac{\lambda d_1}{1-\gamma} + \frac{\lambda d_2 - \lambda d_1}{1-\gamma} = \frac{\lambda \min(d_1, d_2)}{1-\gamma}.$$

Hence $\bar{H}_\gamma = \min(d_1, d_2)/(1-\gamma)$. □

The function H_γ allows us to define another class of valid inequalities.

(D) *Superadditive method.* Given two valid inequalities $c^i x + hy \leq c_0^i$ for $i = 1, 2$, for P^+, use the function H_γ with $\gamma = c_0^2 - c_0^1 - \lfloor c_0^2 - c_0^1 \rfloor$ to obtain the valid inequality

$$\sum_{j \in N} H_\gamma(c_j^1, c_j^2) x_j + \sum_{j \in J} \bar{H}_\gamma(h_j, h_j) y_j \leq H_\gamma(c_0^1, c_0^2) \tag{5}$$

for T^+.

Proposition 5. *The superadditive inequality* (5) *is equal to or dominates the MIR inequality* (2c) *for T^+.*

Proof. Suppose the inequalities (2a) and (2b) are valid for P^+ with $\pi = c^2 - c^1 \in \mathbb{Z}^n$, $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$ and $\gamma = c_0^2 - c_0^1 - \pi_0$. We have

$$H_\gamma(c_j^1, c_j^2) = c_j^1/(1-\gamma) + F_\gamma(c_j^2 - c_j^1) = c_j^1/(1-\gamma) + \pi_j,$$

$$\bar{H}_\gamma(h_j, h_j) = h_j/(1-\gamma)$$

and

$$H_\gamma(c_0^1, c_0^2) = c_0^1/(1-\gamma) + F_\gamma(c_0^2 - c_0^1) = c_0^1/(1-\gamma) + \pi_0.$$

As the terms on the right hand side are the coefficients of (2c) in the MIR method, the claim follows. □

Now we show how the functions F_γ and H_γ also give descriptions of Gomory's mixed integer cuts.

Given P^+ and T^+, let

$$T^+(u)=\{(x,y,s)\in\mathbb{Z}_+^n\times\mathbb{R}_+^p\times\mathbb{R}_+^m:\ uAx+uGy+us=ub\}.$$

$T^+(u)$ represents a row of an optimal LP tableau after adding slack variables and aggregating the rows with multipliers $u\in\mathbb{R}^m$. The corresponding Gomory's mixed integer cut is

$$\sum_{\{j\in N: f_j\leqslant f_0\}} f_jx_j+\frac{f_0}{1-f_0}\sum_{\{j\in N: f_j>f_0\}}(1-f_j)x_j+\sum_{j\in J^+}(ug_j)y_j$$
$$-\frac{f_0}{1-f_0}\sum_{j\in J^-}(ug_j)y_j+\sum_{i\in M^+}u_is_i-\frac{f_0}{1-f_0}\sum_{i\in M^-}u_is_i\geqslant f_0 \tag{6}$$

where $f_j=ua_j-\lfloor ua_j\rfloor$, $f_0=ub-\lfloor ub\rfloor$, $J^+=\{j\in J: ug_j\geqslant 0\}$, $J^-=J\backslash J^+$, $M=\{1,\ldots,m\}$, $M^+=\{i\in M: u_i\geqslant 0\}$ and $M^-=M\backslash M^+$.

Proposition 6. *Let $\gamma=f_0=ub-\lfloor ub\rfloor$. The inequality (6) is equivalent to the superadditive valid inequality*

$$\sum_{j\in N}F_\gamma(ua_j)x_j+\sum_{j\in J}\bar F_\gamma(ug_j)y_j+\sum_{i\in M}\bar F_\gamma(u_i)s_i\leqslant F_\gamma(ub). \tag{7}$$

Proof. We subtract (7) from $uAx+uGy+us=b$. It is easily shown that

$$ua_j-F_\gamma(ua_j)=\begin{cases} f_j & \text{if } f_j\leqslant f_0,\\ f_0(1-f_j)/(1-f_0) & \text{if } f_j>f_0,\end{cases}$$

$$ug_j-\bar F_\gamma(ug_j)=\begin{cases} ug_j & \text{if } g_j\geqslant 0,\\ -f_0(ug_j)/(1-f_0) & \text{if } g_j<0,\end{cases}$$

$$u_i-\bar F_\gamma(u_i)=\begin{cases} u_i & \text{if } u_i\geqslant 0,\\ -f_0u_i/(1-f_0) & \text{if } u_i<0,\end{cases}$$

and $ub-F_\gamma(ub)=f_0$. □

Now, by using the function H_γ, we obtain the Gomory cut in the form of a superadditive inequality (3) in (x,y)-space. Let u^+ be the vector with components $\max(0,u_i)$ and u^- be the vector with components $\min(0,u_i)$.

Proposition 7. *Let $\gamma=ub-\lfloor ub\rfloor$, and $F:\mathbb{R}^m\to\mathbb{R}^1$ be defined by*

$$F(d)=H_\gamma(-u^-d,\ u^+d).$$

The inequality (6) is equivalent to the superadditive valid inequality

$$\sum_{j\in N}F(a_j)x_j+\sum_{j\in J}\bar F(g_j)y_j\leqslant F(b). \tag{8}$$

Proof. We show that the inequality (8) is equivalent to (7).

$$F(d)=H_\gamma(-u^- d, u^+ d)=-u^- d/(1-\gamma)+F_\gamma(ud),$$

$$\bar{F}(d)=\bar{H}_\gamma(-u^- d, u^+ d)=-u^- d/(1-\gamma)+\bar{F}_\gamma(ud).$$

Therefore

$$\sum_{j\in N} F(a_j)x_j+\sum_{j\in J}\bar{F}(g_j)y_j-F(b)$$

$$=\sum_{j\in N} F_\gamma(ua_j)x_j+\sum_{j\in J}\bar{F}_\gamma(ug_j)y_j-F_\gamma(ub)$$

$$-\frac{1}{1-\gamma}\left[\sum_{j\in N}(u^- a_j)x_j+\sum_{j\in J}(u^- g_j)y_j-u^- b\right]$$

$$=\sum_{j\in N} F_\gamma(ua_j)x_j+\sum_{j\in J}\bar{F}_\gamma(ug_j)y_j-F_\gamma(ub)+\frac{u^- s}{1-\gamma}$$

(because $u^- Ax+u^- Gy+u^- s=u^- b$)

$$=\sum_{j\in N} F_\gamma(ua_j)x_j+\sum_{j\in J}\bar{F}_\gamma(ug_j)y_j+\sum_{i\in M}\bar{F}_\gamma(u_i)s_i-F_\gamma(ub)$$

(because $\bar{F}_\gamma(u_i)=0$ if $u_i>0$ and $\bar{F}_\gamma(u_i)=u_i/(1-\gamma)$ if $u_i<0$). □

4. Recursive procedure for generating inequalities

Here we consider a recursive procedure for generating valid inequalities for T (or T^1), i.e. a valid inequality is added to P giving a new polyhedron P' with $T=P'\cap(\mathbb{Z}^n\times\mathbb{R}^p)$ and then the procedure is repeated.

We will use a special case of the disjunctive method called the $\mathscr{D}$-method.

$\mathscr{D}$-method. If

$$cx+hy-\alpha x_k\leqslant c_0 \tag{9a}$$

and

$$cx+hy+\beta(x_k-1)\leqslant c_0 \tag{9b}$$

are valid inequalities for P, where $\alpha,\beta\in\mathbb{R}^1_+$, then $cx+hy\leqslant c_0$ is a valid inequality for $P\cap\{(x,y):x_k\in\mathbb{Z}^1\}$.

We say that $cx+hy\leqslant c_0$ is a *$\mathscr{D}$-inequality for T with respect to P* if (a) $cx+hy\leqslant c_0$ is valid for P or if (b) for some $c_0'\leqslant c_0$, $cx+hy\leqslant c_0'$ is obtained from a finite recursion of the $\mathscr{D}$-method.

Consider the polyhedron

$$\mathrm{PB}=\{x\in\mathbb{R}^n,\ y\in\mathbb{R}^p: Ax+Gy\leqslant b,\ 0\leqslant x\leqslant 1\},$$

and let $\mathrm{TB}=\mathrm{PB}\cap\mathbb{Z}^n\times\mathbb{R}^p$. The main result of this section is that every valid inequality for TB is a $\mathscr{D}$-inequality with respect to PB.

This result leads to a constructive proof of the superadditive duality theorem for feasible 0–1 mixed integer programs.

Proposition 8. *If* TB $\neq \emptyset$ *and if* $\pi x + \mu y \leqslant \pi_0$ *is a valid inequality for* TB, *there exists an* $\omega \geqslant 0$ *such that for all partitions* N^0, N^1 *of* N, *the inequality*

$$\sum_{j\in N} \pi_j x_j - \sum_{j\in N^0} \omega x_j - \sum_{j\in N^1} \omega(1-x_j) + \mu y \leqslant \pi_0 \tag{10}$$

is valid for PB.

Proof. Since TB $\neq 0$ and $\pi x + \mu y$ is bounded over TB, it also is bounded over PB. Thus it suffices to show that (10) is satisfied for all extreme points of PB. Let $\{x^k, y^k\}$ for $k \in K$ be the extreme points of PB. If $x^k \in \mathbb{Z}^n$, then (10) is satisfied for all $\omega \geqslant 0$ since $\pi x + \mu y \leqslant \pi_0$ is valid for TB. So suppose $x^k \notin \mathbb{Z}^n$. Then since

$$\rho^k = \min_{\{N^0:\, N^0 \cup N^1 = N\}} \left(\sum_{j\in N^0} x_j^k + \sum_{j\in N^1} (1 - x_j^k) \right) > 0$$

and $\pi x + \mu y$ is bounded over PB, it follows that (10) is valid for PB for all suitably large values of ω. □

Theorem 9. *Every valid inequality for* TB $\neq \emptyset$ *is a* $\mathcal{D}$*-inequality.*

Proof. Proposition 8 has established that the inequality (10) is valid for PB for all $N^0 \cup N^1 = N$ and hence is a $\mathcal{D}$-inequality.

Now suppose the inequality (10) is a $\mathcal{D}$-inequality for $(N^0 \cup \{t+1\}, N^1)$ and $(N^0, N^1 \cup \{t+1\})$ where $N^0 \cup N^1 = \{1, \ldots, t\}$. Applying the $\mathcal{D}$-method to these inequalities establishes that (10) is a $\mathcal{D}$-inequality for (N^0, N^1).

Using backward induction on t from $t = n, \ldots, 0$, we obtain that (10) is a $\mathcal{D}$-inequality when $N^0 = N^1 = \emptyset$, i.e. $\pi x + \mu y \leqslant \pi_0$ is a $\mathcal{D}$-inequality as claimed. □

Now we observe that since every application of the $\mathcal{D}$-method can be represented by the superadditive method, it is possible by defining superadditive functions recursively to represent any valid inequality by a superadditive function. First we verify that superadditive functions can be defined recursively in an appropriate manner.

Proposition 10. *Given two superadditive functions* $F_1, F_2 : \mathbb{R}^m \to \mathbb{R}^1$, *then the function* $H : \mathbb{R}^m \to \mathbb{R}^1$ *given by* $H = F_1/(1-\gamma) + F_\gamma(F_2 - F_1)$ *is*

(a) *superadditive,*

(b) *nondecreasing if* F_1 *and* F_2 *are nondecreasing.*

If $\bar{F}_1$ *and* $\bar{F}_2$ *exist and are finite,* $\bar{H} = \min(\bar{F}_1, \bar{F}_2)/(1-\gamma)$.

Proof. By Proposition 2, F_γ is superadditive and nondecreasing. Hence (a) and (b) follow from Proposition 3. It remains to show the form of $\bar{H}$.

Using only the property that when $\bar{F}$ exists $F(d) \geq \bar{F}(d)$ for all d, we first establish that $H(\lambda d)/\lambda \geq \min(\bar{F}_1(d), \bar{F}_2(d))/(1-\gamma)$ for all $\lambda \geq 0$.

$$\begin{aligned}
\frac{H(\lambda d)}{\lambda} &= \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{\lambda} F_\gamma(F_2(\lambda d) - F_1(\lambda d)) \\
&\geq \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{\lambda} \bar{F}_\gamma(F_2(\lambda d) - F_1(\lambda d)) \\
&= \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{1-\gamma}\frac{1}{\lambda} \min(F_2(\lambda d) - F_1(\lambda d), 0) \\
&= \frac{1}{1-\gamma} \min\left(\frac{F_1(\lambda d)}{\lambda}, \frac{F_2(\lambda d)}{\lambda}\right) \\
&\geq \frac{1}{1-\gamma} \min(\bar{F}_1(d), \bar{F}_2(d)).
\end{aligned}$$

Now we show the converse. As $\bar{F}_1$ and $\bar{F}_2$ exist, given d and $\varepsilon > 0$, there exists λ^* such that for all $0 < \lambda \leq \lambda^*$, $F_i(\lambda d)/\lambda \leq \bar{F}_i(d) + \varepsilon$ for $i = 1, 2$. Hence

$$\begin{aligned}
\frac{H(\lambda d)}{\lambda} &= \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{\lambda} F_\gamma(F_2(\lambda d) - F_1(\lambda d)) \\
&\leq \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{\lambda} F_\gamma(\lambda(\bar{F}_2(d) - \bar{F}_1(d) + \varepsilon))
\end{aligned}$$

as F_γ is nondecreasing and $F_1(\lambda d) \geq \bar{F}_1(\lambda d) = \lambda \bar{F}_1(d)$. Now for λ sufficiently small, $F_\gamma(\lambda x) = \min(0, \lambda x)/(1-\gamma)$. So

$$\begin{aligned}
\frac{H(\lambda d)}{\lambda} &\leq \frac{1}{1-\gamma}\frac{F_1(\lambda d)}{\lambda} + \frac{1}{1-\gamma} \min(0, \bar{F}_2(d) - \bar{F}_1(d) + \varepsilon) \\
&= \frac{1}{1-\gamma} \min\left(\frac{F_1(\lambda d)}{\lambda}, \bar{F}_2(d) + \frac{F_1(\lambda d)}{\lambda} - \bar{F}_1(d) + \varepsilon\right) \\
&\leq \frac{1}{1-\gamma} \min\left(\frac{F_1(\lambda d)}{\lambda}, \bar{F}_2(d) + 2\varepsilon\right).
\end{aligned}$$

Now in the limit as $\lambda \downarrow 0$, we have

$$\frac{1}{1-\gamma} \min(\bar{F}_1(d), \bar{F}_2(d)) \leq \lim_{\lambda \downarrow 0_+} \frac{H(\lambda d)}{\lambda} \leq \frac{1}{1-\gamma} \min(\bar{F}_1(d), \bar{F}_2(d)).$$

Hence $\bar{H} = \min(\bar{F}_1, \bar{F}_2)/(1-\gamma)$. □

Now by Propositions 1, 5 and 10 and Theorem 9, we obtain the following.

Theorem 11. *Let*

$$\mathrm{PB}^+ = \{x \in \mathbb{R}^n,\ y \in \mathbb{R}^p_+ : Ax + Gy \leq b,\ 0 \leq x \leq 1\}$$

and

$$\mathrm{TB}^+ = \mathrm{PB}^+ \cap \{x \in \mathbb{Z}^n, y \in \mathbb{R}^p\} \neq \emptyset.$$

Then every valid inequality $\pi x + \mu y \leq \pi_0$ *for* TB *is equal to or dominated by an* *inequality of the form*

$$\sum_{j\in N} F(a_j)x_j + \sum_{j\in J} \bar{F}(g_j)y_j \leq F(b).$$

Proof. We use the proof of Theorem 9. Suppose F_1 is a superadditive function producing an inequality equal to or dominating the $(N^0\cup\{k\}, N^1)$ inequality (10), and F_2 produces an inequality equal to or dominating the $(N^0, N^1\cup\{k\})$ inequality (10). Writing the former as $cx+hy-\omega x_k \leq c_0$ and the latter as $cx+hy+\omega x_k\leq c_0+\omega$, it follows from Propositions 1, 5 and 10 and the fact that F_1, F_2 and $H_{1/2}$ are nondecreasing, that the inequality produced by $H=\omega H_{1/2}(F_1/(2\omega), F_2/(2\omega))$ is equal to or dominates $cx+hy\leq c_0$, or the (N^0, N^1) inequality (10).

Moreover, for all (N^0, N^1) with $N^0\cup N^1=N$, we know from Proposition 8 that there exist nonengative dual variables generating inequalities which are equal to or dominate (10). In other words, there exists a nondecreasing linear function F_{N^0,N^1} which produces an inequality equal to or dominating (10). The main statement now follows by induction. □

The validity of this result for general mixed integer programs has previously been established by Jeroslow [7] and Johnson [8]. However their proofs are essentially nonconstructive.

It follows from Theorem 11 that the function F can be constructed iteratively using nonnegative linear functions and the function $H_{1/2}$ a finite number of times. Furthermore, as the procedure starts with linear functions and $\bar{H}_{1/2}$ is the minimum of linear functions, the corresponding function $\bar{F}$ is the minimum of a finite number of linear functions and is therefore piecewise linear and concave.

Example.

$$\text{TB}=\{x\in\mathbb{Z}^2,\ y\in\mathbb{R}^2_+ : y_1+y_2\leq 7,\ y_j\leq 5x_j,\ 0\leq x_j\leq 1,\ j=1,2\}.$$

We construct the function representing the valid inequality

$$y_1+y_2-2x_1-2x_2\leq 3.$$

Consider the enumeration tree shown in Figure 2.

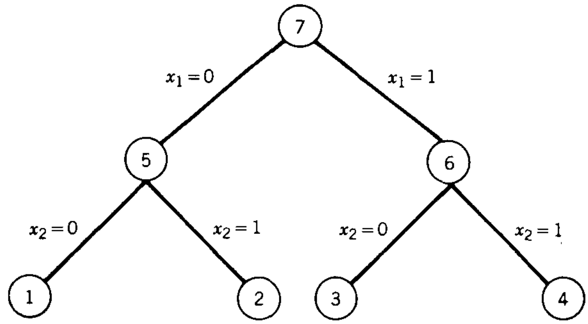

Fig. 2. Enumeration tree.

Let the linear constraints be given in matrix form by

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ -5 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \leqslant \begin{bmatrix} 7 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad x, y \geqslant 0.$$

At each node (N^0, N^1) with $N^0 \cup N^1 = N$, we use a linear function to construct an inequality dominating the inequality

$$-3 \sum_{j \in N^0} x_j - 3 \sum_{j \in N^1} (1 - x_j) - 2x_1 - 2x_2 + y_1 + y_2 \leqslant 3. \tag{11}$$

1. $N^0 = \{1, 2\}$, $N^1 = \emptyset$. $F_1(d) = (0\ 1\ 1\ 0\ 0)d$ gives $-3x_1 - 3x_2 + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 0$,

which is stronger than (11).

2. $N^0 = \{1\}$, $N^1 = \{2\}$. $F_2(d) = (0\ 1\ 1\ 0\ 6)d$ gives $-3x_1 - 3(1 - x_2) + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 3$.

3. $N^0 = \{2\}$, $N^1 = \{1\}$. $F_3(d) = (0\ 1\ 1\ 6\ 0)d$ gives $-3(1 - x_1) - 3x_2 + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 3$.

4. $N^0 = \emptyset$, $N^1 = \{1, 2\}$. $F_4(d) = (1\ 0\ 0\ 1\ 1)d$ gives $-3(1 - x_1) - 3(1 - x_2) + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 3$.

Now to obtain the inequalities that dominate (11) for the sets (N^0, N^1) with $N^0 \cup N^1 = \{1\}$, we combine the superadditive functions generating the above inequalities as explained in the proof of Theorem 11.

Combining the function F_1 generating the $N^0 = \{1, 2\}$ and $N^1 = \emptyset$ inequality and the function F_2 generating the $N^0 = \{1\}$ and $N^1 = \{2\}$ inequality yields

5. $N^0 = \{1\}$, $N^1 = \emptyset$. $F_5 = 3H_{1/2}(\frac{1}{6}F_1, \frac{1}{6}F_2)$ gives $-3x_1 + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 3$.

Combining F_3 and F_4 yields

6. $N^0 = \emptyset$, $N^1 = \{1\}$. $F_6 = 3H_{1/2}(\frac{1}{6}F_3, \frac{1}{6}F_4)$ gives $-3(1 - x_1) + (-2x_1 - 2x_2 + y_1 + y_2) \leqslant 3$.

To obtain the inequality at the root, we combine F_5 and F_6.

7. $N^0 = \emptyset$, $N^1 = \emptyset$. $F_7 = 3H_{1/2}(\frac{1}{6}F_5, \frac{1}{6}F_6)$ gives $-2x_1 - 2x_2 + y_1 + y_2 \leqslant 3$.

5. Conclusions

The results of this paper suggest the possibility that Gomory's mixed integer cutting plane algorithm may be finite for 0–1 mixed integer programs. On the other hand, the recent results of Cook et al. indicate that for arbitrary mixed integer programs the recursive use of the disjunctive method is insufficient to generate all valid inequalities unless it is combined with a discretization step, based on the size of numbers. These remarks motivate the following problems: (a) give a finite cutting plane algorithm for 0-1 mixed integer programs; (b) determine a superadditive function corresponding to the discretization step.

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