Minimum Sizes of Identifying Codes in Graphs Differing by One Vertex

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Abstract

Let G be a simple, undirected graph with vertex set V. For $v \in V$ and $r \geq 1$, we denote by $B_{G,r}(v)$ the ball of radius r and centre v. A set $\mathcal{C} \subseteq V$ is said to be an r-*identifying code* in G if the sets $B_{G,r}(v) \cap \mathcal{C}$, $v \in V$, are all nonempty and distinct. A graph G admitting an r-identifying code is called r-*twin-free*, and in this case the size of a smallest r-identifying code in G is denoted by $\gamma_r(G)$.

We study the following structural problem: let G be an r-twin-free graph, and G^* be a graph obtained from G by adding or deleting a vertex. If G^* is still r-twin-free, we compare the behaviours of $\gamma_r(G)$ and $\gamma_r(G^*)$, establishing results on their possible differences and ratios.

Key Words: Graph Theory, Twin-Free Graphs, Identifiable Graphs, Identifying Codes.

1 Introduction

We introduce basic definitions and notation for graphs, for which we refer to, e.g., [1] and [8], and for identifying codes (see [14] and the bibliography at [17]).

We shall denote by $G = (V, E)$ a simple, undirected graph with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is indifferently denoted by $\{x, y\}$, $\{y, x\}$, xy or yx. The *order* of a graph is its number of vertices $|V|$.

A *path* $P_n = x_1 x_2 \dots x_n$ is a sequence of n distinct vertices x_i, $1 \le i \le n$, such that $x_i x_{i+1}$ is an edge for $i \in \{1, 2, \dots, n-1\}$. The *length* of P_n is its number of edges, $n - 1$. A *cycle* $C_n = x_1 x_2 \dots x_n$ is a sequence of n distinct vertices x_i, $1 \le i \le n$, where $x_i x_{i+1}$ is an edge for $i \in \{1, 2, \dots, n-1\}$, and $x_n x_1$ is also an edge; its length is n.

A graph G is called *connected* if for any two vertices x and y, there is a path between them. It is called *disconnected* otherwise. In a connected graph G, we can define the *distance* between any two vertices x and y, denoted by $d_G(x, y)$, as the length of any shortest path between x and y, since such a path exists. This definition can be extended to disconnected graphs, using the convention that $d_G(x, y) = +\infty$ if there is no path between x and y.

For any vertex $v \in V$ and integer $r \ge 1$, the *ball of radius* r *and centre* v, denoted by $B_{G,r}(v)$, is the set of vertices within distance r from v:

$$B_{G,r}(v) = \{x \in V : d_G(v, x) \le r\}.$$

Two vertices x and y such that $B_{G,r}(x) = B_{G,r}(y)$ are called (G, r)*-twins*; if G has no (G, r)-twins, that is, if

$$\forall x, y \in V \text{ with } x \neq y, \;\; B_{G,r}(x) \neq B_{G,r}(y),$$

then we say that G is r*-twin-free*.

Whenever two vertices x and y are within distance r from each other in G, i.e., $x \in B_{G,r}(y)$ and $y \in B_{G,r}(x)$, we say that x and y r*-cover* each other. When three vertices x, y, z are such that $x \in B_{G,r}(z)$ and $y \notin B_{G,r}(z)$, we say that z r*-separates* x and y in G. A set is said to r-separate x and y in G if it contains at least one vertex which does.

A *code* $\mathcal{C}$ is simply a subset of V, and its elements are called *codewords*. For each vertex $v \in V$, the r*-identifying set* of v, with respect to $\mathcal{C}$, is the set of codewords r-covering v, and is denoted by $I_{G,\mathcal{C},r}(v)$:

$$I_{G,\mathcal{C},r}(v) = B_{G,r}(v) \cap \mathcal{C}.$$

We say that $\mathcal{C}$ is an r*-identifying* code [14] if all the sets $I_{G,\mathcal{C},r}(v)$, $v \in V$, are nonempty and distinct: in other words, every vertex is r-covered by at least one codeword, and every pair of vertices is r-separated by at least one codeword.

It is quite easy to observe that a graph G admits an r-identifying code if and only if G is r-twin-free; this is why r-twin-free graphs are also sometimes called *r-identifiable*.

When G is r-twin-free, we denote by $\gamma_r(G)$ the cardinality of a smallest r-identifying code in G. The search for the smallest r-identifying code in given graphs or families of graphs is an important part of the studies devoted to identifying codes.

In this paper and in the forthcoming [4], we are interested in the following issue: let G be an r-twin-free graph, and G^* be a graph obtained from G by adding or deleting one vertex, or by adding or deleting one edge. Now, if G^* is still r-twin-free, what can be said about $\gamma_r(G)$ compared to $\gamma_r(G^*)$? More specifically, we shall study their difference and, when appropriate, their ratio,

$$\gamma_r(G) - \gamma_r(G^*) \text{ and } \frac{\gamma_r(G)}{\gamma_r(G^*)},$$

as functions of the order of the graph G, and r.

Note that a partial answer to the issue of knowing the conditions for which an r-twin-free graph remains so when one vertex is removed was given in [3] and [5]: any 1-twin-free graph with at least four vertices always possesses at least one vertex whose deletion leaves the graph 1-twin-free; for any $r \geq 1$, any r-twin-free tree with at least $2r+2$ vertices always possesses at least one vertex whose deletion leaves the graph r-twin-free; on the other hand, for any $r \geq 3$, there exist r-twin-free graphs such that the deletion of any vertex makes the graph not r-twin-free. The case $r = 2$ remains open.

Of what interest this study is, can be illustrated by the watching of a museum: we place ourselves in the case $r = 1$ and assume that we have to protect a museum, or any other type of premises, using smoke detectors. The museum can be viewed as a graph, where the vertices represent the rooms, and the edges, the doors or corridors between rooms. The detectors are located in some of the rooms and give the alarm whenever there is smoke in their room or in one of the adjacent rooms. If there is smoke in one room and if the detectors are located in rooms corresponding to a 1-identifying code, then, only by knowing which detectors gave the alarm, we can identify the room where someone is smoking.

Of course we want to use as few detectors as possible. Now, what are the consequences, beneficial or not, of closing or opening one room or one door? This is exactly the object of our investigation, in the more general case when r can take values other than 1.

In the conclusion of [18], it is already observed, somewhat paradoxically, that a cycle with one vertex less can require more codewords/detectors.

We shall exhibit examples of large variations for the minimum size of an identifying code.

A related issue is that of *t-edge-robust* identifying codes, which remain identifying when at most t edges are added or deleted, in any possible way; see, e.g., [11]–[13], [15] or [16].

In this paper, we focus on the addition or deletion of one vertex, whereas in [4] we study the consequences of adding or removing one edge. We shall consider two cases,

(i) both graphs G and G^* are connected,

(ii) the graph with one vertex less may be disconnected,

and observe one significant difference in our results.

Before we proceed, we still need some additional definitions and notation, and we also give two lemmata which, although trivial, will prove useful in the sequel.

For a graph $G = (V, E)$ and a vertex $v \in V$, we denote by G_v the graph with vertex set V' and edge set E', where

$$V' = V \setminus \{v\},\ E' = \{xy \in E : x \in V', y \in V'\}.$$

If $G = (V, E)$ is a graph and $\mathcal{S}$ is a subset of V, we say that two vertices $x \in V$ and $y \in V$ are $(G, \mathcal{S}, r)$*-twins* if

$$I_{G,\mathcal{S},r}(x) = I_{G,\mathcal{S},r}(y).$$

In other words, x and y are not r-separated by $\mathcal{S}$ in G. By definition, if $\mathcal{C}$ is r-identifying in G, then no $(G, \mathcal{C}, r)$-twins exist.

Lemma 1 *[$(G, \mathcal{S}, r)$-twin transitivity] In a graph $G = (V, E)$, if x, y, z are three distinct vertices, if $\mathcal{S}$ is a subset of V, if x and y are $(G, \mathcal{S}, r)$-twins and if y and z are $(G, \mathcal{S}, r)$-twins, then x and z are $(G, \mathcal{S}, r)$-twins.* $\triangle$

Lemma 2 *If $\mathcal{C}$ is an r-identifying code in a graph $G = (V, E)$, then so is any set $\mathcal{S}$ such that*

$$\mathcal{C} \subseteq \mathcal{S} \subseteq V.$$

$\triangle$

We present our main results in the following way. In Section 2 we consider the case $r = 1$: we study how *large* $\gamma_1(G_x) - \gamma_1(G)$ and $\gamma_1(G_x)/\gamma_1(G)$ can be (Proposition 3), then Theorem 4 states exactly how *small* the difference can be (namely, -1).

In Section 3, we study how *large* the difference $\gamma_r(G_x) - \gamma_r(G)$ can be, in the following three cases: (i) $r \geq 2$, r is even and the graphs are connected (Proposition 8); (ii) $r \geq 3$, r is odd and the graphs are connected (Proposition 10); (iii) $r \geq 2$ and the graph G_x is disconnected (Proposition 12).

Then we consider how *large* the ratio $\gamma_r(G_x)/\gamma_r(G)$ can be (Proposition 14), and it so happens that the graphs we use are connected.

Finally, we study how *small* $\gamma_r(G_x)-\gamma_r(G)$ can be for any $r \geq 2$ (Proposition 16) and how *small* $\gamma_r(G_x)/\gamma_r(G)$ can be for any $r \geq 2$ (Proposition 17), and again it so happens that the graphs we use are connected.

In these sections, the number n represents the order of either G or G_x, or an approximation. A general conclusion recapitulates our results in a Table.

2 The case $r = 1$

Note that we obtain the following result with connected graphs: we found no better with disconnected graphs.

Proposition 3 *Let $k \geq 1$ be an arbitrary integer. There exist two (connected) 1-twin-free graphs G and G_x, where G has $2k + \lceil \log_2(k+1) \rceil + 2$ vertices, such that $\gamma_1(G) \leq \lceil \log_2(k+1) \rceil + 2$ and $\gamma_1(G_x) \geq k$.*

Proof. We put the cart before the horse and, before defining G, we describe G_x (see Figure 4 with $r = 1$): we begin by choosing k vertices $x_1, \ldots,$ x_k, none of them adjacent with each other, and then build a graph G_x with a "small" 1-identifying code in the following way: we take $s = \lceil \log_2(k+1) \rceil + 1$ auxiliary vertices $a_1, \ldots, a_s$. We first connect each x_i to a_1; then we connect each x_i to the vertices of a unique nonempty subset A_i of the set $A = \{a_2, \ldots, a_s\}$. The sets A_i can indeed be chosen in this way, because there are $2^{s-1}-1$ nonempty subsets of A, and $s-1 = \lceil \log_2(k+1) \rceil$. Without loss of generality, we can choose the sets A_i in such a way that the graph constructed so far is connected.

Clearly the auxiliary vertices form a 1-identifying code in this graph: the 1-identifying set of each auxiliary vertex is a singleton consisting of the vertex itself; and for all the vertices x_i, the 1-identifying set contains a_1 and at least one more vertex, and no two of these sets are the same by the construction.

As the next step, we take another set of k vertices, $y_1, \ldots, y_k$, none of them adjacent with each other, and each y_i connected to exactly the same auxiliary vertices a_j as x_i. In this new graph G_x, which is connected, every 1-identifying code must contain at least one of the vertices x_i and y_i for each i: otherwise we cannot 1-separate between x_i and its "copy" y_i. But certainly if for each i we take at least one of x_i and y_i into the code and take all the auxiliary vertices a_j into the code, then the code is 1-identifying, and G_x is 1-twin-free. All in all, for this graph G_x, the smallest 1-identifying code has size at least k.

However, if we add one more vertex x, and connect it to each x_i (but not to any y_i nor any a_j), then in the resulting graph G the set consisting of x and all the auxiliary vertices a_j is a 1-identifying code.

Therefore, $\gamma_1(G) \leq \lceil \log_2(k+1) \rceil + 2$ and $\gamma_1(G_x) \geq k$. $\triangle$

Remark. The difference $\gamma_1(G_x) - \gamma_1(G)$ and ratio $\gamma_1(G_x)/\gamma_1(G)$ can be made arbitrarily large:

$$\gamma_1(G_x) - \gamma_1(G) \geq k - \lceil \log_2(k+1) \rceil - 2, \tag{1}$$

$$\frac{\gamma_1(G_x)}{\gamma_1(G)} \geq \frac{k}{\lceil \log_2(k+1) \rceil + 2}. \tag{2}$$

In terms of $n = 2k + \lceil \log_2(k+1) \rceil$, which is the approximate order of G and G_x, we can approximate these two lower bounds by $\frac{n}{2} - \frac{3}{2} \log_2 n$ and $\frac{n}{2 \log_2 n}$, respectively.

An open question is whether these difference or ratio can be made substantially larger.

Theorem 4 *Let $G = (V, E)$ be any 1-twin-free graph with at least three vertices. For any vertex $x \in V$ such that G_x is 1-twin-free, we have:*

$$\gamma_1(G_x) \geq \gamma_1(G) - 1. \tag{3}$$

Proof. Cf. [9, Prop. 3]. For completeness, we still give a proof. Let $x \in V$ be such that G_x is 1-twin-free. Let $\mathcal{C}_x$ be a minimum 1-identifying code in G_x: $|\mathcal{C}_x| = \gamma_1(G_x)$. There are two cases: either (a) x is not 1-covered (in G) by any codeword of $\mathcal{C}_x$, or (b) x is 1-covered (in G) by at least one codeword of $\mathcal{C}_x$.

(a) In this case, let $\mathcal{C} = \mathcal{C}_x \cup \{x\}$. Then $\mathcal{C}$ is clearly 1-identifying in G (in particular, thanks to Lemma 2); therefore, $\gamma_1(G) \leq \gamma_1(G_x) + 1$.

(b) x is 1-covered by $y \in \mathcal{C}_x$. If $\mathcal{C}_x$ is 1-identifying in G, then $\gamma_1(G) \leq \gamma_1(G_x)$, and we are done. So we assume that $\mathcal{C}_x$ is not 1-identifying in G. This means that either (i) at least one vertex in G is not 1-covered by $\mathcal{C}_x$, or (ii) at least two vertices in G are not 1-separated by $\mathcal{C}_x$.

(i) Since $\mathcal{C}_x$ 1-covers any vertex in G_x and x is linked to $y \in \mathcal{C}_x$, this case is impossible.

(ii) Let $u, v \in V$ be two distinct vertices which are not 1-separated by $\mathcal{C}_x$. One of them is necessarily x, and without loss of generality, we assume that $x = u$.

Now, v is unique by Lemma 1: $\mathcal{C}_x$ is not 1-identifying in G only because *one* pair of vertices, x and v, is not 1-separated by $\mathcal{C}_x$.

Since G is 1-twin-free, there is a vertex z which 1-covers exactly one of the vertices v and x. We set $\mathcal{C} = \mathcal{C}_x \cup \{z\}$, and we obtain a 1-identifying code in G, so $\gamma_1(G) \leq \gamma_1(G_x) + 1$. $\triangle$

Corollary 5 *If $\gamma_1(G_x) \leq a$ and $\gamma_1(G) \geq a + 1$, then $\gamma_1(G_x) = a$ and $\gamma_1(G) = a + 1$.* $\triangle$

Note that we made no assumption on the connectivity of G or G_x. Examples where $\gamma_1(G_x) = \gamma_1(G) - 1$, or $\gamma_1(G_x) = \gamma_1(G)$, are numerous and easy to find.

Conclusion 6 *Provided that the graphs considered are* 1*-twin-free, we can see, using Proposition 3 and Theorem 4, that* $\gamma_1(G_x) - \gamma_1(G)$ *cannot be smaller than* -1*, but examples exist where it can be as large as, approximately,* $\frac{n}{2} - \frac{3}{2}\log_2 n$*, and where the ratio* $\frac{\gamma_1(G_x)}{\gamma_1(G)}$ *can be as large as, approximately,* $\frac{n}{2\log_2 n}$*. This can even be obtained with connected examples.*

3 The case $r \geq 2$

Things are different for $r \geq 2$, since we can exhibit pairs of graphs (G, G_x) proving that $\gamma_r(G_x) - \gamma_r(G)$ and $\gamma_r(G_x)/\gamma_r(G)$ can be arbitrarily large or small.

We first give a result with $\gamma_r(G_x) - \gamma_r(G)$ arbitrarily large. We start with connected graphs, and have two subcases, r even and r odd. In both cases, we shall use the following result on cycles of even length.

Theorem 7 *[2] For all* $r \geq 1$ *and for all even* n*,* $n \geq 2r + 4$*, we have:*

$$\gamma_r(C_n) = \frac{n}{2}.$$

△

• (i) Case of a connected graph G_x and $r \geq 2$, r even

Proposition 8 *There exist two (connected)* r*-twin-free graphs* G *and* G_x*, with* $n+1$ *and* n *vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \geq \frac{n}{4} - (r+1), \tag{4}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{2n}{n + 4r + 4}. \tag{5}$$

Remark preceding the proof. The lower bound (4) is equivalent to $n/4$ when n increases with respect to r. An open question is whether this can be improved. The lower bound (5) is equivalent to 2, but will be strongly improved in Proposition 14.

Proof of Proposition 8. Let $r \geq 2$ be an even integer, and n be an (even) integer such that $n = k \cdot 2r$, $k \geq 2$; let $G_x = C_n = x_1 x_2 \ldots x_n$ be the cycle of length n and G be the graph obtained from G_x by adding the vertex x and linking it to the k vertices $x_{j\cdot 2r}$, $1 \leq j \leq k$. See Figure 1, which illustrates the case $r = 6, k = 4, n = 48$ and G has 49 vertices.

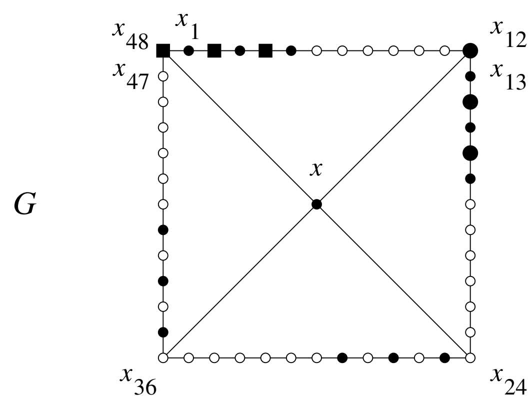

Figure 1: Graph G in Proposition 8, for $r = 6$ and $k = 4$. Squares and circles, white or black, small or large, are vertices. The 19 black vertices constitute a 6-identifying code in G.

We know by Theorem 7 that $\gamma_r(G_x) = \frac{n}{2}$, and we claim that

$$\gamma_r(G) \leq 1 + (k+2)\frac{n}{4k} = \frac{n}{4} + r + 1,$$

from which (4) and (5) follow. Proving this claim, by exhibiting an r-identifying code for G, is tedious and of no special interest; therefore, we content ourselves with showing how it works in the case $r = 6$, $n = 48$, hoping that this will help the reader to gain an insight into the general case. We consider a first set

$$\mathcal{S} = \{x, x_1, x_3, x_5, x_{13}, x_{15}, x_{17}, x_{25}, x_{27}, x_{29}, x_{37}, x_{39}, x_{41}\},$$

see the small black circles in Figure 1. It is now quite straightforward to observe that the pairs $\{x_{48}, x_1\}$, $\{x_2, x_3\}$ and $\{x_4, x_5\}$ are pairs of $(G, \mathcal{S}, 6)$-twins, as well as $\{x_{12}, x_{13}\}$, $\{x_{14}, x_{15}\}$, $\{x_{16}, x_{17}\}$, $\{x_{24}, x_{25}\}$, $\{x_{26}, x_{27}\}$, $\{x_{28}, x_{29}\}$, $\{x_{36}, x_{37}\}$, $\{x_{38}, x_{39}\}$ and $\{x_{40}, x_{41}\}$, for reasons of symmetry, and that they are the only ones.

Let us consider the first three pairs, $\{x_{48}, x_1\}$, $\{x_2, x_3\}$, $\{x_4, x_5\}$. Using edges going through x, they can be 6-separated, for instance, by the vertices x_{16}, x_{14} and x_{12} (see the large black circles), and these three vertices also 6-separate the other pairs of $(G, \mathcal{S}, 6)$-twins, except for $\{x_{12}, x_{13}\}$, $\{x_{14}, x_{15}\}$, $\{x_{16}, x_{17}\}$. These three pairs can however be 6-separated by three more codewords, for instance x_4, x_2 and x_{48}, see the black squares in Figure 1. Now the code

$$\mathcal{C} = \mathcal{S} \cup \{x_{12}, x_{14}, x_{16}, x_{48}, x_2, x_4\}$$

is 6-identifying in G and has $1 + (4 \times 3) + (2 \times 3) = 19$ codewords.

In the general case,

$$\mathcal{S} = \{x\} \cup \{x_{1+j\cdot 2r}, x_{3+j\cdot 2r}, \ldots, x_{r-1+j\cdot 2r} : 0 \leq j \leq k-1\},$$

there are $k \times \frac{r}{2}$ pairs of $(G, \mathcal{S}, r)$-twins, and $\mathcal{C}$ can be chosen, for instance, as

$$\mathcal{C} = \mathcal{S} \cup \{x_n, x_2, \ldots, x_{r-2}\} \cup \{x_{2r}, x_{2r+2}, \ldots, x_{2r+(r-2)}\},$$

which shows that the cardinality of $\mathcal{C}$ is

$$1 + (k \times \frac{r}{2}) + (2 \times \frac{r}{2}) = 1 + (k+2)\frac{n}{4k},$$

and so $\gamma_r(G) \leq 1 + (k+2)\frac{n}{4k}$. △

Conclusion 9 *When r is even, Proposition 8 gives pairs of connected graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n}{4}$.*

• **(ii) Case of a connected graph G_x and $r \geq 3$, r odd**

Proposition 10 *There exist two (connected) r-twin-free graphs G and G_x, with $n+1$ and n vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \geq \frac{n(3r-1)}{12r} - r, \tag{6}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{6nr}{n(3r+1) + 12r^2}. \tag{7}$$

Remark preceding the proof. An open question is whether the first lower bound, which is equivalent to $\frac{n(3r-1)}{12r}$ when r is fixed and n goes to infinity, can be improved. The second lower bound, equivalent to $\frac{6r}{3r+1}$, will be improved in Proposition 14.

Proof of Proposition 10. Let $r \geq 3$ be an odd integer, and n be an (even) integer such that $n = k \cdot 2r$, where $k \geq 3$ is a multiple of 3; let $G_x = C_n = x_1 x_2 \ldots x_n$ be the cycle of length n and G be the graph obtained from G_x by adding the vertex x and linking it to the k vertices $x_{j\cdot 2r}$, $1 \leq j \leq k$. See Figure 2, which illustrates the case $r = 5, k = 6, n = 60$ and G has 61 vertices.

We know by Theorem 7 that $\gamma_r(G_x) = \frac{n}{2}$, and we claim that

$$\gamma_r(G) \leq \frac{n}{4} + \frac{n}{12r} + r,$$

from which (6) and (7) follow. Again, proving this claim is of no interest here, and we just show how it works in the case $r = 5$, $n = 60$. We consider a first set

$$\mathcal{S} = \{x, x_1, x_3, x_{11}, x_{13}, x_{21}, x_{23}, x_{31}, x_{33}, x_{41}, x_{43}, x_{51}, x_{53}\},$$

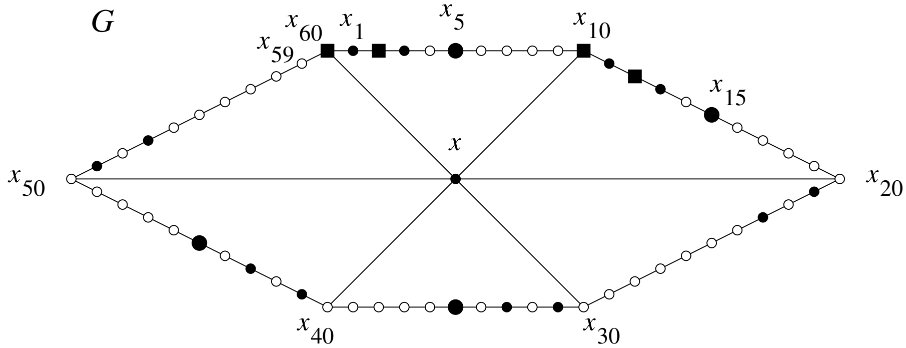

Figure 2: Graph G in Proposition 10, for $r = 5$ and $k = 6$. Squares and circles, white or black, small or large, are vertices. The 21 black vertices constitute a 5-identifying code in G.

see the small black circles in Figure 2. It is straightforward to see that only the following sets of $(G, \mathcal{S}, 5)$-twins exist:

• (i) $\{x, x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60}\}$,
• (ii) $\{x_{59}, x_1, x_2\}$ together with the five symmetrical sets $\{x_9, x_{11}, x_{12}\}, \ldots$,
• (iii) $\{x_3, x_4\}$ together with the five symmetrical sets $\{x_{13}, x_{14}\}, \ldots$

The first two cases are annoying and will be "expensive" because they present symmetries with respect to x. Define the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \mathcal{S} \cup \{x_5, x_{15}, x_{35}, x_{45}\},$$

see the large black circles in Figure 2. Now in Case (i), all the vertices are 5-separated by the vertices in $\mathcal{T} \setminus \mathcal{S}$, and so are x_{59} on the one hand and x_1, x_2 on the other hand, as well as their symmetrical counterparts from Case (ii). The remaining pairs of $(G, \mathcal{T}, 5)$-twins are $\{x_1, x_2\}$, $\{x_3, x_4\}$ and the 10 pairs obtained by symmetry. As in the proof of Proposition 8, these handle very economically: the vertex x_{60} 5-separates the 5 pairs $\{x_{13}, x_{14}\}$, $\ldots$, $\{x_{53}, x_{54}\}$, and so does x_2 for $\{x_{11}, x_{12}\}$, $\ldots$, $\{x_{51}, x_{52}\}$; finally, $\{x_1, x_2\}$ and $\{x_3, x_4\}$ can be 5-separated, for instance, by x_{10} and x_{12}, see the black squares in Figure 2:

$$\mathcal{C} = \mathcal{T} \cup \{x_{60}, x_2, x_{10}, x_{12}\}$$

is a 5-identifying code in G and has $1 + (6 \times 2) + (4 \times 1) + (2 \times 2) = 21$ codewords. In the general case,

$$\mathcal{S} = \{x\} \cup \{x_{1+j \cdot 2r}, x_{3+j \cdot 2r}, \ldots, x_{r-2+j \cdot 2r} : 0 \leq j \leq k-1\}$$

contains $1 + (k \times \frac{r-1}{2})$ vertices; then

$$\mathcal{T} = \mathcal{S} \cup \{x_{r+j \cdot 2r} : 0 \leq j \leq k-1, j \text{ not congruent to 2 modulo 3}\}$$

contains $|\mathcal{S}| + \frac{2k}{3}$ elements, and finally we take

$$\mathcal{C} = \mathcal{T} \cup \{x_n, x_2, \ldots, x_{r-3}\} \cup \{x_{2r}, x_{2r+2}, \ldots, x_{2r+(r-3)}\},$$

which shows that

$$\gamma_r(G) \leq 1 + (k \times \frac{r-1}{2}) + \frac{2k}{3} + (2 \times \frac{r-1}{2}) = \frac{n}{4} + \frac{n}{12r} + r.$$

$\triangle$

Conclusion 11 *When $r \geq 3$ and r is odd, Proposition 10 gives pairs of connected graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n(3r-1)}{12r}$.*

If we do not require to consider a connected graph G_x, then we can obtain a larger difference or ratio than in (4)-(7), we need consider only one case, whatever the parity of r is, and moreover the construction is easy to understand; see next paragraph.

• (iii) Case of a disconnected graph G_x and $r \geq 2$, r even or odd

Proposition 12 *There exist two graphs G and G_x, with $p(2r+1)+1$ and $n = p(2r+1)$ vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \geq \frac{n(2r-2)}{2r+1} - 2r, \tag{8}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{nr}{n + 4r^2 + 2r}. \tag{9}$$

Remark preceding the proof. Can the first lower bound, equivalent to $\frac{n(2r-2)}{2r+1}$, be improved? The second bound, equivalent to r, is still improved in Proposition 14.

Proof of Proposition 12. Let $r \geq 2$ and $p \geq 3$ be integers; the graph G_x consists of p copies of the path P_{2r+1}, and G is obtained by adding the vertex x and linking it to all the middle vertices of the path copies, see Figure 3. We claim that: (a) $\gamma_r(G_x) = 2pr$ and (b) $\gamma_r(G) \leq 2p + 2r$, from which (8) and (9) follow.

Proof of (a). The result comes from the obvious fact that $\gamma_r(P_{2r+1}) = 2r$.
Proof of (b). It is not difficult to check that

$$\mathcal{C} = \{x\} \cup \{v_{i,1}, v_{i,2r+1} : 1 \leq i \leq p-1\} \cup \{v_{p,j} : 1 \leq j \leq 2r+1\}$$

(see the black circles in Figure 3) is indeed r-identifying in G. Note however that, for simplicity, we chose to give the bound $2p+2r$, when actually, with a little more care, $2p+2r-3$ can be reached, which would improve only slightly on (8) and (9). $\triangle$

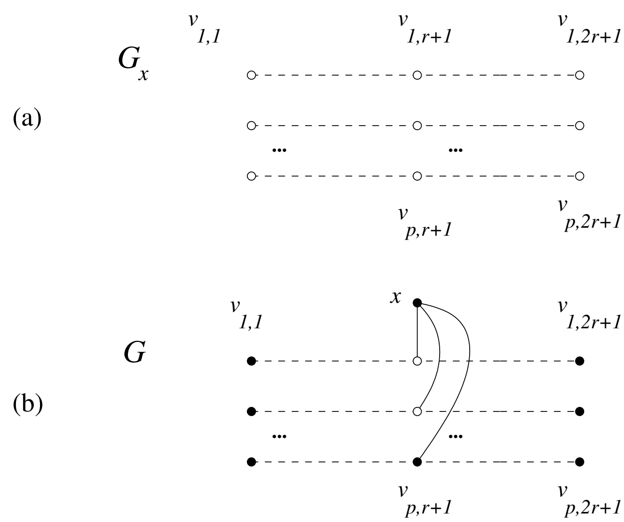

Figure 3: The graphs G_x and G in Proposition 12.

Conclusion 13 *Proposition 12 gives pairs of graphs* (G, G_x), *where* G_x *is not connected, proving that* $\gamma_r(G_x) - \gamma_r(G)$ *can be, asymptotically, as large as approximately* $\frac{n(2r-2)}{2r+1}$.

Finally, we give a construction (obtained with connected graphs) with a ratio $\gamma_r(G_x)/\gamma_r(G)$ arbitrarily large, but where the difference $\gamma_r(G_x) - \gamma_r(G)$ is not as large as in (4) and (6).

Proposition 14 *Let* $k \geq 2$ *be an arbitrary integer. There exist two (connected)* r*-twin-free graphs* G *and* G_x, *where* G *has* $2rk+r\lceil\log_2(k+1)\rceil+r+1$ *vertices, such that*

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{k}{r\lceil\log_2(k+1)\rceil + r + 1}. \tag{10}$$

Proof. The construction is a straigthforward generalization to any $r \geq 2$ of the one used in the proof of Proposition 3, see Figure 4; the basic idea is similar, but the implementation becomes somewhat more involved.

We consider, for each i between 1 and k, the paths $x_i(1)x_i(2)\dots x_i(r)$, and $y_i(1)y_i(2)\dots y_i(r)$. We need also some auxiliary vertices. Denoting again $s = \lceil\log_2(k+1)\rceil + 1$, for each $j = 1, 2, \dots, s$, we consider the path $a_j(1)a_j(2)\dots a_j(r)$; we denote the set of these sr auxiliary vertices by A. We say that the vertices $x_i(-h)$, $y_i(-h)$ and $a_j(h)$ are on the h-th level (cf. Figure 4).

We now imitate the proof of Proposition 3, and for each $i \in \{1, \dots, k\}$ choose a unique nonempty subset A_i of the set $\{a_2(1), \dots, a_s(1)\}$ and connect $x_i(1)$ and $y_i(1)$ by an edge to the vertices $a_j(1)$ for which $j \in \{1\} \cup A_i$.

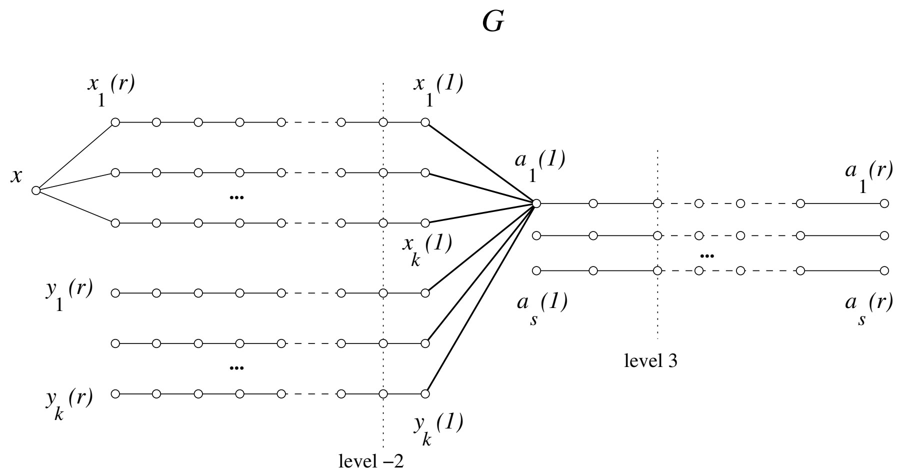

Figure 4: A partial representation of the graph G in Proposition 14: more edges exist between the vertices $x_i(1)$ and $y_i(1)$ on the one hand, and the vertices $a_j(1)$ on the other hand. The case $r = 1$ can be used to illustrate Proposition 3.

In the resulting graph G_x, we first take all the vertices in A as codewords. Then we observe that for an arbitrary, unknown vertex v,

- $B_r(v)$ contains at least two vertices $a_j(r)$ if v is on the level -1;
- $B_r(v)$ does not contain any vertices $a_j(r)$ if v is on the h-th level for some $h \leq -2$; and
- $B_r(v)$ contains exactly one $a_j(r)$ if $v \in A$.

From the last case we see that we can uniquely tell whether or not $v \in A$ simply by looking which vertices of A are in $B_r(v)$. We can in fact do even more: if j is the only index for which $a_j(r)$ is in $B_r(v)$, then v is one of the vertices $a_j(h)$ for some $h = 1, 2, \ldots, r$. We know that $a_j(1)$ is connected to at least one $x_i(1)$ (as we chose s to be as small as possible) and $x_i(1)$ is connected to at least one $a_{j'}(1)$ with $j' \neq j$. Then exactly $r - h$ of the vertices $a_{j'}(1), \ldots, a_{j'}(r)$ are in $B_r(v)$, and this uniquely identifies v.

Assume now that we already know that $v \notin A$. Let h be the highest level for which some $a_j(h)$ belongs to $B_r(v)$. Then v must be one of the vertices $x_i(r+1-h)$ or $y_i(r+1-h)$, and moreover, we can uniquely tell i by looking at the indices j for which $a_j(h)$ belong to $B_r(v)$, because by the construction $\{j : a_j(h) \in B_r(v)\} = \{1\} \cup A_i$ (as we can only reach these vertices from v by going from v to $x_i(1)$ or $y_i(1)$ and from it directly to those $a_j(1)$ to which $x_i(1)$ or $y_i(1)$ was connected to by an edge).

In conclusion, by only looking at which auxiliary vertices are in $B_r(v)$ we can "almost" identify v: we find indices i and m such that v is either

$x_i(m)$ or $y_i(m)$. This implies that the graph is clearly r-twin-free. Indeed, if all the vertices are in the code, then the only remaining task, i.e., separating each $x_i(m)$ from $y_i(m)$, becomes easy: if $x_i(r)$ is in $B_r(v)$ then $v = x_i(m)$; if not then $v = y_i(m)$.

Moreover, every r-identifying code must contain at least one element of the set $\{x_i(1), x_i(2), \dots, x_i(r), y_i(1), y_i(2), \dots, y_i(r)\}$: otherwise we cannot r-separate $x_i(1)$ and $y_i(1)$. Consequently, any r-identifying code in this graph has size at least k.

We now add one more vertex x, and connect it by an edge to each $x_i(r)$. We claim that the vertex x together with all the vertices in A form an r-identifying code. By the construction, the set $B_r(v)$, $v \neq x$, contains exactly the same vertices of A as before adding the vertex x (and the set $B_r(x)$ contains none), so the only thing to check is that $x_i(m)$ and $y_i(m)$ can now be r-separated: but this is indeed done by x. △

Remark. In terms of $n = 2rk + r\lceil \log_2(k+1) \rceil$, the lower bound (10) can be approximated by $\frac{n}{2r^2 \log_2 n}$ and is open to improvements.

Conclusion 15 *Proposition 14 gives pairs of (connected) graphs proving that $\gamma_r(G_x)/\gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n}{2r^2 \log_2 n}$.*

Then we turn to examples where $\gamma_r(G) - \gamma_r(G_x)$ is arbitrarily large. Note that we obtain this result with connected graphs.

Proposition 16 *There exist two (connected) r-twin-free graphs G_x and G, with $n = pr + 1$ and $pr + 2$ vertices respectively, such that*

$$\gamma_r(G_x) = p + 2r - 3 = \frac{n + 2r^2 - 3r - 1}{r} \quad \textit{and} \quad \gamma_r(G) = r(p-1) + 1 = n - r,$$

where p is any integer greater than or equal to 3.

Proof. Let $r \geq 2$ and $p \geq 3$ be integers; before defining G, we describe G_x in the following informal way, illustrated in Figure 5(a): G_x consists of p copies of the path P_r, and in each copy the last vertex is linked to v. This graph has $n = pr + 1$ vertices. Next, we construct the graph G consisting of G_x to which we add one vertex x, linked to each first vertex of all the copies of P_r. See Figure 5(b). We claim that: (a) $\gamma_r(G_x) = p + 2r - 3$, and (b) $\gamma_r(G) = r(p-1) + 1$, from which (13) and (14) follow.

Proof of (a). The code

$$\mathcal{C} = \{v_{1,i} : 2 \leq i \leq r\} \cup \{v_{2,i} : 1 \leq i \leq r\} \cup \{v_{j,1} : 3 \leq j \leq p\},$$

i.e., the code consisting of all the vertices of the first two copies of P_r, except $v_{1,1}$, and the first vertex of each of the following copies, is r-identifying in G_x; this it is straightforward to check. So $\gamma_r(G_x) \leq (r-1) + r + (p-2) =$

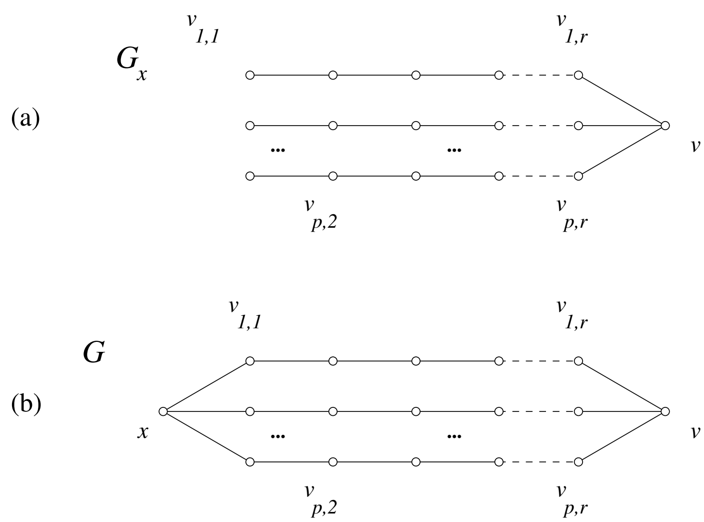

Figure 5: The graphs G_x and G in Proposition 16.

$p+2r-3$. We now prove that $\gamma_r(G_x) \geq p+2r-3$. The following two observations will be useful. For $1 \leq i \leq p$ and $2 \leq k \leq r$, we have:

$$B_{G_x,r}(v_{i,r-k+1}) \Delta B_{G_x,r}(v_{i,r-k+2}) = \{v_{j,k} : 1 \leq j \leq p, j \neq i\}, \tag{11}$$

where Δ stands for the symmetric difference, and for $1 \leq i < j \leq p$:

$$B_{G_x,r}(v_{i,r}) \Delta B_{G_x,r}(v_{j,r}) = \{v_{i,1}, v_{j,1}\}. \tag{12}$$

The consequences are immediate. First, in order to have the vertices $v_{i,r}$, $1 \leq i \leq p$, pairwise r-separated in G_x, we see by (12) that we need at least $p-1$ codewords among the p vertices $v_{i,1}$; second, for k fixed between 2 and r, we see, using (11), that we need at least two codewords among the p vertices $v_{i,k}$. So $\gamma_r(G_x) \geq (p-1)+2(r-1) = p+2r-3$, and Claim (a) is proved.

Proof of (b). Note that in G, for i and j such that $1 \leq i < j \leq p$, the set of vertices

$$\{x\} \cup \{v_{i,k} : 1 \leq k \leq r\} \cup \{v\} \cup \{v_{j,k} : 1 \leq k \leq r\}$$

forms the cycle C_{2r+2}, which is r-twin-free and is denoted by $C(i,j)$. On such a cycle, we say that the vertex z is the *opposite* of the vertex y if z is the (only) vertex at distance $r+1$ from y.

We claim that, for k fixed between 1 and r, among the p vertices $v_{i,k}$, at least $p-1$ of them belong to any r-identifying code $\mathcal{C}$ in G. Indeed, assume on the contrary that two vertices, say $v_{1,k}$ and $v_{2,k}$, are not in $\mathcal{C}$; then their opposite vertices in $C(1,2)$, $v_{2,r-k+1}$ and $v_{1,r-k+1}$ respectively, cannot be r-separated by $\mathcal{C}$.

Finally, the fact that $B_{G,r}(v)\Delta B_{G,r}(x)=\{v,x\}$ shows that v or x belong to $\mathcal{C}$, and finally $\gamma_r(G)\geq(p-1)r+1$. On the other hand,

$$\{v\}\cup\{v_{i,k}:2\leq i\leq p,1\leq k\leq r\}$$

is an r-identifying code in G, with size $(p-1)r+1$, thus Claim (b) is proved. Observe that this code contains all the vertices in G, except the $r+1$ vertices x and $v_{1,k}$, $1\leq k\leq r$. △

Note that we could have contented ourselves with the inequalities $\gamma_r(G_x)\leq p+2r-3$ and $\gamma_r(G)\geq r(p-1)+1$, so as to obtain $\gamma_r(G)-\gamma_r(G_x)\geq p(r-1)-3r+4$ and $\frac{\gamma_r(G)}{\gamma_r(G_x)}\geq\frac{r(p-1)+1}{p+2r-3}$.

Remark. The difference

$$\gamma_r(G)-\gamma_r(G_x)=p(r-1)-3r+4$$

can be made arbitrarily large; in terms of n, the number of vertices of G_x, we can see that we have:

$$\gamma_r(G)-\gamma_r(G_x)=\frac{(n-3r)(r-1)+1}{r}, \tag{13}$$

which is equivalent to $\frac{n(r-1)}{r}$ when r is fixed and n goes to infinity. As far as the ratio given by Proposition 16 is concerned, we have:

$$\frac{\gamma_r(G)}{\gamma_r(G_x)}=\frac{r(n-r)}{n+2r^2-3r-1}, \tag{14}$$

which is equivalent to r when we increase n. This can be improved, with a ratio which becomes arbitrarily large; again, it so happens that the graphs are connected:

Proposition 17 *Let $k\geq 2$ be an arbitrary integer.*
There exist two (connected) 2-twin-free graphs G and G_x, where G has $3k+2\lceil\log_2(k+2)\rceil+4$ vertices, such that

$$\frac{\gamma_2(G)}{\gamma_2(G_x)}\geq\frac{k}{2\lceil\log_2(k+2)\rceil+3}. \tag{15}$$

Let $r\geq 3$. There exist two (connected) r-twin-free graphs G and G_x, where G has $(r+1)k+r\lceil\log_2(k+2)\rceil+2r+1$ vertices, such that

$$\frac{\gamma_r(G)}{\gamma_r(G_x)}\geq\frac{k}{r\lceil\log_2(k+2)\rceil+r+3}. \tag{16}$$

Proof. We first deal with the general case $r\geq 3$. We construct the graph G for a given $k\geq 2$ in the following way, see Figure 6: G consists of the paths $x_i(0)x(1)x(2)\dots x(r-2)x(r-1)x$ and $y_i(0)y_i(1)\dots y_i(r-1)$, for $i=1,\dots,k$,

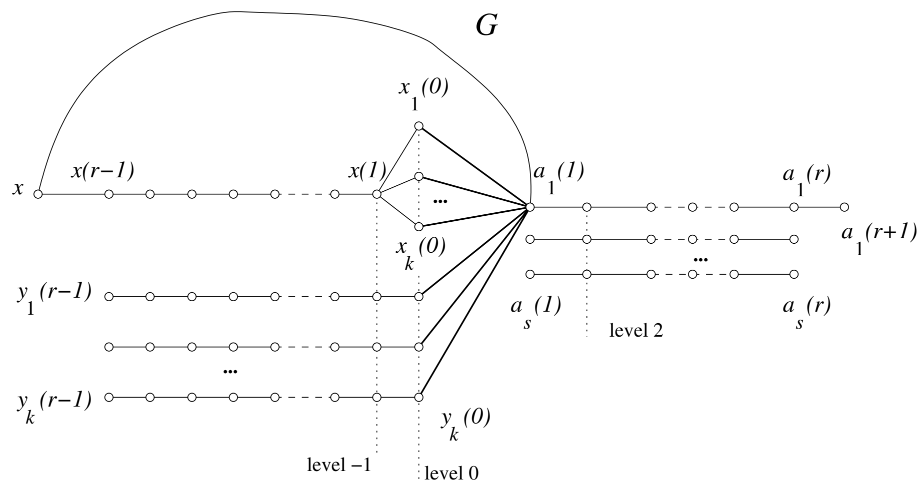

Figure 6: A partial representation of the graph G in Proposition 17, in the general case $r \geq 3$: more edges exist between the vertices $x_i(0)$ and $y_i(0)$ on the one hand, and the vertices $a_j(1)$ on the other hand.

of the path $a_1(1) \dots a_1(r+1)$, of the paths $a_j(1) \dots a_j(r)$ for $j = 2, \dots, s$, where $s = 1 + \lceil \log_2(k+2) \rceil$, plus the edge $xa_1(1)$ and the following edges, joining exclusively the vertices $x_i(0)$ and $y_i(0)$ on the one hand, and the vertices $a_j(1)$ on the other hand: for each i we choose a unique nonempty proper subset A_i of the set $A = \{2, 3, \dots, s\}$, and connect every $x_i(0)$ and every $y_i(0)$ to every vertex $a_j(1)$ for which $j \in A_i$. Moreover, we connect every $x_i(0)$ and every $y_i(0)$ to $a_1(1)$. The sets A_i can indeed be chosen in this way, because there are $2^{s-1} - 2$ proper nonempty subsets of A, and $s - 1 = \lceil \log_2(k+2) \rceil$. Without loss of generality, we can choose the sets A_i in such a way that each $a_j(1)$ has degree at least two, and so the graph constructed is connected, as will be G_x.

We say that the vertices $x(-h)$, $x_i(-h)$, $y_i(-h)$ and $a_j(h)$ are on the h-th level, cf. Figure 6 (and x is not given any level). Let

$$\mathcal{A} = \{a_j(h) : 1 \leq j \leq s, 1 \leq h \leq r\} \cup \{a_1(r+1)\}.$$

Let us first consider G_x, and let $\mathcal{C} = \mathcal{A} \cup \{x_1(0), x(r-1)\}$. We show that $\mathcal{C}$ is r-identifying, so that $\gamma_r(G_x) \leq sr + 3$. The argument is very similar to the first part of the proof of Proposition 14: let v be an arbitrary, unknown vertex in G_x.

If v belongs to $\mathcal{A}$, then v is r-covered by exactly one codeword $a_j(r)$, whereas every vertex of level 0 is r-covered by at least two codewords of level r, and no vertex with negative level is r-covered by any codeword of level r; if $v \in \mathcal{A}$ is r-covered by $a_j(r)$, we know moreover that $v = a_j(h)$ for some h between 1 and $r+1$. If $h < r$, then h is given by the highest level ℓ of any codeword $a_{j'}(\ell)$ r-covering $a_j(h)$, with $j' \neq j$ (such a j' exists because $a_j(1)$ is connected to at least one $x_i(0)$, which in turn is connected to at least one $a_{j'}(1)$). If $j \neq 1$ and $h = r$, then h is given by the fact that no codeword $a_{j'}(\ell)$ ($j' \neq j$) r-covers $a_j(h)$. And if $j = 1$ and $h \in \{r, r+1\}$, then the codeword $x_1(0)$ tells whether $h = r$ or $h = r+1$. This means that we can determine first that $v \in \mathcal{A}$, then on which path and at which level it is located.

If $v \notin \mathcal{A}$, then its level can be determined by the highest level, say ℓ, of the codewords in $\mathcal{A}$ which r-cover it. Then the codeword $x(r-1)$ tells if v is of type x or y; and finally, if $v = x_i(0)$ or $v = y_i(h)$ for some h between 0 and $r-1$, then we can uniquely tell i by looking at the indices j for which $a_j(\ell) \in B_r(v)$, because by the construction $\{j : a_j(\ell) \in B_r(v)\} = \{1\} \cup A_i$. This ends the study of G_x.

We now consider the graph G, and prove that it is r-twin-free. Comparing with the previous graph G_x, it is still true that every vertex in $\mathcal{A}$ is r-covered by exactly one vertex $a_j(r)$, whereas every vertex of level 0 is r-covered by at least two vertices of level r, and no vertex with negative level is r-covered by any vertex of level r – and note that x is r-covered by exactly one $a_j(r)$, namely $a_1(r)$; it is still true that no two vertices inside $\mathcal{A}$ are r-twins, that one vertex in $\mathcal{A}$ and one vertex of type y or x (except maybe x itself) are not r-twins, and that no two vertices of type y are r-twins; also, thanks to the vertices $y_i(r-1)$, no vertex of type y can be r-twin with a vertex of type x; but we have to see what happens with the vertices of type x between themselves, and with the vertex x and one vertex in $\mathcal{A}$.

Now x is not r-twin with any $a_j(h)$, $j > 1$, thanks to $a_j(r)$, and not either with any $a_1(h)$, thanks to $a_1(r+1)$ – note in particular that $a_1(r+1)$ is the only vertex r-separating x and $a_1(2)$. Assume finally that v is of type x, $v \neq x$. If $v = x_i(0)$ for some i, the set of indices j for which $a_j(r) \in B_r(v)$ equals $\{1\} \cup A_i$, has size at least two, and identifies v. So assume that v is not on level 0, and denote by $h \in \{1, 2, \ldots, r-1\}$ the largest level for which at least one $a_j(h)$ belongs to $B_r(v)$. If the **only** shortest path between v and $a_1(1)$ goes via x, then $\{j : a_j(h) \in B_r(v)\} = \{1\}$; if there is a shortest path between v and $a_1(1)$ that goes via one (and hence all) $x_i(0)$, then $\{j : a_j(h) \in B_r(v)\} = \{1, 2, \ldots, s\}$: in both cases, h uniquely identifies v.

Ultimately, what is the smallest size of an r-identifying code in G? For a given i between 1 and k, it is easy to see that we have:

$$B_r(y_i(0)) = B_r(x_i(0)) \cup \{y_i(r-1)\}, \tag{17}$$

where the right-hand side is a disjoint union; this shows that any r-identifying code in G contains at least k elements, and ends the case $r \geq 3$. Note that if we had considered this construction for $r = 2$, then (17) would not be true, since $x(1)$ would be in $B_2(x_i(0)) \setminus B_2(y_i(0))$.

When $r = 2$, the previous construction does not work, as we have just seen,

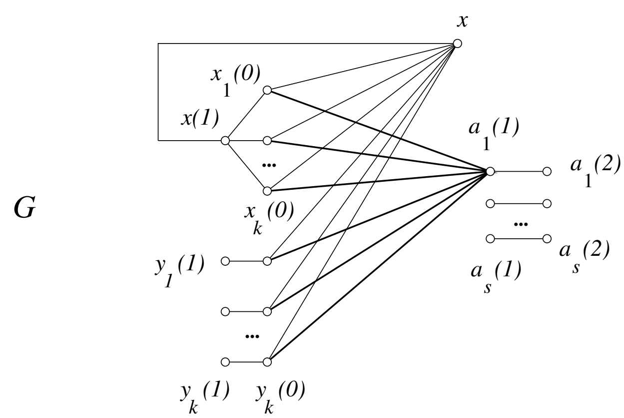

Figure 7: A partial representation of the graph G in Proposition 17, in the particular case $r = 2$: more edges exist between the vertices $x_i(0)$ and $y_i(0)$ on the one hand, and the vertices $a_j(1)$ on the other hand.

but the following does: the x-paths are again $x_i(0)x(1)x$, and the y-paths are $y_i(0)y_i(1)$ as before; the vertex $a_1(3)$ is removed, and, keeping all the edges between the vertices of level 1 in $\mathcal{A}$ and the vertices of level 0 as before, we add all the edges between x and the vertices of level 0; see Figure 7.

It is then rather straightforward, using the same kind of argument as in the general case, to check that $\mathcal{C} = \{a_j(h) : 1 \leq j \leq s, 1 \leq h \leq 2\} \cup \{x(1)\}$ is 2-identifying in G_x, that G is 2-twin-free, and that any 2-identifying code in G needs at least k codewords. △

Remark. In terms of $n = (r+1)k + r\lceil \log_2(k+2) \rceil$, the approximate order of G and G_x, we can approximate the lower bounds in (15) and (16) by $\frac{n}{r(r+1)\log_2 n}$. Again, can the bounds given in (13), (15) and (16) be significantly improved?

Conclusion 18 *When $r \geq 2$, Propositions 16 and 17 provide pairs of graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as small as approximately $-\frac{n(r-1)}{r}$, and $\frac{\gamma_r(H_x)}{\gamma_r(H)}$ can be, asymptotically, as small as approximately $\frac{r(r+1)\log_2 n}{n}$, and both can even be obtained with connected examples.*

4 General conclusion

Table 1 recapitulates the results obtained in the previous sections, using in particular the Conclusions 6, 9, 11, 13, 15 and 18; these are stated for n large with respect to r, where n is the approximate order of G or of G_x; when using $\gtrapprox X$ (respectively, $\lessapprox X$), we mean that we have a lower bound (respectively, an upper bound), for the difference or ratio, which is approximately X . We only consider the difference $\gamma_r(G_x) - \gamma_r(G)$ and the ratio $\frac{\gamma_r(G_x)}{\gamma_r(G)}$.

r	r	comment	$\gamma_r(G_x)-\gamma_r(G)$	$\frac{\gamma_r(G_x)}{\gamma_r(G)}$	reference
$r=1$		impossible to have	<-1		Concl. 6
		(connected) graphs	$\gtrapprox \frac{n}{2}-\frac{3}{2}\log_2 n$	$\gtrapprox \frac{n}{2\log_2 n}$	
≥ 2	even	connected graphs	$\gtrapprox \frac{n}{4}$		Concl. 9
	odd	connected graphs	$\gtrapprox \frac{n(3r-1)}{12r}$		Concl. 11
	any	graphs	$\gtrapprox \frac{n(2r-2)}{2r+1}$		Concl. 13
		(connected) graphs		$\gtrapprox \frac{n}{2r^2\log_2 n}$	Concl. 15
≥ 2	any	(connected) graphs	$\lessapprox -\frac{n(r-1)}{r}$	$\lessapprox \frac{r(r+1)\log_2 n}{n}$	Concl. 18

Table 1: The difference $\gamma_r(G_x)-\gamma_r(G)$ and ratio $\frac{\gamma_r(G_x)}{\gamma_r(G)}$, as functions of n and r.

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