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FULL LENGTH PAPER

"Facet" separation with one linear program

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Abstract Given polyhedron P and and a point x^*, the separation problem for polyhedra asks to certify that $x^* \in P$ and if not, to determine an inequality that is satisfied by P and violated by x^*. This problem is repeatedly solved in cutting plane methods for Integer Programming and the quality of the violated inequality is an essential feature in the performance of such methods. In this paper we address the problem of finding efficiently an inequality that is violated by x^* and either defines an improper face or a facet of P. We show that, by solving a single linear program, one almost surely obtains such an improper face or facet.

Keywords Integer programming · Separation problem · Polyhedra · Extended formulations · Facets · Cutting plane algorithm · Split inequalities

Mathematics Subject Classification 90C27 · 90C57

1 Introduction

Given a polyhedron P and a point x^*, the *separation problem* asks to either certify that $x^* \in P$ or to find an inequality $\gamma x \leq \gamma_0$ that is valid for P such that $\gamma x^* > \gamma_0$. The

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separation problem is solved at each iteration of a cutting plane phase of algorithms for pure or mixed integer programming and is therefore crucial for the performance of such methods.

A most important consequence of the Ellipsoid method for linear programming states that, assuming P satisfies some natural conditions, the separation problem can be efficiently solved if and only if one can efficiently optimize a linear function over P. In this paper we focus on the following aspect of this problem:

If $x^* \notin P$, we insist on finding an inequality $\gamma x \leq \gamma_0$ that is violated by x^* and has "strong" geometric properties: Either $\gamma x = \gamma_0$ holds for all $x \in P$ (i.e. $\gamma x \leq \gamma_0$ induces an improper face of P) or $\gamma x \leq \gamma_0$ induces a facet of P.

Of course, the above problem can be efficiently solved by inspection if P is described by an irredundant system of equations and inequalities and this system is small. However we investigate here the case in which either P or the cone $\Gamma(P)$ that contains the vectors (γ, γ_0) that define inequalities that are valid for P is described in an extended space by a system of inequalities that uses additional variables, or in the original space but with redundant inequalities. The former is most often the case for instance, in cutting plane methods based on split or lift-and-project cuts and in Benders' method. There is an extensive literature on the merits and weakness of the cuts generated and the need to find the "correct" normalization (if it exists) when reducing optimization over a cone to a bounded linear program. We believe our result is relevant to such approaches.

We show that with an appropriate normalization of the cone $\Gamma(P)$ the position of P with respect to the origin is crucial to the solution of the above problem with a single LP. Motivated by a simple geometric interpretation, our main result, presented in Sect. 3, is to show that with an appropriate normalization a single LP suffices when the origin is in the relative interior of P whatever the position of x^*. In Sect. 4 we prove a slightly more general version of this result and also argue that the LP almost surely generates a violated facet or an improper face of P. In Sect. 5 we discuss how to find a point in the relative interior of P, again by solving a single LP, and what can happen when the origin is not on the relative interior of P.

In Sect. 6 we illustrate our approach on two examples that arise in Integer Programming: split disjunctions and fixed charge networks. We conclude in Sect. 7 with further remarks and some questions suggested by some preliminary computations.

1.1 Relation to earlier work

Walter [27] in his doctoral dissertation, addresses the problem of finding a facet of polytope P that is violated by a given point outside P. He recognizes the importance of placing the origin in the interior of P in formulating an LP whose basic optimal solution gives a violated facet. He extends this result to the case when P is not full-dimensional.

Balas [2] introduced the concept of disjunctive programming in the early 70's, which studies optimization over the union of polyhedra. Split disjunctions and the lift-and-project approach developed by Balas et al. [4] study optimization over the union of two polyhedra, which in the 0–1 case are two faces of the same polyhedron. More generally, the theory of multi-row cuts derived from lattice-free convex sets introduced by Andersen et al. [1] fits this framework.

Balas and Perregard [5] survey the area of disjunctive programming, and among others discuss the question of different normalizations of the cone of valid inequalities in order to obtain a "good" valid inequality by linear programming for the lift-and-project approach. This question is also addressed in Fischetti et al. [14]. They show that even when $P \subset \mathbb{R}^2$, the inequalities generated by using different normalizations can be very weak. They also propose a different normalization and compare its behavior to some of the alternatives. A core point in the relative interior of P is used in Fischetti and Salvagnin [15] in an attempt to speed up the convergence of the cutting plane algorithm.

Benders' algorithm [7] can be viewed as a separation algorithm in which an extended formulation of P is given explicitly. Magnanti and Wong [19] present a modification designed to generate "pareto-optimal" cuts, see also Papadakos [22]. Their approach is also based on the use of a core point in the relative interior of P. Fischetti et al. [16] discuss the normalization problem as it arises in Benders' algorithm.

We finally point out that the separation problem for convex sets is a central problem in Convex Analysis, see e.g. Chapter A4 in [18] and Convex Programming, see e.g. [8]. In this context, quality of separation is mostly measured in terms of maximizing a given norm.

Cornuéjols and Lemaréchal [12] study the problem of separation of the origin from a closed convex set Q from a convex analysis perspective and apply it to the polyhedral case, most notably to the disjunctive programming case. They point out the relevance of the reverse polar introduced by Balas, that contains the inequalities that are valid for Q and are violated by the origin (the point to be cut off, in this case).

Cadoux [10] takes a convex analysis/geometric approach concerning the depth/ strength of a cut. He shows how a cut that maximizes a given norm can be decomposed into a conic combination of facet-inducing inequalities by solving a series of linear programs.

2 Discussion of the separation problem for polyhedra

In this section we discuss the properties of the cone of valid inequalities $\Gamma(P)$ and the problem of its normalization. We then provide a small instance of the lift-and-project approach indicating what happens when using some standard normalizations.

Given a polyhedron $P \subseteq \mathbb{R}^n$, $\Gamma(P)$, called the the f-cone of P in [11], is the polyhedral cone

$$\Gamma(P) = \left\{ \begin{pmatrix} \gamma \\ \gamma_0 \end{pmatrix} : \gamma x \le \gamma_0 \ \forall x \in P \right\}.$$

(We use basic results from the theory of polyhedra, which may be found e.g. in Ch. 3 of [11]. Given a polyhedron P, we denote by $\dim(P)$, $\mathrm{aff}(P)$, $\mathrm{lin}(P)$, $\mathrm{rec}(P)$, $\mathrm{int}(P)$, $\mathrm{relint}(P)$, $\mathrm{bd}(P)$ its dimension, affine hull, lineality space, recession cone, interior, relative interior and boundary). The following proposition, see e.g. Proposition 5.1, Chapter I.4 in [21], gives a minimal set of generators of $\Gamma(P)$:

Proposition 1 *Let $P \subset \mathbb{R}^n$ be a nonempty polyhedron and let $\gamma^i x = \gamma_0^i, i \in I^=$ and $\gamma^i x \leq \gamma^i, i \in I^<$ be respectively irredundant representations of* aff(P) *and of the set of facets of P. Then*

$$\Gamma(P) = \text{cone}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\gamma^i\\\gamma_0^i\end{pmatrix}, i \in I^<\right) + \text{lin}\left(\begin{pmatrix}\gamma^i\\\gamma_0^i\end{pmatrix}, i \in I^=\right). \tag{1}$$

Furthermore $\begin{pmatrix}\gamma^i\\\gamma_0^i\end{pmatrix}$ $i \in I^<$ and $\begin{pmatrix}\gamma^i\\\gamma_0^i\end{pmatrix}$ $i \in I^=$ are always necessary in the above representation.

The following proposition gives a minimal set of equations and inequalities that defines $\Gamma(P)$.

Proposition 2 *Given a nonempty polyhedron $P \subset \mathbb{R}^n$, let $(b_k, k \in B)$ be a basis of* lin(P)*, $(v_i, i \in V)$ and $(r_j, j \in R)$ be the vertices and extreme rays of the projection of P on the orthogonal complement of* lin(P)*. Then*

$$\Gamma(P) = \left\{\begin{pmatrix}\gamma\\\gamma_0\end{pmatrix} : \gamma v_i - \gamma_0 \leq 0, i \in V;\ \gamma r_j \leq 0, j \in R;\ \gamma b_k = 0, k \in B\right\} \tag{2}$$

Furthermore the above linear system provides an irredundant representation of $\Gamma(P)$.

Remark 1 It follows from Propositions 1 and 2 that $\Gamma(P)$ is a pointed cone if and only if $\dim(P) = n$ and $\dim(\Gamma(P)) = n$ if and only if P is pointed.

Whenever $P \neq \emptyset$ is represented by a linear system, $\Gamma(P)$ can be expressed using the Farkás multipliers that certify the validity for P of an inequality. More precisely, if $P = \{x : \exists y \text{ s.t. } Ax + By \leq d\}$, then

$$\Gamma(P) = \left\{\begin{pmatrix}\gamma\\\gamma_0\end{pmatrix} : \exists u \geq 0 \text{ s.t. } \gamma = uA, 0 = uB, \gamma_0 \geq ud\right\}. \tag{3}$$

In this paper we address the following *separation problem* for P that we (informally) state as follows:

Given a criterion and a point x^*, certify that either $x^* \in P$ or select an inequality $\binom{\gamma}{\gamma_0} \in \Gamma(P)$ with $\gamma x^* > \gamma_0$ that optimizes the given criterion.

Even if this "criterion" is a linear function, this is a challenging problem, first because $\Gamma(P)$ is a cone and also because, in the applications we consider, $\Gamma(P)$ is represented as in (1), (2) or (3). We now elaborate on these points and survey the existing literature.

Since $\Gamma(P)$ is a cone, the maximum of a linear function over $\Gamma(P)$ is either 0 or ∞. Usually this problem is overcome by *normalizing* $\Gamma(P)$, i.e. adding inequalities that make sure the given objective is bounded. Ideally, one would like to *truncate* $\Gamma(P)$:

that is add a set of inequalities that transforms $\Gamma(P)$ into a polytope. A most desirable truncation is one using a single inequality. Note that such a truncation exists if and only if $\Gamma(P)$ is a pointed cone.

When $\Gamma(P)$ is truncated by adding a single inequality, the set of extreme rays of $\Gamma(P)$ is in one-to-one correspondence with the newly created vertices, and this is a desirable feature. However, the added inequality introduces a ranking of the vertices that determines which one optimizes the objective. Indeed one can choose any extreme ray r of $\Gamma(P)$ and construct a truncation so that the vertex that corresponds to r is optimal with respect to the given linear criterion.

There is a further, important issue: the truncation with a single inequality is usually performed on a cone C that projects onto $\Gamma(P)$, such as the cone described by the inequalities in (3). Since $\Gamma(P)$ is the projection of C in the (γ, γ_0)-space, even if C is truncated with a single inequality, the projected truncation of C does not usually result in the truncation of $\Gamma(P)$ with a single inequality and a ray that is extreme in C, but whose projection is not extreme in $\Gamma(P)$, may be truncated to a vertex of the projected polyhedron that optimizes a given linear objective function, but whose corresponding inequality is very weak.

Finally, even if P is represented by a linear system with few inequalities, P may have exponentially many vertices and extreme rays. So the size of the representation of $\Gamma(P)$ in (2) is too large.

In this paper we address the following criterion:

When $x^* \notin P$, select an inequality $\binom{\gamma}{\gamma_0} \in \Gamma(P)$ with $\gamma x^* > \gamma_0$ that defines an improper face or a facet of P.

The importance of finding violated facets has been recognized by various authors. Cadoux's approach, cited above, involves the solution of a series of linear programs and Cornuéjols and Lemaréchal present an approach due to Bonami [9] that also involves solving several linear programs. Padberg et al. in their work on 0–1 integer programs and on combinatorial optimization problems, such as the travelling salesman problem, argued strongly for the importance of violated facet-defining inequalities, see for example [13,23].

2.1 Disjunctive programming, lift-and-project, split inequalities

We illustrate the above questions on a problem that is of importance in integer programming, see e.g. [4], or Chapter 5 in [11]. Given polyhedron $P = \{x : Ax \leq b\}$ and $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$, we define

$$P^0 := P \cap \{x : \pi x \leq \pi_0\} \quad P^1 := P \cap \{x : \pi x \geq \pi_0 + 1\}.$$

Let $P^{(\pi,\pi_0)} := conv(P^0 \cup P^1)$. Then $P^{(\pi,\pi_0)} \subseteq P$ and $P \cap \mathbb{Z}^n = P^{(\pi,\pi_0)} \cap \mathbb{Z}^n$. Furthermore (assuming P pointed) if x^* is a vertex of P and $\pi_0 < \pi x^* < \pi_0 + 1$, then $x^* \notin P^{(\pi,\pi_0)}$. In this case $P^{(\pi,\pi_0)}$ is a better approximation than P of $conv(P \cap \mathbb{Z}^n)$. In the lift-and-project method with $P \subset [0, 1]^n$, π is a unit vector and $\pi_0 = 0$.

By Lemma 4.45 in [11], $P^{(\pi,\pi_0)}$ is a polyhedron which can be obtained as the projection in the x-space of the polyhedron defined by the following system:

$$\begin{aligned} Ax^0 &\le b\lambda \\ \pi x^0 &\le \pi_0\lambda \\ Ax^1 &\le b(1-\lambda) \\ \pi x^1 &\ge (\pi_0+1)(1-\lambda) \\ x^0+x^1 &= x \\ 0 \le \lambda &\le 1 \end{aligned} \tag{4}$$

An inequality is valid for $P^{(\pi,\pi_0)}$ if and only if it is valid for both P^0 and P^1, so assuming that P^0, P^1 are both nonempty, let C be the cone defined by the following system:

$$\begin{aligned} uA + u_0\pi &= \gamma \\ vA - v_0\pi &= \gamma \\ ub + u_0\pi_0 &\le \gamma_0 \\ vb - v_0(\pi_0+1) &\le \gamma_0 \\ u, u_0, v, v_0 &\ge 0 \end{aligned} \tag{5}$$

Then by (3), we have that $\Gamma(P^{(\pi,\pi_0)}) = \{(\gamma,\gamma_0) : \exists(u,u_0,v,v_0) \text{ s.t. } (u,u_0,v,v_0,\gamma,\gamma_0) \in C\}$.

Since $\max\{x^*\gamma - \gamma_0 : (u,u_0,v,v_0,\gamma,\gamma_0) \in C\} = \infty$ if and only if $x^* \notin P^{(\pi,\pi_0)}$, some normalizations of C have been introduced and studied in the literature.

- In the context of lift-and-project, Balas et al. [4] present three normalizations $\gamma_0 \in \{-1,1\}$, $-1 \le \gamma_i \le 1$ for $i \in 1,\dots,m$ and $\sum_{i=1}^m |\gamma_i| \le 1$. Citing Balas [2,3], they observe that with the normalization $\gamma_0 \in \{-1,1\}$ the separation problem over the polyhedron (5) is finite valued if and only if $\lambda x^* \in P^{(\pi,\pi_0)}$ for some $\lambda > 0$ and use the other two normalizations in their computational study.
- Balas and Perregaard [6] mention $\gamma_0 \in \{-1,1\}$ as an effective normalization for covering and packing problems. The normalization $\gamma_0 \in [-1,1]$ is also studied and will be important in this paper. $\sum_{i=1}^n |\gamma_i| = 1$ is a normalization that needs the 2^n inequalities that define the octahedron and may introduce additional vertices. They also introduce a generic normalization with a single inequality $a\gamma \le b$ and mention that when P is full-dimensional and the separation LP produces a finite solution which is extreme, then a violated facet is produced.

- $u_0+v_0 = 1$ Balas and Perregaard [5,6] show that finding a most violated inequality (i.e. one that maximizes the objective function) can be reformulated as an LP on n variables and corresponds to a (possibly infeasible) basis of the system $Ax \le b$ defining P, see also [11] Ch.5.

  Fischetti et al. (Theorem 2 in [14]) show that if x^* is a vertex of P, a most violated inequality can be read from a basis defining x^*.

- $\mathbf{1}u + u_0 + \mathbf{1}v + v_0 = 1$ This normalization produces a truncation of C and has been introduced by Balas and has been studied by Balas and Perregaard among others. Fischetti et al. report better computational results than for the previous normalization. However, the performance is obviously tied to the scaling of the constraints defining P.

Although a cut-generating LP is solved, the cut obtained fails to have strong geometric properties. It is neither an implicit equation nor a facet of $P^{(\pi,\pi_0)}$. To illustrate this point, the following 2-variable example is taken from Fischetti et al. [14].

Example 1 The problem is $\max\{x_1 + 2x_2 : x \in P \cap \mathbb{Z}^2\}$ where $P \subset [0, 1]^2$ is the polytope:

$$\begin{aligned} -4x_1 + 4x_2 &\leq 2 \\ 2x_1 + 2x_2 &\leq 3 \\ -8x_1 + 4x_2 &\leq 1 \\ x_1 &\leq 1 \\ x_2 &\leq 1 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0. \end{aligned}$$

The unique optimal LP solution is $x^* = (0.5, 1)$. Taking $(\pi, \pi_0) = ((1, 0), 0)$, namely the disjunction $x_1 \leq 0$ or $x_1 \geq 1$, one obtains $P_0 = \{x : x_1 = 0, 0 \leq x_2 \leq 0.25\}$ and $P_1 = \{x : x_1 = 1, 0 \leq x_2 \leq 0.5\}$ and $P^{(\pi,\pi_0)} = \{x : 0 \leq x_1 \leq 1, x_2 \geq 0, -x_1 + 4x_2 \leq 1\}$.

Solving the cut generating LP $\max\{\gamma x^* - \gamma_0 : (5)\}$ with either the normalization $\mathbf{1}u + u_0 + \mathbf{1}v + v_0 = 1$ or $u_0 + v_0 = 1$ leads to the violated inequality $x_2 \leq 0.5$. If the constraint $x_2 \leq 1$ is rescaled as $kx_2 \leq k$ with $k \geq 8$ both normalizations give the violated inequality $-x_1 + 2x_2 \leq 1$. Neither of these inequalities defines a facet of $P^{(\pi,\pi_0)}$.

On the other hand we will show below why, given the relative positions of P, 0 and the point x^* to be cut off, using the normalization $-1 \leq \gamma_0 \leq 1$, the cut generation LP produces the only facet $-x_1 + 4x_2 \leq 1$ of $P^{(\pi,\pi_0)}$ that is violated by x^*.

3 Outline of the main results

We summarize here the main results in this paper. Given polyhedron P and x^*, we consider the "separation LP"

$$\max\{x^*\gamma - \gamma_0 \text{ s.t. } (\gamma, \gamma_0) \in \Gamma(P), -1 \leq \gamma_0 \leq 1\}$$

whose "dual LP" is the 1-dimensional LP

$$\min\{|\psi - 1| \text{ s.t. } Ax^* \leq b\psi,\ \psi \geq 0\}$$

where $Ax \leq b$ is a system defining P. In Propositions 4 and 5 below we characterize feasibility, boundedness and the optimal solutions of the above LPs.

If P contains the origin 0 in its relative interior and $\binom{\gamma}{\gamma_0} \in \Gamma(P)$, then $\gamma_0 \geq 0$. Furthermore $\gamma_0 = 0$ if and only if $\gamma x = \gamma_0$ for all $x \in P$ and $\gamma_0 > 0$ if $\gamma x < \gamma_0$ for some $x \in P$.

Therefore given x^*, the above "separation LP" is unbounded (and the "dual LP" is infeasible) if and only if x^* is not in affine hull of P. Any unbounded ray yields an improper face of P whose corresponding hyperplane does not contain x^*.

In this case, the "dual LP" can be interpreted as follows: If x^* lies in the affine hull of P, then the segment S connecting 0 to x^* traverses the boundary of P in a point y^* that is in the relative interior of S and also in the relative interior of a proper face f of P and almost surely f is a facet of P. This is Theorem 1 and the "almost surely" is argued in Proposition 6. This is illustrated in Fig. 1.

Getting 0 in the relative interior of P (with a linear transformation) is crucial to the above approach. In Sect. 5.1 we discuss how to achieve this in an efficient manner and in Sect. 5.2 we look at alternative strategies for the placement of 0.

4 Analysis of the separation LPs

Given polyhedron P, let $\Gamma^{<>}(P) = \{\binom{\gamma}{\gamma_0} \in \Gamma(P), -1 \leq \gamma_0 \leq 1\}$ and given x^*, consider the LP

$$\begin{aligned} \zeta = \max\ & x^*\gamma - \gamma_0 \\ & \begin{pmatrix} \gamma \\ \gamma_0 \end{pmatrix} \in \Gamma^{<>}(P). \end{aligned} \tag{6}$$

Let $Ax \leq b$ be a system of inequalities defining P . Then, using (3), we have that program (6) is equivalent to the following:

$$\begin{aligned} \zeta = \max\ & x^*\gamma - \gamma_0 \\ & uA - \gamma = 0 \\ & ub - \gamma_0 \leq 0 \\ & -1 \leq \gamma_0 \leq 1 \\ & u \geq 0. \end{aligned} \tag{7}$$

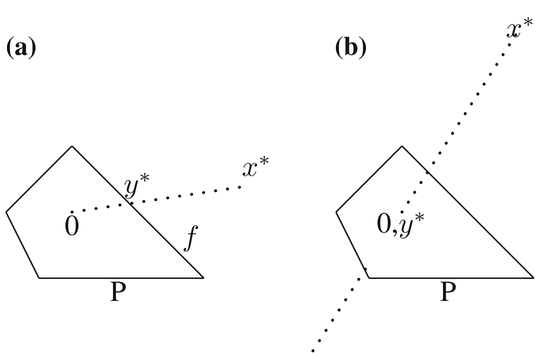

Fig. 1 **a** Line joining the origin and x^* intersects boundary(P) in y^*, **b** $x^* \notin \mathrm{aff}(P)$. The line hits $\mathrm{aff}(P)$ at $y^* = 0$

The dual of (7) is

$$\begin{aligned} \min |\psi - 1| \\ Ax^* \leq b\psi \\ \psi \geq 0. \end{aligned} \tag{8}$$

Since the LP (7) is always feasible, it admits a finite optimum if and only if (8) is feasible. We characterize when this happens. We assume P is nonempty, and consider the following cones.

- $hom^e(P) := \{(x, \psi) : \ Ax \leq b\psi, \ \psi \geq 0\}$.
- $cone^e(P) := \{(x, \psi) : \exists z \in P \text{ s.t. } x = \psi z, \ \psi \geq 0\}$.

Proposition 3 *Given a nonempty polyhedron P, let $hom^e(P)$, $cone^e(P)$ be the cones defined above:*

1. *$cone^e(P) \subseteq hom^e(P)$ and $hom^e(P) \setminus cone^e(P) = (\mathrm{rec}(P), 0) \setminus (0, 0)$*
2. *$(\mathrm{rec}(P), 0)$ is the face of $hom^e(P)$ induced by $\psi = 0$*
3. *$hom^e(P) = cl(cone^e(P))$, where $cl(\cdot)$ denotes the closure operator.*

Proof Let $(x, \psi) \in cone^e(P)$. Then $x = \psi z$, where $\psi \geq 0$ and $Az \leq b$. Therefore $Ax \leq b\psi$ and this shows $cone^e(P) \subseteq hom^e(P)$. Let now $(x, \psi) \in hom^e(P) \setminus cone^e(P)$. Then $\psi = 0$, since otherwise $z := \frac{x}{\psi} \in P$ and $x = \psi z$. Furthermore $x \neq 0$, since otherwise $P \neq \emptyset$, $x = 0z$ for any $z \in P$. This shows that $(x, 0) \in (\mathrm{rec}(P), 0) \setminus (0, 0)$ and this proves 1.

Since $P \neq \emptyset$, $(\mathrm{rec}(P), 0) = (\{r : \ Ar \leq 0\}, 0)$ and this proves 2.

By 1. and 2., we have that $cone^e(P) \subseteq hom^e(P)$ and $hom^e(P) \setminus cone^e(P) \subseteq \mathrm{bd}(hom^e(P))$. Since $hom^e(P)$ is a closed set, this proves 3. □

Corollary 1 *Given a nonempty polyhedron P, let*

$$hom(P) := \{x : \ \exists \psi \ s.t. \ (x, \psi) \in hom^e(P)\},$$
$$cone(P) := \{x : \ \exists \psi \ s.t. \ (x, \psi) \in cone^e(P)\}.$$

Then $hom(P) = cone(P) \cup \mathrm{rec}(P)$.

Proposition 4 *Given nonempty polyhedron P, the LP (8) is feasible if and only if $x^* \in hom(P)$. Let ψ^* be an optimal solution of (8). Then*

1. *$\psi^* = 1$ if and only if $x^* \in P$*
2. *$\psi^* = 0$ if and only if $x^* \notin P$ and $x^* \in (\mathrm{rec}(P) \setminus cone(P)) \cup \{0\}$*
3. *$\psi^* \neq 0, 1$ if and only if $x^* \in cone(P) \setminus (P \cup \{0\})$ and:*
   - *If $\{\frac{x^*}{\psi}, \psi > 1\} \cap P \neq \emptyset$, then $1 < \psi^* = \min\{\psi : \ \frac{x^*}{\psi} \in P\}$*
   - *If $\{\frac{x^*}{\psi}, \psi < 1\} \cap P \neq \emptyset$, then $1 > \psi^* = \max\{\psi : \ \frac{x^*}{\psi} \in P\}$.*

Proof By Corollary 1, (8) is feasible if and only if $x^* \in \mathrm{hom}(P)$.

Now 1. is immediate, so we assume $x^* \notin P$. If $x^* \in (\mathrm{rec}(P) \setminus cone(P)) \cup \{0\}$, since $x^* \notin P$, by Proposition 3, $\psi = 0$ is the only value for which $(x^*, \psi) \in hom(P)$ and this proves 2.

Assume finally $x^* \in cone(P) \setminus (P \cup \{0\})$. Then $\psi^* \neq 0, 1$. Therefore exactly one of $\{\frac{x^*}{\psi}, \psi < 1\} \cap P$, $\{\frac{x^*}{\psi}, \psi > 1\} \cap P$ is nonempty and we have 3. □

Proposition 5 *Assume* (8) *admits a finite optimimum* $\psi^* \neq 0, 1$. *Then* $y^* := \frac{x^*}{\psi^*} \in bd(P)$ *and* $F := \{\binom{\gamma}{\gamma_0} \in \Gamma^{<>}(P) : \gamma y^* = \gamma_0\}$ *is the optimal face of the LP* (7).

Furthermore $\gamma_0 = 1$ *for every* $\binom{\gamma}{\gamma_0} \in F$ *if* $\psi^* > 1$ *and* $\gamma_0 = -1$ *for every* $\binom{\gamma}{\gamma_0} \in F$ *if* $\psi^* < 1$.

Proof By Proposition 4 , (8) admits a finite optimum $\psi^* \neq 0, 1$ if and only if $x^* \in cone(P) \setminus (P \cup \{0\})$, and case 3. of the same proposition applies. Let ψ^* be the optimal value of ψ in (8) and let

$$F := \left\{ \begin{pmatrix} \gamma \\ \gamma_0 \end{pmatrix} \in \Gamma^{<>}(P) : \gamma x^* = \gamma_0 + |\psi^* - 1| \right\}$$

be the optimal face of the LP (7). Since $max\{0, \psi^* - 1\}$ and $max\{0, -(\psi^* - 1)\}$ are the optimal values taken by the dual variables associated with constraints $\gamma_0 \le 1$ and $\gamma_0 \ge -1$ in (7) respectively, then complementary slackness shows that $\gamma_0 = 1$ for every $\begin{pmatrix} \gamma \\ \gamma_0 \end{pmatrix} \in F$ when $\psi^* > 1$ and $\gamma_0 = -1$ for every $\binom{\gamma}{\gamma_0} \in F$ when $\psi^* < 1$.

Assume $\psi^* > 1$. Then $\gamma x^* = \gamma_0 + \psi^* - 1$ for every $\binom{\gamma}{\gamma_0} \in F$ and $\gamma x^* < \gamma_0 + \psi^* - 1$ for every $\binom{\gamma}{\gamma_0} \in \Gamma^{<>}(P) \setminus F$. Since $\gamma_0 = 1$ for every $\binom{\gamma}{\gamma_0} \in F$ and $x^* = \psi^* y^*$, then $F := \{\binom{\gamma}{\gamma_0} \in \Gamma^{<>}(P) : \gamma y^* = \gamma_0\}$. When $\psi^* < 1$, the proof is the same. □

Now we state and prove our main result.

Theorem 1 *If* $0 \in \text{relint}(P)$, *then:*

1. *The LP* (8) *is infeasible if and only if* $x^* \notin \text{aff}(P)$. *In this case, let* $\binom{\gamma}{\gamma_0}$ *be an unbounded ray of* (7). *Then* $\gamma_0 = 0$, $\gamma x^* > 0$ *and* $\gamma x = 0$ *for all* $x \in P$.
2. *If the LP* (8) *is feasible, let* ψ^* *be a optimal solution of* (8). $\psi^* = 1$ *if and only if* $x^* \in P$ *and* $\psi^* > 1$ *if and only if* $x^* \in \text{aff}(P) \setminus P$. *In this latter case, let* $F := \{\binom{\gamma}{\gamma_0} \in \Gamma^{<>}(P) : \gamma x^* = \gamma_0 + \psi^* - 1\}$ *be the optimal face of the LP* (7) *and let* $y^* = \frac{x^*}{\psi^*}$. *Then* $y^* \in bd(P)$ *and* $F := \{\binom{\gamma}{1} \in \Gamma^{<>}(P) : \gamma y^* = 1\}$

Proof Since $0 \in \text{relint}(P)$, we have that $\text{aff}(P) = hom(P)$. Therefore Proposition 4 shows that (8) is infeasible (and (7) is unbounded) if and only if $x^* \notin \text{aff}(P)$. Assume this is the case and let $\binom{\gamma}{\gamma_0}$ an unbounded ray of $\Gamma^{<>}(P)$. Then $\gamma x^* > \gamma_0$ and

$\binom{\lambda\gamma}{\lambda\gamma_0} \in \Gamma^{<>}(P)$ for every $\lambda > 0$. Therefore $\gamma_0 = 0$. Since $0 \in \text{relint}(P)$ and $\gamma 0 = 0$, then $\gamma x = 0$ for all $x \in P$. This proves 1.

Assume now that (8) is feasible, i.e. $x^* \in \text{aff}(P)$. By Proposition 4 $\psi^* = 1$ iff $x^* \in P$. Assume $x^* \in \text{aff}(P) \setminus P$. Since $0 \in \text{relint}(P)$, then $\text{rec}(P) \subseteq P$ and since $x^* \notin P$, $\psi^* > 0$. Therefore case 3. of Proposition 4 applies and again since $0 \in \text{relint}(P)$, we must have $\psi^* > 1$. Therefore $|\psi^* - 1| = \psi^* - 1$. Now 2. follows from Proposition 5. □

4.1 Obtaining a facet-inducing inequality almost surely

Proposition 6 *When the LP* (8) *admits a finite optimum* $\psi^* \neq 0, 1$, *then an optimal solution of the LP* (7) *almost surely defines a facet-inducing inequality when* $x^* \in \text{aff}(P) \setminus P$ *and almost surely defines an improper face of P when* $x^* \notin \text{aff}(P)$.

Proof By Proposition 4 we have that (8) admits a finite optimum and $\psi^* \neq 0, 1$ if and only if $x^* \in cone(P) \setminus (P \cup \{0\})$, so case 3. of the same proposition applies. Let $y^* := \frac{x^*}{\psi^*}$. Since for the Lebesgue measure restricted to the boundary of P the set of all faces with dimension at most $dim(P)-2$ is negligible when P is full-dimensional, and the set of all faces with dimension at most $dim(P) - 1$ is negligible when P is not full-dimensional, a random point on the boundary of P almost surely lies in the relative interior of a facet of P when P is full-dimensional and lies in the relative interior of P when P is not full-dimensional.

By Proposition 5 $\{\binom{\gamma}{\gamma_0} \in \Gamma^{<>}(P) : \gamma y^* = \gamma_0\}$ is the optimal face of the LP (7). Therefore when $x^* \notin \text{aff}(P) \cup \{0\}$, almost surely $y^* \in \text{relint}(P)$ and F contains vectors $\binom{\gamma}{\gamma_0}$ that represent improper faces of P that are violated by x^*.

When $x^* \in \text{aff}(P) \setminus (P \cup \{0\})$, we may assume P full-dimensional. Almost surely $y^* \in \text{relint}(f)$, where f is a facet of P and in this case F contains all vectors $\binom{\gamma}{\gamma_0}$ that represent f. □

Remark 2 When the description $\{x : Ax \leq b\}$ of P contains an inequality $a^i x \leq b_i$ that is not facet-defining, there exist points x^* for which at least one of the optimal solutions of the separation LP (7) gives the face $f = \{x \in P : a^i x = b_i\}$, that is not a facet of P.

Remark 3 Walter (Theorem 2.5.4 in [27]) provides an LP whose basic optimal solution gives a violated inequality that induces a facet or an improper face.

5 Further aspects

Here we discuss how to obtain a point $\hat{x} \in \text{relint}(P)$. We then consider what happens using the separation LPs when the origin does not lie in $\text{relint}(P)$.

5.1 Computing $\mathrm{aff}(P)$ and $x \in \mathrm{relint}(P)$

Theorem 1 requires that $0 \in \mathrm{relint}(P)$. This poses the following question:

Let $Ax \leq b$ be a system of inequalities that defines polyhedron P. Determine whether $P = \emptyset$ and if $P \neq \emptyset$, compute $\mathrm{aff}(P)$ and $\hat{x} \in \mathrm{relint}(P)$.

When $\hat{x} \in \mathrm{relint}(P)$ is found, we can apply a linear transformation that maps $\hat{x}$ into 0. Freund, Roundy and Todd [17] show that this question can be answered by solving a single LP:

Proposition 7 *Given $P = \{x \in R^n : Ax \leq b\}$, consider the following LP:*

$$\begin{aligned} \max \quad & \mathbf{1}t \\ & Ay + t \leq b\lambda \\ 0 \leq{} & t \leq \mathbf{1} \\ & \lambda \geq 1 \end{aligned} \tag{9}$$

1. *The LP (9) is bounded and is feasible if and only if $P \neq \emptyset$.*
2. *Let $\begin{pmatrix} \hat{y} \\ \hat{t} \\ \hat{\lambda} \end{pmatrix}$ be an optimal solution to (9). Then $\hat{x} := \frac{\hat{y}}{\hat{\lambda}} \in \mathrm{relint}(P)$, and $\hat{t} \in \{0, 1\}^m$.*

   Furthermore the i^{th} constraint in the system $Ax \leq b$ defines an improper face of P if and only if $\hat{t}_i = 0$.

Proof 1. is straightforward. We prove 2. Let $\begin{pmatrix} \hat{y} \\ \hat{t} \\ \hat{\lambda} \end{pmatrix}$ be an optimal solution to (9). Since $\hat{t} \geq 0$, we have that $\hat{x} := \frac{\hat{y}}{\hat{\lambda}} \in P$ and since $\mathbf{1}t$ is maximized, then $\hat{t}_i = 0$ if and only if $\hat{x}$ satisfies the ith constraint in $Ax \leq b$ at equality. Furthermore, since λ is unbounded from above and $0 \leq t \leq \mathbf{1}$, then $\hat{t}_i = 1$ if and only if $\hat{x}$ satisfies the ith constraint with strict inequality. Therefore the LP (9) finds a point $\hat{x} \in P$ that satisfies strictly the largest number of inequalities in $Ax \leq b$, namely a point in $\mathrm{relint}(P)$. □

There are alternative linear programming methods that solve the above problem. These methods typically find a strictly complementary solution or need the computation of a tolerance ε that depends on the system defining P, see e.g. [20]. The LP (9) avoids these issues.

Remark 4 If $P := proj_x(Q)$ and $Q := \{(x, y) : Ax + By \leq d\}$, then applying the LP (9) to the system defining Q, one gets a point $(\hat{x}, \hat{y}) \in \mathrm{relint}(Q)$ and a system $A^= x + B^= y = d^=$ of equations defining $\mathrm{aff}(Q)$. Then $\hat{x} \in \mathrm{relint}(P)$ and $\mathrm{aff}(P) = \{x : uA^= x = ud^= \ \forall u \in N(B)\}$, where $N(B)$ is a basis of the space $\{u : uB = 0\}$.

Furthermore given x^*, one can decide if $x^* \in \mathrm{aff}(P)$ or find an improper face of P that is violated by x^* by checking $uA^= x^* = ud^= \ \forall u \in N(B)$. Therefore by solving a single LP to determine $\mathrm{aff}(Q)$ and eventually finding a basis of a linear space (when $Q \neq P$) one can check whether $x^* \in \mathrm{aff}(P)$.

5.2 Keeping the origin, or placing 0 outside relint(P)

Here we consider what happens if one maintains the normalization $(\gamma, \gamma_0) \in \Gamma^{<>}(P)$ but does not impose $0 \in \text{relint}(P)$. This may arise when either one does not wish to change the origin or some other choice appears appropriate, thus we are interested in sufficient conditions that guarantee that the LP (8) is feasible and $\psi^* \neq 0, 1$.

Given polyhedron P and x^*, the *penumbra* of P with respect to x^* is the set

$$Sh(P, x^*) = \{x^* + \lambda(P - x^*),\ \lambda \geq 1\}. \tag{10}$$

The *reverse cone* of P with respect to x^* is the set

$$Rcone(P, x^*) = \{x^* - \lambda(P - x^*),\ \lambda \geq 0\}. \tag{11}$$

Proposition 8 *Given a nonempty polyhedron P and $x^* \notin P$, if $0 \in \text{relint}(Sh(P, x^*)) \bigcup \text{relint}((Rcone(P, x^*))$, then the LP (8) is feasible and $\psi^* \neq 0, 1$.*

Proof Assume $0 \in \text{relint}(Sh(P, x^*))$. Then $0 = x^* + \lambda(y - x^*)$ where $y \in P$ and $\lambda > 1$. Let $\psi := \frac{\lambda}{\lambda - 1}$. Then $\psi > 1$ and $y = \frac{x^*}{\psi} \in P$. Therefore $x^* \in \text{cone}(P)$ and, since $x^* \notin P \cup \{0\}$, case 3. of Proposition 4 applies. When $0 \in \text{relint}(Rcone(P, x^*))$, the proof is similar. $\square$

In Fig. 2 we show the behaviour of the separation LP when $0 \notin \text{relint}(P)$. In Fig. 3, we illustrate the different regions arising in Proposition 8 and the behavior of the LP (8) as a function of the position of 0.

Note that the definition of penumbra and reverse cone depend on the relative position of x^* with respect to P and Proposition 8 requires $0 \in \text{relint}(Sh(P, x^*)) \bigcup \text{relint}((Rcone(P, x^*))$. On the other hand Theorem 1 shows that when $0 \in \text{relint}(P)$ and $x^* \notin P$, the separation LP (6) almost surely returns an inequality that is violated by x^* and it either induces a facet or an improper face of P.

Returning to Example 1, remark that $0 \in \text{relint}(Sh(P, x^*))$. The line from 0 to x^* traverses the boundary of P in the interior of the facet $-x_1 + 4x_2 \leq 1$ of $P^{(\pi, \pi_0)}$, so by Proposition 8, $\psi^* > 1$ and the optimal solution of the LP (6) generates this facet.

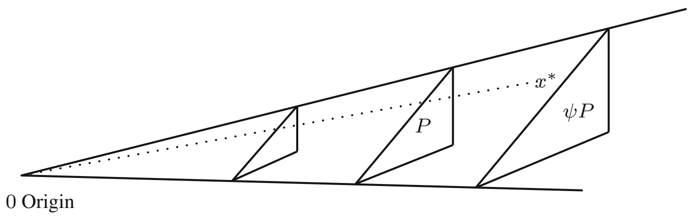

Fig. 2 Scaling P to include x^*. The case when $0 \notin P$, but $x^* \in cone(P)$

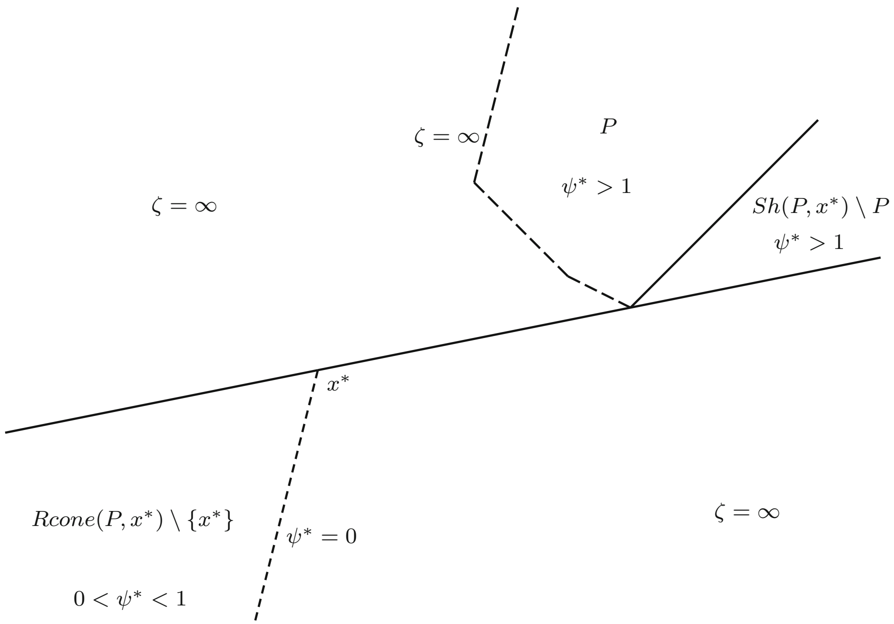

Fig. 3 Behaviour of the separation LP (8) as a function of the position of the origin ($\zeta = \infty$ implies infeasible)

6 Applications in integer programming

Here we present the potential interest of our approach in cutting plane methods for solving structured and unstructured integer programs.

- For 0–1 integer or mixed integer programs, one approach is to use cutting planes based on split disjunctions (or more generally unions of polyhedra) of which an example was presented above.
- For uncapacitated fixed charge network flow problems, there is multi-commodity extended formulation that can be used to generate cutting planes in the space of the original arc variables.

For both applications we consider the form taken by the separation LP (6) and present a small example.

6.1 Split disjunctions

Given polyhedron $P = \{x : Ax \le b\}$ and $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, let P^0, P^1 and $P^{(\pi,\pi_0)}$ be defined as in Sect. 2.1. Given $x^* \notin P^{(\pi,\pi_0)}$, when $0 \in \text{relint}(P^{(\pi,\pi_0)})$ by Theorem 1, the LP (6) returns an inequality that almost surely either induces an improper face or a facet of $P^{(\pi,\pi_0)}$.

As this problem is repeatedly solved in a cutting plane algorithm, and at each iteration P and (π, π_0) change, one wants to find a point $\hat{x}$ that can be mapped into 0 and does not depend on the current P and (π, π_0), so that the LP (9) does not have to be solved repeatedly.

When $conv(P \cap \mathbb{Z}^n)$ is not a face of P, the ideal choice is $\hat{x} \in \text{relint}(conv(P \cap \mathbb{Z}^n))$ and for some structured problems such a point can be found by inspection, without solving the LP (9). For instance, this is easily achieved for a polytope P of the submissive type, such as the knapsack polytope. As 0 and the n unit vectors typically belong to $P \cap \mathbb{Z}^n$, we have that $\hat{x} := (\frac{1}{n}, \ldots, \frac{1}{n}) \in \text{relint}(conv(P \cap \mathbb{Z}^n))$.

When such a point $\hat{x}$ is found and mapped into 0, the separation LP (6) for $P^{(\pi,\pi_0)}$ becomes

$$\begin{aligned} \zeta = \max \gamma(x^* - \hat{x}) &- \gamma_0 \\ uA + u_0\pi &= \gamma \\ vA - v_0\pi &= \gamma \\ u(b - A\hat{x}) + u_0(\pi_0 - \pi\hat{x}) &\le \gamma_0 \\ v(b - A\hat{x}) - v_0(\pi_0 + 1 - \pi\hat{x}) &\le \gamma_0 \\ -1 \le \gamma_0 &\le 1 \\ u, u_0, v, v_0 &\ge 0 \end{aligned}$$

This LP is used to treat the following example.

Example 2 (Example 2 from Fischetti et al. [14]).
$\max\{x_1 + 2x_2 : x \in P \cap \mathbb{Z}^2\}$ where $P \subset [0, 1]^2$ is the polytope:

$$\begin{aligned} -2x_1 + 2x_2 &\le 1 \\ 2x_1 + 2x_2 &\le 3 \\ -4x_1 - 4x_2 &\le -3 \\ x_1 &\le 1 \\ x_2 &\le 1 \\ -x_1 &\le 0 \\ -x_2 &\le 0 \end{aligned}$$

The unique optimal solution of the LP relaxation is $x^* = (0.5, 1)$. Let $(\pi, \pi_0) = ((1, 0), 0)$. In this case P_0 and P_1 are the faces of P defined by the inequalities $x_1 \ge 0$ and $x_1 \le 1$ respectively. Since $P_0 = \emptyset$ and $P_1 = \{x : x_1 = 1, 0 \le x_2 \le 1/2\}$, we have that $P^{(\pi,\pi_0)} = conv(P_0 \cup P_1) = P_1$.

Consider the transformation that maps $\hat{x} = (1, 0.25) \in \text{relint}(P^{(\pi,\pi_0)})$ into 0. Since $(0.5, 1) \notin \text{aff}(P^{(\pi,\pi_0)})$, by Theorem 1, the separation LP (6) is unbounded and the unique unbounded ray produces the inequality $-x_1 \le -1$ that defines $\text{aff}(P^{(\pi,\pi_0)})$.

Consider the penumbra $Sh(P^{(\pi,\pi_0)}, x^*)$ and the linear transformation that maps $\hat{x} = (1.01, 0.25) \in \text{relint}(Sh(P^{(\pi,\pi_0)}, x^*))$ into 0. By Proposition 8, the LP (6) has a finite optimal solution and the optimal face produces the inequality $-x_1 \le -1$. The same happens when $\hat{x} = (0, 1.75) \in \text{relint}(Rcone(P, x^*))$ is mapped into 0.

On the other hand, if 0 is not changed, the LP (6) is unbounded and the unbounded ray produces the inequality $-x_1 + 0.5x_2 \le 0$ that is valid, but not facet defining for $P^{(\pi,\pi_0)}$.

6.2 Multi-commodity reformulation of incapacitated fixed charge network flow

We consider a single source uncapacitated network flow problem for which there is a multicommodity extended formulation, see Rardin and Choe [25]. We present the original and extended formulations, demonstrate in detail how to derive the separation LP (6) and then present a small example indicating the behavior of the facet separation algorithm.

Given a network $D = (V, A)$ where $V = \{0, 1, \dots, n\}$ and $A = \{1, \dots, m\}$, source node 0 and demands $b_i \geq 0$ for $i \in V \setminus \{0\}$ where $b_0 = -\sum_{i \in V \setminus \{0\}} b_i$, and for each $e \in A$ we are given a unit flow cost p_e and a fixed cost f_e per (large capacity M) arc. The Fixed Charge Network Flow Problem is to find a feasible flow of minimum cost (i.e. flow costs plus fixed cost).

The following is a mixed integer programming formulation. We define x_e to be the flow value in arc e, y_e to be the binary variable indicating if arc e carries positive flow.

$$\begin{aligned} \min \textstyle\sum_{e \in A} p_e x_e + \sum_{e \in A} f_e y_e & \\ \textstyle\sum_{e: h(e)=i} x_e - \sum_{e: t(e)=i} x_e &= b_i \;\; i = 1, \dots, n \\ x_e &\leq M y_e \;\; e = 1, \dots, m \\ x &\in \mathbb{R}^m_+, \, y \in \mathbb{Z}^m. \end{aligned} \tag{12}$$

where M is a very large number. Assuming $p_e, f_e \geq 0$ for all $e \in A$, one notes that there is always an optimal solution with $y_e \in \{0, 1\}$ and that the big M can be replaced by $-b_0$.

To obtain the multi-commodity reformulation, let $K = \{i \in V \setminus \{0\} : b_i > 0\}$ be the set of terminals. We represent the flow as the sum of $|K|$ distinct flows from 0 to k for $k \in K$. We define the variable w_e^k to be the flow in arc e with destination node k. it can be shown that the following is an equivalent MIP formulation:

$$\begin{aligned} \min \textstyle\sum_{e \in A} p_e x_e + \sum_{e \in A} f_e y_e & \\ \textstyle\sum_{e: h(e)=i} w_e^k - \sum_{e: t(e)=i} w_e^k &= 0 \;\; i = 1, \dots, n, k \in K, k \neq i \\ \textstyle\sum_{e: h(e)=k} w_e^k - \sum_{e: t(e)=k} w_e^k &= b_k \;\; k \in K \\ w_e^k &\leq b_k y_e \;\; e = 1, \dots, m, k \in K \\ \textstyle\sum_k w_e^k &\leq x_e \;\; e = 1, \dots, m, \\ w &\in \mathbb{R}_+^{m(|K|)}, \, y \in \mathbb{Z}^m, x \in \mathbb{R}^m. \end{aligned} \tag{13}$$

Let Q be the polyhedron defined by the constraints of (13) by dropping the integrality requirement on y and let $P = \{(x, y) : \exists w \text{ s.t. } (x, y, w) \in Q\}$.

It is known that the linear programming relaxation of (12) is in general considerably weaker than that obtained from solving the linear program over P. In particular for uncapacitated lot-sizing with or without backlogging [24], that is a special case of the fixed charge network flow problem, P is known to provide the convex hull of the feasible solutions of (12). Though a complete description of P in the original space is not known in general, Rardin and Wolsey [26] show that all the facet-defining inequalities of P belong to a class of “dicut” inequalities.

The separation LP for fixed charge network flow

Typically a point $(\hat{x}, \hat{y}, \hat{w})$ in the relative interior of Q is easy to construct. Let $\hat{w}$ be a feasible flow in which $b_k > \hat{w}_e^k > 0$ unless $w_e^k = 0$ or $w_e^k = b_k$ for all feasible solutions in which case $\hat{w}_e^k = 0$ or $\hat{w}_e^k = b_k$ respectively. Set $\hat{x}_e > \sum_k \hat{w}_e^k$ for all e, $\hat{y}_e > 1$ if $w_e^k = b^k$ for some k and $\hat{y}_e = 1$ otherwise.

By applying a linear transformation that maps $(\hat{x}, \hat{y}, \hat{w})$ into 0, the constraints of Q become:

$$\begin{aligned}
&\sum_{e:h(e)=i} \tilde{w}_e^k - \sum_{e:t(e)=i} \tilde{w}_e^k = -\sum_{e:h(e)=i} \hat{w}_e^k + \sum_{e:t(e)=i} \hat{w}_e^k \quad i = 1, \ldots, n, k \in K, && \mu_i^k \\
&\sum_{e:h(e)=k} \tilde{w}_e^k - \sum_{e:t(e)=k} \tilde{w}_e^k = b_k - \sum_{e:h(e)=k} \hat{w}_e^k + \sum_{e:t(e)=k} \hat{w}_e^k \quad k \in K && \mu_k^k \\
&\tilde{w}_e^k - b_k \tilde{y}_e \le b_k \hat{y}_e - \hat{w}_e^k \quad e = 1, \ldots, m, k \in K && v_e^k \\
&\sum_k \tilde{w}_e^k - \tilde{x}_e \le -\sum_k \hat{w}_e^k + \hat{x}_e \quad e = 1, \ldots, m, && \pi_e \\
&-\tilde{w}_e^k \le \hat{w}_e^k \quad e = 1, \ldots, m, k \in K && \phi_e^k \\
&\tilde{w} \in \mathbb{R}^{m(|K|)}, \tilde{y} \in \mathbb{R}^m, \tilde{x} \in \mathbb{R}^m
\end{aligned}$$

where $x_e = \tilde{x}_e + \hat{x}_e$, $w_e^k = \tilde{w}_e^k + \hat{w}_e^k$, $y_e = \tilde{y}_e + \hat{y}_e$.

The separation LP (6) now takes the form:

$$\begin{aligned}
\max &\sum_e (\gamma_e^x (x_e^* - \hat{x}_e) + \gamma_e^y (y_e^* - \hat{y}_e)) - \gamma_0 \\
&\gamma_e^x + \pi_e = 0 \quad e \in 1 \ldots, m \\
&\gamma_e^y + \sum_{k \in K} b_k v_e^k = 0 \quad e \in 1 \ldots, m \\
&\pi_e + v_e^k - \phi_e^k + \sum_{i:h(e)=i} \mu_i^k - \sum_{i:t(e)=i} \mu_i^k = 0 \quad e \in 1 \ldots, m, k \in K \\
&\gamma_0 - \sum_k \mu_k^k b_k - \sum_{i,k} \mu_i^k (-\sum_{e:h(e)=i} \hat{w}_e^k + \sum_{e:t(e)=i} \hat{w}_e^k) \\
&\sum_e \pi_e (-\sum_k \hat{w}_e^k + \hat{x}_e) - \sum_{e,k} v_e^k (b_k \hat{y}_e - \hat{w}_e^k) - \sum_{k,e} \phi_e^k \hat{w}_e^k \ge 0 \\
&-1 \le \gamma_0 \le 1 \\
&\gamma^x, \gamma^y \in \mathbb{R}^m, \gamma_0 \in \mathbb{R}^1, \ \mu \in \mathbb{R}^{nK}, \ v, \pi, \phi \ge 0
\end{aligned} \tag{14}$$

As we always work with the same polyhedron, the set of constraints of the LP (14) does not change at every iteration. The only change is in the objective function that is defined by the current point to be separated. So one only needs to compute a point in relint(P) once (Fig. 4).

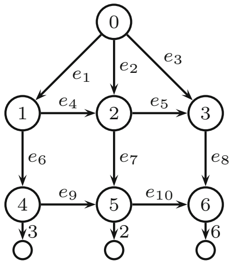

Fig. 4 The fixed charge network

A small example in detail

Example 3 Here we consider an instance with $n = 6$ nodes other than the root node 0 and $m = 10$ arcs. The network is shown in Fig. 4 along with the demands at nodes 4,5,6, corresponding to commodities 1,2,3. The variable costs are $p = (4, 2, 2, 1, 1, 2, 2, 2, 1, 1)$ and the fixed costs $f = (25, 42, 17, 36, 18, 34, 25, 48, 37, 46)$.

Consider the point

e	1	2	3	4	5	6	7	8	9	10
$\hat{y}_e$	1.1	1	1	1	1	1.1	1	1	1	1
$\hat{w}_e^1$	3	0	0	0	0	3	0	0	0	0
$\hat{w}_e^2$	3/2	1/2	0	1/2	0	1	1	0	1	0
$\hat{w}_e^3$	3	3/2	3/2	3/2	3/2	3/2	3/2	3	3/2	3

with $\hat{x}_e = \sum_k \hat{w}_e^k + 0.1$.

Since $(\hat{x}, \hat{y}) \in \text{relint}(P)$, we apply the linear transformation that maps this point into 0. Since the separation LP is solved with $(x^*, y^*) \in \text{aff}(P)$, it follows from Theorem 1 that the LP (6) is always bounded.

Below we show the results obtained using this facet separation LP. The optimal value of the LP relaxation of (12) is 113.727. The cuts added at each iteration and the resulting LP values are shown below. The algorithm terminates after generating eight inequalities that are facet inducing for P.

Cut	*Inequality*	*LPval*
1	$y_6 \geq 1$	138.455
2	$y_1 \geq 1$	156.636
3	$6y_8 + x_{10} \geq 6$	178.455
4	$y_7 + y_9 \geq 1$	195.636
5	$x_8 + 6y_{10} \geq 6$	198.909
6	$6y_3 + x_5 + x_{10} \geq 6$	206.636
7	$y_2 + y_4 + y_9 \geq 1$	216.455
8	$2y_7 + x_9 \geq 2$	217

7 Final remarks

It appears that the normalization $-1 \leq \gamma_0 \leq 1$ together with a linear transformation that maps 0 into a point in the relative interior of the feasible region is a good strategy for generating cuts that are facet-defining. This and some very limited computational experience raise some practical questions:

– The separation LP (8) can be viewed as a one dimensional LP. Can one devise efficient methods that take advantage of this viewpoint?
– When based on an extended formulation Q, solving a large separation LP (7) or its dual at each iteration can be costly. Can methods, such as variable fixing and cut-lifting, be used to reduce the size of such LPs?
– Does the generation of facet-cuts reduce the tailing-off effect and the numerical instability that typically affect pure cutting-plane algorithms?
– Benders' algorithm applies to problems of the form $\max\{cx + hy : Ax + By \leq d, x \in X\}$ where typically in mixed integer programming $X \subseteq \mathbb{Z}^n$. This is equivalent to the problem $\max\{\eta : (\eta, x) \in P, x \in X\}$, where $Q := \{(\eta, x, y) : \eta - cx - hy \leq 0, Ax + By \leq d\}$ and $P := \{(\eta, x) : \exists y \text{ s.t. } (\eta, x, y) \in Q\}$. Would it be effective to modify Benders' algorithm with the approach described above?

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