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THE VALUE FUNCTION OF AN INTEGER PROGRAM

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We consider integer programs in which the objective function and constraint matrix are fixed while the right-hand side varies. The value function gives, for each feasible right-hand side, the criterion value of the optimal solution. We provide a precise characterization of the closed-form expression for any value function.

The class of Gomory functions consists of those functions constructed from linear functions by taking maximums, sums, non-negative multiples, and ceiling (i.e., next highest integer) operations.

The class of Gomory functions is identified with the class of all possible value functions by the following results: (1) for any Gomory function g, there is an integer program which is feasible for all integer vectors v and has g as value function; (2) for any integer program, there is a Gomory function g which is the value function for that program (for all feasible right-hand sides); (3) for any integer program there is a Gomory function f such that $f(v) \leq 0$ if and only if v is a feasible right-hand side. Applications of (1)–(3) are also given.

Key words: Integer Programming, Cutting-Planes, Subadditive Duals.

1. Introduction

The value function of the pure integer program

$$\begin{aligned} \min \quad & cx, \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0,\ x \text{ integer}, \end{aligned} \tag{1.1}$$

provides the sensitivity analysis of (1.1) to changes in the right-hand-side b. Specifically, it is the function G such that $G(b)$ is the optimal value of (1.1). When (1.1) is inconsistent (i.e. when there is no $x \geq 0$, x integer, with $Ax = b$) we put $G(b) = +\infty$. We also allow values $G(b) = -\infty$ if no lower bound can be put on cx over the set of solutions to the constraints. We shall assume throughout the

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paper that

$$A,\ b \text{ and } c \text{ are rational matrices and vectors, and } G(0) > -\infty. \tag{1.2}$$

The hypothesis $G(0) > -\infty$ implies that $G(v) > -\infty$ for all v.

This paper provides an exact description of the class of value functions, by showing how they are iteratively constructed by simple operations, and by showing also that all functions thus constructed are value functions. In order to give the intuitive content of our results, we provide this verbal sketch of the class of functions involved: they are exactly the functions (which we call 'Gomory functions' in Section 2 below) which are obtained by starting with the linear functions λb, and finitely often repeating the operations of sums, maxima and nonnegative multiples of functions already obtained, and rounding up to the nearest integer. Thus, for example the Gomory function $G(b_1, b_2) = \max\{-3b_1 + \frac{1}{2}b_2,\ b_1 + b_2 + \lceil \frac{1}{3}b_2 \rceil\}$ is the value function of some two-constraint pure integer program, where $\lceil r \rceil$ denotes the least integer which is greater than or equal to the real number r.

Perhaps the main deficiency of our intuitive summary is that it ignores the domain of definition of the value function, which, as it turns out, is defined by the vectors for which a second Gomory function is not positive (see Theorem 3.13 and Theorem 5.2 below). In Section 2 we give precise definitions for the terms to be used later on, further motivation and discussion of related literature, and some preliminary results.

Our intuitive summary shows that, once the 'technology matrix' A and 'criterion function' c are fixed in the integer program (1.1), there is a simple (although perhaps lengthy) closed form expression for the value of the solution in terms of the right-hand-side (r.h.s.) b. This result is in exact analogy to the similar result for a linear program: in fact, the value functions of linear programs are built up precisely in the same way, *except* that the rounding-up operation is not used. The characterization of linear programming value functions does not require the rationality hypotheses in (1.2).

This paper is a continuation of our earlier investigations (Blair and Jeroslow, 1977, 1979; Jeroslow, 1979). We extend work of Gomory (1963), particularly from the perspective of Chvátal (1973), and we have benefited from Schrijver (1979) and Wolsey (1979, 1981). These are the most immediate influences on our results here, and recent related work has been done by Edmonds and Giles (1977). The literature on this topic, which is part of the theory of cutting-planes, is extensive and partially summarized in the references of the survey of Jeroslow (1978a).

This completes our introductory remarks. The plan of the remainder of the paper is as follows. Section 2 defines the Gomory functions and establishes some of their important properties. In Section 3, we show that Gomory functions provide value functions, by means of the monoid basis results of Jeroslow

(1978b). Section 4 is devoted to the proof of some elementary principles which are used later, and seem to have some interest in their own right. In Section 5 we prove that value functions are Gomory functions. Section 6 is devoted to the proof of two results (Theorems 6.2 and 6.3) which are closely related to our study of the value function, the first of which (Theorem 6.2) is a result announced in Wolsey (1979). In Section 7 we work an example to illustrate our characterization of the value function.

We conclude this section with some notational issues. In (1.1), A is an m by n matrix with columns denoted by a_j: $A = [a_j]$ [cols]. Also b is an m by one vector, c is one by n, and x is n by one; for components we write $c = (c_j) = (c_1, ..., c_n)$, $b = (b_i) = (b_1, ..., b_m)$ and $x = (x_j) = (x_1, ..., x_n)$. With this notation, $Ax = b$ can also be written $\sum_{j=1}^{n} a_j x_j = b$, and we use the second form generally when some specific column of A has to be identified (as in Section 5 below).

All variables, such as the x_j, are understood as continuous throughout, which here means rational; if a variable is to be restricted to be integer this will be explicitly stated. In many contexts below, it does not actually matter whether our continuous variables are rationals or reals, but we shall not treat the latter distinction. We let Q denote the rationals. If v and w are vectors we will use vw for the inner product.

2. Chvátal functions and Gomory functions; general background

The class $\mathscr{C}$ of Chvátal functions consists of essentially the Gomory functions built up without taking maximums. The exact definition follows.

Definition 2.1. The *class $\mathscr{C}_m$ of m-dimensional Chvátal functions* is the smallest class $\mathscr{C}$ of functions with these properties;

(i) $f \in \mathscr{C}$ if $f(v) = \lambda v$ and $\lambda \in Q^m$ (here $v = (v_1, ..., v_m)$);

(ii) $f, g \in \mathscr{C}$ and $\alpha, \beta \geq 0$ with $\alpha, \beta \in Q$, implies $\alpha f + \beta g \in \mathscr{C}$;

(iii) $f \in \mathscr{C}$ implies $\lceil f \rceil \in \mathscr{C}$, where $\lceil f \rceil$ is the function defined by the condition

$$\lceil f \rceil(v) = \lceil f(v) \rceil. \tag{2.1}$$

Definition 2.2. The *class $\mathscr{C}$ of Chvátal functions* is defined by

$$\mathscr{C} = \bigcup \{\mathscr{C}_m \mid m \geq 1, \ m \text{ integer}\}. \tag{2.2}$$

Note that, while non-negative multipliers α, $\beta \geq 0$ occur in clause (ii) of Definition 2.1, the vector $\lambda \in Q^m$ of clause (i) is unrestricted in sign.

We similarly obtain an exact definition of the class of Gomory functions.

Definition 2.3. The *class $\mathscr{G}_m$ of m-dimensional Gomory functions* is the smallest class $\mathscr{C}$ of functions with the properties (i)–(iii) of Definition 2.1, and also this fourth property:

(iv) $f, g \in \mathscr{C}$ implies $\max\{f, g\} \in \mathscr{C}$.

Definition 2.4. The *class $\mathscr{G}$ of Gomory functions* is defined by

$$\mathscr{G} = \bigcup \{\mathscr{G}_m \mid m \geq 1,\ m \text{ integer}\}, \tag{2.3}$$

In Definitions 2.1 and 2.3 the function notation is understood in the usual way. For example, the function $\alpha f + \beta g$ of Definition 2.1(ii) is defined by the condition:

$$(\alpha f + \beta g)(v) = \alpha f(v) + \beta g(v) \quad \text{for all } v \in Q^m. \tag{2.4}$$

Similarly, the defining condition for $\max\{f, g\}$ in Definition 2.3(iv) is $\max\{f, g\}(v) = \max\{f(v), g(v)\}$. Note that function $f \in \mathscr{G}_m$ or $f \in \mathscr{C}_m$ are defined for all $v \in Q^m$, although in several instances below, we shall have occasion to restrict their domains to smaller sets, as e.g. integer vectors $v \in Z^m$.

Of course, the device of phrasing $\mathscr{C}_m$ and $\mathscr{G}_m$ in terms of smallest classes of functions, which contain the linear function and have certain closure properties, is equivalent to saying that these classes are built up from the linear functions by iterative finite application of the operations defined in the closure properties. Our next definition makes the concept of 'iterative application' exact.

Definition 2.5 A function f has *pre-rank* zero if it is a linear function. It has pre-rank $(r+1)$ exactly if there are functions g, h of pre-rank $\leq r$ which satisfy at least one of these conditions:

(i) $f = \alpha g + \beta h$ for some rational scalars α, $\beta \geq 0$; or

(ii) $f = \max\{g, h\}$ or

(iii) $f = \lceil g \rceil$.

In general, a function has several pre-ranks.

Definition 2.6. If f has at least one pre-rank, its *rank* is its least pre-rank.

We can now state and prove the equivalence of e.g. Definition 2.3 with one by iterative application.

Proposition 2.7. *For an m-dimensional function f, $f \in \mathscr{G}_m$ if and only if f has a pre-rank.*

Proof. Let $\mathscr{K}$ be the class of all m-dimensional functions f which have a pre-rank. If $f \in \mathscr{K}$, one proves $f \in \mathscr{G}_m$ by induction on the rank of f. Thus $\mathscr{K} \subseteq \mathscr{G}_m$. Conversely it is easy to prove that $\mathscr{K}$ satisfies (i) to (iv) of Definition 2.3. Therefore $\mathscr{G}_m \subseteq \mathscr{K}$, hence $\mathscr{G}_m = \mathscr{K}$.

Many results about Chvátal and Gomory functions are most easily proven by induction on rank. We will sometimes use the phrase 'induction on the formation of f' to mean induction on the rank of f.

We next define a class of functions which we shall need in Section 5, to discuss the components of an optimal solution to (1.1).

Definition 2.8. The class $\mathcal{G}_m^{\pm}$ of *unrestricted m-dimensional Gomory functions* is the smallest class $\mathcal{K}$ with properties (i) and (iii) of Definition 2.1, and (iv) of Definition 2.3 and also this property:

(ii) $f, g \in \mathcal{K}$ and $\alpha, \beta \in Q$ implies $\alpha f + \beta g \in \mathcal{K}$.

The class $\mathcal{G}^{\pm}$ is defined by

$$\mathcal{G}^{\pm} = \bigcup_m \{\mathcal{G}_m^{\pm} \mid m \geq 1, m \text{ integer}\}.$$

We remark that the composition of unrestricted Gomory functions is an unrestricted Gomory function.

The rounding-up operation $\lceil r \rceil$ (actually, truncation $\lfloor r \rfloor$, but $\lceil r \rceil = -\lfloor -r \rfloor$) occurs in Gomory's 'method of integer forms'. It also occurs in the following 'rule of deduction' which is due to Chvátal (1973), which we here adapt to non-negative (rather than unconstrained) integer variables:

If the inequality $\pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_n x_n \geq \pi_0$ is valid, and if the x_j are non-negative integers, then the inequality $\lceil \pi_1 \rceil x_1 + \lceil \pi_2 \rceil x_2 + \cdots + \lceil \pi_n \rceil x_n \geq \lceil \pi_0 \rceil$ is also valid. (2.6)

For example, if $\frac{1}{3}x_1 \geq \frac{1}{6}$ (i.e. $x_1 \geq \frac{1}{2}$) is valid, and if x_1 is a non-negative integer, then $x_1 \geq 1$ is valid.

Chvátal's rule can be justified in two steps. For if its hypothesis is valid, then by adding suitable multiples of the non-negativities $x_j \geq 0$, we see that the weaker statement

$$\lceil \pi_1 \rceil x_1 + \lceil \pi_2 \rceil x_2 + \cdots + \lceil \pi_n \rceil x_n \geq \pi_0 \tag{2.7}$$

is valid. Since the left-hand side of (2.7) is an integer for integral x_j, and is not less than π_0, it also is not less than $\lceil \pi_0 \rceil$. This justifies Chvátal's rule.

Chvátal and Hoffman observed (see Chvátal (1973)) that Gomory's algorithm proceeds by certain instances of the rule (2.6). The precise mode of its implementation of (2.6) is affected by the way it introduces variables for cuts, and in its given form Gomory's algorithm is not convenient for analysis. If the Chvátal operations is repeatedly applied, and is viewed as parametric in the right-hand side, it constructs a Chvátal function (see Wolsey (1981)).

The Chvátal functions are essentially the discrete analogue of linear functions. We will see below that their carrier is linear and that they are pointwise close to it (Definition 2.9 and Proposition 2.10). Now if this analogy holds true, just as the value functions of linear programs are the finite maximum of linear functions, the value function of an integer program should be a finite maximum of Chvátal functions. That is why one might conjecture that value functions are Gomory functions, at least on their domain of definition.

The technical difficulties toward establishing the equivalence of Gomory functions and integer value functions should be clear enough. For one thing, further operations, beyond maxima, might be necessary. For another, it is conceivable that infinitely many different Chvátal functions occur for the infinitely many possible right-hand-sides b. In fact, our result, that the value function G is a Gomory function, can be construed as a 'hyper-finiteness' result concerning Gomory-type algorithms based on the Chvátal operation (2.6).

We establish as a consequence of our work, that not only can such algorithms be designed to be finitely convergent, but one uniform finite upper bound on the number of cuts needed is valid for all r.h.s. (once A and c are fixed in (1.1)).

We associate with each Gomory function $f \in \mathcal{G}$ a set of homogeneous polyhedral functions called 'carriers', in our next definition. The carrier will turn out to be unique.

Definition 2.9. To every $f \in \mathcal{G}_m$ we assign a set $S(f)$ of functions inductively as follows:

(i) If $f \in \mathcal{G}_m$ is linear (i.e. $f(v) = \lambda v$ for some $\lambda \in Q^m$), then $f \in S(f)$.

(ii) If $f \in \mathcal{G}_m$ can be written as $f = \alpha g + \beta h$ with α, $\beta \in Q$ non-negative and g, $h \in \mathcal{G}_m$, and if $g' \in S(g)$ and $h' \in S(h)$, then $\alpha g' + \beta h' \in S(f)$.

(iii) If $f \in \mathcal{G}_m$ can be written as $f = \lceil g \rceil$ with $g \in \mathcal{G}_m$, and if $g' \in S(g)$, then $g' \in S(f)$.

(iv) If $f \in \mathcal{G}_m$ can be written as $f = \max\{g, h\}$ with g, $h \in \mathcal{G}_m$ and if $g' \in S(g)$ and $h' \in S(h)$, then $\max\{g', h'\} \in S(f)$.

(v) The sets $S(f)$, $f \in \mathcal{G}_m$, are formed by inductive application of rules (i)–(iv) preceeding.

Because of clause (iii) in Definition 2.9 a carrier, i.e. an element of $S(f)$, of $f \in \mathcal{G}_m$ is trivially obtained by simply deleting the integer round-up operations. For example, if $f(v) = \max\{-b_1 + \frac{3}{4}b_2, 2b_1 + \lceil -b_1 \rceil\}$, then one carrier of f is $\max\{-b_1 + \frac{3}{4}b_2, b_1\}$.

Proposition 2.10. *If $f' \in S(f)$, $f \in \mathcal{G}$, then f' is a homogeneous function iteratively constructed from linear functions by taking sums and maximums, and f' satisfies, for some constant $k \geq 0$ (depending on the formation of f'):*

$$0 \leq f(v) - f'(v) \leq k \quad \text{for all } v \in Q^m. \tag{2.8}$$

Moreover, if $f \in \mathcal{C}$, then f' is linear.

Proof. The nature of f' is evident as the clauses (i)–(iv) of Definition 2.9 do not involve the round-up operation, and such functions f' are easily proven to be homogeneous by induction on their iterative formation.

Similarly, the inequality $f(v) \geq f'(v)$ is easily seen to be preserved in clauses (i)–(iv). For example, if $f = \alpha g + \beta h$, then since $g \geq g'$ and $h \geq h'$, and α, $\beta \geq 0$, we have $f \geq \alpha g' + \beta h' = f'$. We now examine the bound $f(v) - f'(v) \leq k$ of (2.8).

If f' is a carrier of f due to clause (i), $k = 0$ since $f = f'$.

If f' is a carrier of f due to clause (ii), let k_1 and k_2 be such that

$$\begin{aligned} g(v) - g'(v) &\leq k_1 \quad \text{for all } v \in R^m, \\ h(v) - h'(v) &\leq k_2 \quad \text{for all } v \in R^m. \end{aligned} \tag{2.9}$$

k_1 and k_2 exist by induction on the number of steps in the inductive formation of g' and h' under the clauses of Definition 2.9. Then we have, as $f' = \alpha g' + \beta h'$,

$$\begin{aligned} f(v) - f'(v) &\leq \alpha(g(v) - g'(v)) + \beta(h(v) - h'(v)) \\ &\leq \alpha k_1 + \beta k_2, \end{aligned} \tag{2.10}$$

so we may take $k = \alpha k_1 + \beta k_2$.

If f' is a carrier of f due to clause (iii), let k' be such that

$$g(v) - g'(v) \leq k' \quad \text{for all } v \in R^m, \tag{2.11}$$

Then as $f' = g'$, we have

$$f(v) - f'(v) = \lceil g(v) \rceil - g'(v) < k' + 1 \tag{2.12}$$

and we may take $k = k' + 1$.

Clause (iv) formation is handled in a manner similar to clause (ii). For $f \in \mathscr{C}$, f' is linear, since no application of maximums (clause (iv)) occurs.

Corollary 2.11. *For $f \in \mathscr{G}$, $S(f)$ contains exactly one function.*

Proof. Clearly $S(f) \neq \emptyset$ by induction on the rank of f. Let $f_1', f_2' \in S(f)$. If $f_1' \neq f_2'$, let v_0 be such that $f_1'(v_0) \neq f_2'(v_0)$. Let k_1, k_2 be such that, for all v,

$$0 \leq f(v) - f_1'(v) \leq k_1, \qquad 0 \leq f(v) - f_2'(v) \leq k_2. \tag{2.13}$$

For all $\lambda \geq 0$, (2.13) applied to $v = \lambda v_0$ gives

$$\begin{aligned} \lambda |f_1'(v_0) - f_2'(v_0)| &= |f_1'(\lambda v_0) - f_2'(\lambda v_0)| \\ &\leq |f_1'(\lambda v_0) - f(\lambda v_0)| + |f(\lambda v_0) - f_2'(\lambda v_0)| \\ &\leq k_1 + k_2. \end{aligned} \tag{2.14}$$

But (2.14) is impossible for $\lambda > (k_1 + k_2)/(|f_1'(v_0) - f_2'(v_0)|)$, and this contradicts $f_1' \neq f_2'$.

Definition 2.12. A *monoid* is a set M of vectors of Q^m whic forms a semi-group under addition in Q^m. To be precise: (i) $0 \in M$; and (ii) if $v, w \in M$, then $v + w \in M$. The monoid M is *integral* if it contains only integer vectors.

Any monoid $M \neq \{0\}$ contains infinitely many elements. Any set of vectors generates a monoid by taking all non-negative integer combinations of vectors in the set.

A function $f : M \to R$, with M a monoid, is called subadditive if

$$f(v+w) \leq f(v) + f(w) \quad \text{for all } v, w \in M. \tag{2.15}$$

The interest in subadditive functions is that they generate valid cutting-planes, as summarized in our next result.

Proposition 2.13. (Gomory, 1969). *If f is a subadditive function on the monoid generated by the columns of $A = [a_j]$, then the inequality*

$$\sum_{j=1}^{n} f(a_j)x_j \geq f(b) \tag{2.16}$$

is satisfied by all solutions to (1.1).

A converse to Proposition 2.13 is also true.

Proposition 2.14 (Jeroslow, 1979). *Assume that* (1.1) *is consistent. If the inequality*

$$\sum_{j=1}^{n} \Pi_j x_j \geq \Pi_0 \tag{2.17}$$

is satisfied by all solutions to (1.1), *then there is a subadditive function f, defined on the monoid generated by the columns of $A = [a_j]$, which satisfies*

$$f(0) = 0, \qquad f(a_j) \leq \Pi_j \quad \textit{for } j = 1, \dots, n, \qquad f(b) \geq \Pi_0. \tag{2.18}$$

We remark that it is easy to derive (2.17) as a consequence of (2.18) and (2.16), if one simply notes that $x \geq 0$ for all solutions to (1.1).

An alternate form of Propositions 2.13 and 2.14 is the 'subadditive dual' we referred to earlier.

Theorem 2.15 (Jeroslow, 1979). *If* (1.1) *is consistent and has a finite value, then this program has the same finite value*:

$$\begin{array}{rl} \max & f(b), \\ \text{subject to} & f(a_j) \leq c_j, \quad j = 1, \dots, n, \\ & f \text{ subadditive on the monoid generated by the columns of} \\ & A = [a_j]. \end{array} \tag{2.19}$$

Moreover, the value function G is always an optimal solution to (2.19).

We next relate subadditivity to Gomory functions (Proposition 2.17).

Lemma 2.16. *Suppose that f and g are subadditive on M, and α, $\beta \geq 0$. Then the following functions are subadditive on M:*

(i) $\alpha f + \beta g$,

(ii) $\lceil f \rceil$,

(iii) $\max\{f, g\}$.

Proof. Let $v, w \in M$ be given. Then we have

$$\begin{aligned}(\alpha f + \beta g)(v + w) &= \alpha f(v + w) + \beta g(v + w) \\ &\leq \alpha f(v) + \alpha f(w) + \beta g(v) + \beta g(w) \\ &\leq (\alpha f(v) + \beta g(v)) + (\alpha f(w) + \beta g(w)) \\ &= (\alpha f + \beta g)(v) + (\alpha f + \beta g)(w),\end{aligned} \tag{2.20}$$

which establishes (i). Also

$$\begin{aligned}\lceil f \rceil(v + w) &= \lceil f(v + w) \rceil \\ &\leq \lceil f(v) + f(w) \rceil \\ &\leq \lceil f(v) \rceil + \lceil f(w) \rceil = \lceil f \rceil(v) + \lceil f \rceil(w),\end{aligned} \tag{2.21}$$

which establishes (ii). The first inequality in (2.21) is due to the subadditivity of f (see (2.15)) and the fact that $\lceil r \rceil$ is a non-decreasing function of r. The second inequality in (2.21) is due to the easily verified subadditivity of the function $\lceil r \rceil$.

Moreover, for f and g subadditive,

$$\begin{aligned} f(v + w) &\leq f(v) + f(w) \leq \max\{f(v), g(v)\} + \max\{f(w), g(w)\}, \\ g(v + w) &\leq g(v) + g(w) \leq \max\{f(v), g(v)\} + \max\{f(w), g(w)\}, \end{aligned} \tag{2.22}$$

By taking the maximum over both sides in (2.22), we prove (iii).

Proposition 2.17. *All Gomory functions $f \in \mathcal{G}_m$ are subadditive on Q^m.*

Proof. By induction on the rank of $f \in \mathcal{G}_m$.

Thus, Gomory functions can be used to obtain valid cutting-planes (in Proposition 2.13).

The fact that Chvátal functions are subadditive, and usually somewhere strictly subadditive (i.e. in (2.15) there is strict inequality for at least some choice of v, w), shows that the negative of a Chvátal function is not usually subadditive. For example, $-\lceil v \rceil$, is not subadditive (although it is a typical element of $\mathcal{G}^{\pm}$, because $-1 = -\lceil 1 \rceil = -\lceil 0.5 + 0.5 \rceil > -2 = (-\lceil 0.5 \rceil) + (-\lceil 0.5 \rceil)$, which contradicts (2.15).

The following simple result is a 'normal form' for Gomory functions.

Proposition 2.18. *Every Gomory function $f \in \mathcal{G}_m$ is a maximum of finitely many Chvátal functions:*

$$f = \max\{g_1, \ldots, g_t\} \quad \textit{all } g_i \in \mathcal{C}_m. \tag{2.23}$$

Proof. By induction on the rank of f. If f is a linear function the result is immediate.

Suppose that $f = \alpha g + \beta h$ where $\alpha, \beta \geq 0$ are rational and g and h are of lower rank than f. We write

$$g = \max_{i \in I}\{g_i\}, \qquad h = \max_{j \in J}\{h_j\}, \tag{2.24}$$

for finite non-empty index sets I and J, where g_i and h_j are Chvátal functions. Then one easily verifies that

$$f = \max_{\substack{i \in I \\ j \in J}}\{\alpha g_i + \beta h_j\}. \tag{2.25}$$

Suppose $f = \ulcorner g \urcorner$, where g has lower rank than f. We may again assume (2.24) holds for g, and we can conclude

$$f = \max_{i \in I}\{\ulcorner g_i \urcorner\}. \tag{2.26}$$

Suppose that $f = \max\{g, h\}$, where g and h have lower rank than f. We again may assume (2.24), and we have

$$f = \max\left\{\max_{i \in I}\{g_i\}, \max_{j \in J}\{h_j\}\right\}, \tag{2.27}$$

so that again the inductive hypothesis is preserved.

3. Gomory functions are value functions

Just as we have been using small letters $f, g, h, \ldots$ for Chvátal and Gomory functions we shall reserve capital letters $F, G, H, \ldots$ for value functions.

In this section, we derive sufficient closure properties for value functions, to insure that Gomory functions are value functions, at least when their domains are suitably restricted. The issue regarding the domain of definition is, of course, that value functions are defined, i.e. are not $+\infty$, only for certain r.h.s. b in (1.1), while Gomory functions are defined in all Q^m.

In this section, we will confine ourselves to showing how Gomory functions arise in the setting of programs (1.1) with A, b and c integral. The value functions associated with such programs we shall call *integral* value functions. The extension of our work to the rational case (i.e. hypothesis (1.2)) is straightforward (see e.g. Corollary 3.14 below).

We proceed by use of certain results in Jeroslow (1978b), particularly Theorem 3.2 below.

A set $S \subseteq Q^m$ is a *slice* precisely if S has the form

$$S = T + M, \tag{3.1}$$

where $T \neq \emptyset$ is a finite set of integer vectors in Q^m, and M is an integer monoid in Q^m which has a finite set of generators.

A monoid is the discrete analogue of a convex cone with vertex at the origin; a slice is the discrete analogue of a polyhedron. It is trivial for polyhedra that their intersection is a polyhedron. The analogous result is true for slices (but see also Blair and Jeroslow (1979) or Jeroslow (1978b) for a continuous result which has a false integer analogue).

Theorem 3.1 (Jeroslow, 1978b). *If T_1 and T_2 are slices and $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cap T_2$ is a slice.*

Theorem 3.2. *If M_1 and M_2 are integer monoids which are finitely generated, then $M_1 \cap M_2$ is also a finitely generated monoid.*

Proof. It is trivial that $M_1 \cap M_2$ is a monoid.

Since $M_1 \cap M_2 \supseteq \{0\}$, $M_1 \cap M_2$ is a slice:

$$M_1 \cap M_2 = T + M, \tag{3.2}$$

where T is a non-empty finite set of integer vectors, and M is a finitely generated integer monoid. As $M_1 \cap M_2$ is a monoid, so is $T + M$, hence

$$T + T + M = (T + M) + (T + M) = T + M. \tag{3.3}$$

Let $T = \{t_1, \dots, t_a\}$ and let M be generated by $s_1, \dots, s_b$. We claim that $T + M$ is generated by $U = \{t_1, \dots, t_a, s_1, \dots, s_b\}$.

It is clear that any element $t + m \in T + M$ $(t \in T, m \in M)$ is generated by U. Conversely, let v be generated by U:

$$v = \sum_{i=1}^{a} n_i t_i + \sum_{j=1}^{b} m_j s_j. \tag{3.4}$$

One may easily prove, by induction on $\rho = \sum_{i=1}^{a} n'_i$, that any vector of the form $\sum_{i=1}^{a} n'_i t_i$ is an element of $T + M$, using (3.3) for the inductive step, and the fact that $0 \in T + M$ for $\rho = 0$ (the latter by (3.2) and the fact that $0 \in M_1 \cap M_2$).

Thus in (3.4), $\sum_{i=1}^{a} n_i t_i \in T + M$, and as $\sum_{j=1}^{b} m_j s_j \in M$, we have $v \in T + M + M = T + M$. This completes the proof of our claim.

We recall our assumption at the start of the section that A is integer. $G(b)$ will be defined (i.e., $G(b) < +\infty$) only for certain integer vectors b. In what follows, we may interchangeably write row vectors as column vectors, or vice-versa, simply to improve readability.

Lemma 3.3 *If $M_1, \dots, M_r$ are finitely generated integral monoids, so is their Cartesian product $M_1 \times \cdots \times M_r$.*

Proof. Without loss of generality, $r = 2$. Let M_j be generated by $v^{j1}, \dots, v^{jt}$ for $j = 1, 2$ (we may take t to be the same as $0 \in M_j$). Then $M_1 \times M_2$ is generated by

$$\begin{pmatrix} v^{11} \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} v^{1t} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v^{21} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ v^{2t} \end{pmatrix}. \tag{3.5}$$

Lemma 3.4. *If M is a finitely generated integral monoid, then so is the projection*

$$M_1 = \{v^1 \mid \textit{for some } v^2, (v^1, v^2) \in M\}. \tag{3.6}$$

Proof. If M is generated by (v^{1j}, v^{2j}) for $j = 1, \dots, t$, then M_j is generated by v^{1j} for $j = 1, \dots, t$.

Proposition 3.5. *Let G be a function $G : Q^m \to R \cup \{+\infty\} \cup \{-\infty\}$. Then G is the integral value function of some integer program* (1.1) *if and only if the set M defined by*

$$M = \{(z, b) \mid z \textit{ is an integer and } z \geq G(b)\} \tag{3.7}$$

is a finitely generated integer monoid.

Proof. Suppose that M is a finitely generated integer monoid, and let its generators be (c_j, a_j). Then

$$M = \left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} \text{there is an integer vector } x \text{ with} \\ z = cx, \quad Ax = b, x \geq 0 \end{array} \right\}, \tag{3.8}$$

where $A = [a_j]$ (cols) and $c = (c_j)$. Then the value function of the integer program (1.1) for this A and c is

$$\min\{cx \mid Ax = b, x \geq 0 \text{ and } x \text{ integer}) = \min\{z \mid (z, b) \in M\} = G(b). \tag{3.9}$$

Conversely, if $G(b)$ is the integral value function of (1.1), we have

$$\begin{aligned} &\left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} z \text{ is integral} \\ \text{and } z \geq G(b) \end{array} \right\} = \\ &= \left\{ \begin{pmatrix} z \\ b \end{pmatrix} \;\middle|\; \begin{array}{l} \text{for some non-negative integers } x_0, x_1, \dots, x_m, \\ \binom{z}{b} = x_0 \binom{1}{0} + \sum_{j=1}^{n} x_j \binom{c_j}{a_j} \end{array} \right\}. \end{aligned} \tag{3.10}$$

Via the same ideas as in the proof of Proposition 3.5, one easily establishes the following result.

Corollary 3.6. *M is the domain of some integral value function G (i.e. $M = \{b \mid G(b) < +\infty\}$) if and only if M is a finitely generated integer monoid.*

Throughout this paper, the infimum over an empty set is $+\infty$.

Theorem 3.7. *Let H and $H_1, \dots, H_r$ be integral value functions and let Q and*

$Q^1, \dots, Q^r$ be matrices of rationals. Then the function defined by

$$G(b) = \inf\left\{ H(w_1, \dots, w_r) \,\middle|\, \begin{array}{l} w_1, \dots, w_r \text{ and } b \text{ are integral, and there are} \\ \text{integer vectors } b^1, \dots, b^r \text{ such that } Qb + \\ \sum_{j=1}^r Q^j b^j \geq 0 \text{ and moreover all } w_j \geq \\ H_j(b^j),\ j = 1, \dots, r \end{array} \right\} \tag{3.11}$$

is an integral value function.

In (3.11), $w_1, \dots, w_r$ are integers, the vectors $b, b^1, \dots, b^r$ may be of different dimensions, and the matrices $Q, Q^1, \dots, Q^r$ are dimensioned to make all expressions displayed compatible.

Proof. The monoid

$$M = \left\{ \begin{pmatrix} z \\ w'_1 \\ \vdots \\ w'_r \end{pmatrix} \,\middle|\, \begin{array}{l} z \text{ is integer and} \\ z \geq H(w'_1, \dots, w'_r) \end{array} \right\} \tag{3.12}$$

and the monoids

$$M_j = \left\{ \begin{pmatrix} w_j \\ b^j \end{pmatrix} \,\middle|\, \begin{array}{l} w_j \text{ is integer} \\ \text{and } w_j \geq H_j(b^j) \end{array} \right\}, \quad j = 1, \dots, r, \tag{3.13}$$

are all integer monoids with a finite set of generators, by Proposition 3.5. By Lemma 3.3, so is $M \times M_1 \times \cdots \times M_r$.

It is well known that (the result goes back to Hilbert (1890) for one proof, also see Jeroslow (1978b)) any monoid, defined by imposing integrality conditions on the solutions to homogeneous linear inequalities in rationals, has a finite set of generators. In particular, this monoid is finitely generated:

$$P = \left\{ \begin{pmatrix} z \\ w'_1 \\ \vdots \\ w'_r \\ w_1 \\ b^1 \\ \vdots \\ w_r \\ b^r \\ b \end{pmatrix} \,\middle|\, \begin{array}{l} z, w'_1, \dots, w'_r, b^1, \dots, w_r, b^r, \text{ and } b \text{ are} \\ \text{integral, and } w'_j = w_j \text{ for } j = 1, \dots, r \\ \text{and } Qb + \sum_{j=1}^r Q^j b^j \geq 0 \end{array} \right\} \tag{3.14}$$

By Corollary 3.2, the monoid

$$M' = (M \times M_1 \times \cdots \times M_r) \cap P \tag{3.15}$$

has a finite set of generators. Let M^* denote the projection of M' onto its co-ordinates (z, b). By Lemma 3.4, M^* has a finite set of generators. One also checks from (3.11) that

$$(z, b) \in M^* \quad \text{if and only if } z \text{ is integral and } z \geq G(b). \tag{3.16}$$

By Proposition 3.5, G is an integral value function.

In what follows, when we write a composition of functions such as

$$G(b) = H(H_1(b), \dots, H_r(b)) \tag{3.17}$$

we shall understand that $G(b)$ is defined (i.e., $G(b) < +\infty$) exactly if each quantity $w_j = H_j(b) < +\infty$ and also $H(w_1, \dots, w_r) < +\infty$, in which case $G(b) = H(w_1, \dots, w_r)$.

Corollary 3.8. *If H is a monotone non-decreasing integral value function and $H_1, \dots, H_r$ are integral value functions which are nowhere $-\infty$, then the function G in (3.17) is an integral value function.*

Proof. Note that, by the monotonicity of H,

$$G(b) = \inf\left\{ H(w_1, \dots, w_r) \,\middle|\, \begin{array}{c} w_1, \dots, w_r \text{ are integral and, there are integral} \\ b^j = b \text{ with } w_j \geq H_j(b^j) \text{ for } j = 1, \dots, r \end{array} \right\}. \tag{3.18}$$

Theorem 3.7 applies.

Corollary 3.9. *If H_1 and H_2 are integral value functions, n_1 and n_2 are non-negative integers and D is an integer, then the following three functions are integral value functions*:

(i) $G = n_1 H_1 + n_2 H_2$,
(ii) $G = \lceil H_1 / D \rceil$,
(iii) $G = \max\{H_1, H_2\}$.

Proof. In cases (i) and (iii), it suffices to show that $G(b) = H(H_1(b), H_2(b))$, where H is a monotone non-decreasing value function. In case (ii), we show that $G(b) = H(H_1(b))$, where H is a monotone non-decreasing integral value function. Corollary 3.8 then yields the desired result.

For (i), the value function H is that of this two row integer program:

$$\begin{array}{rl} \inf & n_1 x_1 + n_2 x_2, \\ \text{subject to} & x_1 \qquad\;\; = b_1, \\ & \qquad\;\; x_2 = b_2, \\ & x_1, x_2 \text{ integral}, \end{array} \tag{3.19}$$

where we can obtain a formulation in non-negative variables by setting $x_j = x_j' - x_j''$ where x_j' and x_j'' are integral and non-negative. The value function is non-decreasing because $n_1, n_2 \geq 0$.

For (ii), the value function is that of the integer program

$$\begin{array}{rl} \inf & x_1, \\ \text{subject to} & Dx_1 - x_2 = b, \\ & x_2 \geq 0, \\ & x_1, x_2 \text{ integral,} \end{array} \tag{3.20}$$

and again a formulation in non-negative variables easily follows. The function $H(b) = \lceil b/D \rceil$ is clearly non-decreasing.

For (iii), H is the value function of the integer program

$$\begin{array}{rl} \inf & x_1, \\ \text{subject to} & x_1 - x_2 = b_1, \\ & x_1 - x_3 = b_2, \\ & x_2, x_3 \geq 0, \\ & x_1, x_2, x_3 \text{ integer,} \end{array}$$

and the desired properties are easily verified.

Proposition 3.10. *If ρ is an integer vector, then the function $F(v) = \rho v$ is an integral value function.*

Proof. F is the value function of this integer program:

$$\begin{array}{rl} \inf & \rho x, \\ \text{subject to} & Ix = b, \\ & x \text{ integer.} \end{array} \tag{3.22}$$

and by the usual device of setting $x = x' - x''$ with $x', x'' \geq 0$ we can put (3.22) in the form (1.1).

The statement that 'Gomory functions are value functions' has to be properly construed. The domain of a Gomory function g is all of Q^m, while that of a value function G is some subset of the integer vectors Z^m; hence a Gomory function g must first be restricted to Z^m for any such statement to hold. A second issue derives from the fact that a Gomory function g need not have an integer value $g(v)$ even for an integer vector $v \in Z^m$, yet the value $G(v)$ for a value function is always integral, since c is assumed integral in this section. A precise statement follows next.

Theorem 3.11. *If g is a Gomory function, there is an integral value function G and non-negative integer $D \geq 1$ such that*

$$g(v) = G(v)/D \quad \textit{for all } v \in Z^m. \tag{3.23}$$

Proof. By induction on the rank of g. If $g(v) = \lambda v$ for some $\lambda \in Q^m$, write

$\lambda = \rho/D$ for ρ integer and $D \geq 1$ integer. Then $g(v) = \rho v/D$ and the result follows by Proposition 3.10.

If $g = \alpha h_1 + \beta h_2$ where $\alpha = n_1/D_1$ and $\beta = n_2/D_2$ are non-negative rationals, $D_1, D_2 \geq 1$ and h_1 and h_2 are Gomory functions, let D_3 and D_4 be non-negative integers such that

$$\begin{aligned} h_1(v) &= H_1(v)/D_3 \quad \text{for all } v \in Z^m, \\ h_2(v) &= H_2(v)/D_4 \quad \text{for all } v \in Z^m, \end{aligned} \tag{3.24}$$

for value functions H_1 and H_2. Then for $v \in Z^m$,

$$g(v) = \frac{n_1 H_1(v)}{D_1 D_3} + \frac{n_2 H_2(v)}{D_2 D_4} = \frac{(D_2 D_4 n_1) H_1(v) + (D_1 D_3 n_2) H_2(v)}{D_1 D_3 D_2 D_4}. \tag{3.25}$$

Since $n_1' = D_2 D_4 n_1$ and $n_2' = D_1 D_3 D_2$ are non-negative integers, $n_1' H_1 + n_2' H_2$ is a value function by Corollary 3.9(i).

If $g = \lceil h_1 \rceil$, let (3.24) hold. Then for v integer, $g(v) = \lceil H_1(v)/D_3 \rceil$ and $\lceil H_1/D_3 \rceil$ is a value function by Corollary 3.9(ii).

If $g = \max\{h_1, h_2\}$, let (3.24) hold. For $v \in Z^m$ we have

$$\begin{aligned} g(v) &= \max\{H_1(v)/D_3, H_2(v)/D_4\} \\ &= \frac{1}{D_3 D_4} \max\{D_4 H_1(v), D_3 H_2(v)\}. \end{aligned} \tag{3.26}$$

Now $D_4 H_1(v)$ and $D_3 H_2(v)$ are value functions by Corollary 3.9(i), and so is $\max\{D_4 H_1, D_3 H_2\}$ by Corollary 3.9(iii).

We also wish to be able to restrict Gomory functions g by a non-negativity condition $h \leq 0$ on another Gomory function h, and still have a value function. In this context, the domain of g and h will be Q^m, not Z^m, hence some hypothesis on the Gomory function h will be needed. This hypothesis will take the form

$$h(v) > 0 \quad \text{if } v \notin Z^m, \tag{3.27}$$

so that, in essence, the compositely-defined function is $< +\infty$ only for $v \in Z^m$.

We proceed toward our goal in the next two results.

Theorem. 3.12. *Let G and H be integral value functions. Then the function defined by*

$$F(v) = \begin{cases} G(v) & \text{if } H(v) \leq 0, \\ +\infty & \text{if } H(v) > 0, \end{cases} \tag{3.28}$$

is also an integral value function.

Proof. We have $F(v) = K(G(v), H(v))$, where, for $w_1, w_2 \in Z$,

$$K(w) = \begin{cases} w_1 & \text{if } w_2 \leq 0, \\ +\infty & \text{if } w_2 > 0. \end{cases} \tag{3.29}$$

K is non-decreasing and it is the value function of this two-row integer program:

$$\begin{aligned} \inf\quad & x_1, \\ \text{subject to}\quad & x_1 = w_1, \\ & -x_2 = w_2, \\ & x_2 \geq 0, \\ & x_1, x_2 \text{ integer.} \end{aligned} \tag{3.30}$$

Then F is a value function by Corollary 3.8.

Theorem 3.13. *Suppose that g and h are Gomory functions and that h satisfies* (3.27). *Let the function f be defined by*

$$f(v) = \begin{cases} g(v) & \text{if } h(v) \leq 0, \\ +\infty & \text{if } h(v) > 0. \end{cases} \tag{3.31}$$

Then there is an integral value function F and an integer $D \geq 1$ with

$$f(v) = F(v)/D \quad \text{for all } v \in Q^m. \tag{3.32}$$

Proof. By Theorem 3.11 there are value functions G and H and integers $D_1, D_2 \geq 1$ with

$$\begin{aligned} g(v) &= G(v)/D_1 \quad \text{for all } v \in Z^m, \\ h(v) &= H(v)/D_2 \quad \text{for all } v \in Z^m. \end{aligned} \tag{3.33}$$

Note that, by (3.27),

$$h(v) \leq 0 \quad \text{if and only if } H(v) \leq 0 \text{ for all } v \in Q^m. \tag{3.34}$$

From (3.31), we have, using (3.33),

$$D_1 f(v) = \begin{cases} G(v) & \text{if } H(v) \leq 0, \\ +\infty & \text{if } H(v) > 0. \end{cases} \tag{3.35}$$

By Theorem 3.12, $D_1 f(v)$ is a value function.

Corollary 3.14. *Suppose that g and h are Gomory functions and that there is a rational non-singular m by m matrix B such that, for all $v \in Q^m$,*

$$h(v) > 0 \quad \text{if } Bv \notin Z^m. \tag{3.36}$$

Then there is a value function F arising from a program (1.1) *with rational A, c such that*

$$F(v) = \begin{cases} g(v) & \text{if } h(v) \leq 0, \\ +\infty & \text{if } h(v) > 0. \end{cases} \tag{3.37}$$

Proof. Define $h'(v) = h(B^{-1}v)$, $g'(v) = g(B^{-1}v)$, and apply Theorem 3.13 to g', h' to obtain an integer matrix A', an integer vector c' and an integer D, such that it has value function

$$F'(v) = \begin{cases} Dg(B^{-1}v) & \text{if } h(B^{-1}v) \le 0, \\ +\infty & \text{if } h(B^{-1}v) > 0. \end{cases} \tag{3.38}$$

Let $c = c'/D$, $A = B^{-1}A'$. Then the value function G of (1.1) satisfies

$$\begin{aligned} F(v) &= \min\{c'x/D \mid B^{-1}A'x = v, x \ge 0 \text{ and integer}\} \\ &= \frac{1}{D}\min\{c'x \mid A'x = Bv, x \ge 0 \text{ and integer}\} \\ &= \frac{1}{D} F'(Bv) = \begin{cases} g(v) & \text{if } h(v) \le 0, \\ +\infty & \text{if } h(v) > 0. \end{cases} \end{aligned} \tag{3.39}$$

4. Some results on the relation between an integer program and its LP relaxation

We begin with two results showing that if an integer program is inconsistent, then a perturbation of the linear programming relaxation is also inconsistent. Throughout this section $a_1, \dots, a_n \in Q^m$ are fixed, e is a vector with all components equal to one.

Theorem 4.1. *There exists a $k > 0$ such that, for all $v \in R^m$, if there are no integer* x_j such that

$$\sum_1^n a_j x_j \ge v, \tag{4.1}$$

then there are no x_j such that

$$\sum_1^n a_j x_j \ge v + ke. \tag{4.2}$$

Proof. Let k be n times larger than any non-negative component of any a_j. If $x = (x_1, \dots, x_n)$ satisfies (4.2), then replacing each x_i by the next lower integer provides an integer solution that satisfies (4.1). Hence if (4.1) has no integer solution, (4.2) has no continuous solution.

Next we examine the analogous problem for integer programs whose constraints are given as equations rather than inequalities. For $v \in R^m$ define

$$I_v = \{(x_1, \dots x_n) \mid \sum_1^n a_j x_j = v;\ x_j \ge 0;\ x_j \text{ integer}\}. \tag{4.3}$$

Theorem 4.2. *There exists a $K_1 \ge 0$ such that, for all v, either*:

(i) *I_v is non-empty; or*

(ii) *there are no integer x_j (positive or negative) such that $\Sigma a_j x_j = v$; or*

(iii) *there is no $x \ge K_1 e$ such that $\sum_1^n a_j x_j = v$.*

In other words, if an integer program with right-hand-side v is inconsistent, then either it remains inconsistent when the non-negativity constraints are dropped or else the LP relaxation is inconsistent if lower bounds of K_1 are imposed on all the variables.

Proof. Let $S = \{(\alpha_1, \dots, \alpha_n) \mid \sum_1^n \alpha_j a_j = \mathbf{0}\}$. Let $F \subset Z^n$ be a basis for S. Let K_1 be larger than the dimension of S multiplied times the largest non-negative component of any member of F. If v is such that (ii) and (iii) are false then there is an integer $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ such that $\sum_1^n a_j \boldsymbol{x}_j = v$ and scalars α_f, $f \in F$, such that $\boldsymbol{x} + \sum_{f \in F} \alpha_f f \geq K_1 e$. If α'_f is the largest integer $\leq \alpha_f$, then $\boldsymbol{x} + \sum_{f \in F} \alpha'_f f$ is integer, because $F \subset Z^n$, and non-negative because $\sum_{f \in F} (\alpha'_f - \alpha_f) f \geq -K_1 e$. Hence $\boldsymbol{x} + \sum_{f \in F} \alpha'_f f \in I_v$.

Remark. An alternate form of Theorem 4.2 replaces (iii) by

(iii) There is a $1 \leq J \leq n$ dependent on v such that $\{x \mid \sum_1^n a_j x_j = v;\ x \geq 0,\ x_j \geq k\} = \emptyset$.

The k constructed here is n times the K_1 constructed in the proof of Theorem 4.2. We will not use this result later, and omit the detailed proof.

Next we present some results relating the optimal solution to an integer program to the optimal solution to linear programming problems. The results we require later are Theorem 4.6 and Corollary 4.7. These can be deduced from Blair and Jeroslow (1977, 1979), but our presentation is self-contained. Also, we believe the value of the constant K_2 is new.

For $v \in R^m$, define

$$L_v = \left\{x \mid \sum a_j x_j = v,\ x_j \geq 0\right\}, \tag{4.4}$$

$$R_c(v) = \inf\{cx \mid x \in L_v\}, \tag{4.5}$$

$$G_c(v) = \inf\{cx \mid x \in I_v\}, \tag{4.6}$$

Lemma 4.3. *There exists $K_2 > 0$ and a finite $F \subset Z^n$ such that, for every c, if every component of x is either zero or $\geq K_2$, then either*

(*i*) $R_c(\sum a_j x_j) = cx$; *or*

(ii) *there is $y \in F$ such that $\sum a_j(x_j + y_j) = \sum a_j x_j$, $c(x+y) < cx$, and $x + y \geq \mathbf{0}$.*

Proof. For $S \subset \{1, 2, \dots n\}$ let

$$U_S = \left\{x \mid \sum a_j x_j = \mathbf{0} \text{ and } x_j = 0 \text{ if } j \notin S\right\}. \tag{4.7}$$

For each S such that U_S is one-dimensional, let $x^S \in U_S$ be a non-zero integer vector. We take F to be x^S and $-x^S$ for all such S. K_2 is chosen to be as large as any component of any member of F. If x, c are such that (i) is false there is a $z \in R^n$ such that: (α) $\sum a_j z_j = \mathbf{0}$; ($\beta$) $cz < 0$; (γ) $z_j \geq 0$ if $x_j = 0$. Let z^* satisfying (α)–(γ) be such that $\{j \mid z_j^* = 0\}$ is maximal. By definition of F, there is a $w \in F$ such that $w_j = 0$ if $z_j^* = 0$.

We claim that $z^* = \Theta w$ for some scalar Θ. Let Θ be such that $z' = z^* - \Theta w$ satisfies

$$z_j' \geq 0 \quad \text{if } z_j^* \geq 0, \tag{4.8}$$

$$z_j' \leq 0 \quad \text{if } z_j^* \leq 0, \tag{4.9}$$

$$\text{for at least one } j,\ z_j' = 0 \text{ and } z_j^* \neq 0. \tag{4.10}$$

z' satisfies (α) and (γ). (4.10) and the maximality property of z^* imply $cz' \geq 0$. If $z' \neq \mathbf{0}$, we could find a scalar Θ' such that $z^* - \Theta' z'$ satisfies (α)–(γ) and has more zero components than z^*. Since this would contradict the maximality we must have $z' = \mathbf{0}$, $z = \Theta w$, and our claim is established.

Hence there is a $y \in F$ satisfying (α)–(γ) [$y = w$ or $-w$]. If every component of x is zero or $\geq K_2$, then $x + y \geq \mathbf{0}$; hence (ii) holds.

Corollary 4.4. *Let the set F be as in Lemma 4.3. For $x \in Z^n$, $x \geq \mathbf{0}$, define $\bar{x}$ by*

$$\bar{x}_j = \begin{cases} x_j & \text{if } x_j \geq K_2, \\ 0 & \text{otherwise} \end{cases}. \tag{4.11}$$

Then, for every c, either $R_c(\sum a_j \bar{x}_j) = c\bar{x}$ or there exists $y \in F$ such that $x + y \geq \mathbf{0}$ and $c(x + y) < cx$.

Proof. Apply Lemma 4.3 to $\bar{x}$.

Lemma 4.5. *Let c, v be such that $I_v \neq \emptyset$ and $G_c(v) > -\infty$. Then, for every $x \in I_v$, there is an $x^* \in I_v$ such that*

$$cx^* \leq cx, \tag{4.12}$$

$$\{a_i \mid x_i^* \geq K_2\} \quad \text{is linearly independent}, \tag{4.13}$$

$$R_c(\textstyle\sum a_j \bar{x}_j^*) = c\bar{x}^* \quad [\bar{x}^* \text{ defined by } (4.11)]. \tag{4.14}$$

Proof. Apply Corollary 4.4 to x. Either $R_c(\sum a_i \bar{x}_i) = c\bar{x}$ or there is an $x' = x + y \in I_v$ with $cx' \leq cx + \max\{cy \mid y \in F, cy < 0\}$. Then we apply Corollary 4.4 to x' etc. Since $G_c(v) > -\infty$ we must eventually obtain an $x^{(n)}$ such that (4.12) and (4.14) hold. By (4.14) and the complementary slackness theorem there is a $w \in R^m$ such that $wa_j \leq c_j$ for all j and $wa_j = c_j$ if $\bar{x}_j^{(n)} > 0$. If (4.13) fails there is a $y \in F$ such that $cy \leq 0$, $y_j = 0$ if $\bar{x}_j^{(n)} = 0$, and at least one component of y is negative (recall $G_c(v) > -\infty$). For some integer $\Theta > 0$, $x + \Theta y \in I_v$ and $x + \Theta y$ has fewer components $\geq K_2$. This process is repeated until an x^* is obtained such that (4.12) and (4.13) hold, and $\bar{x}_j^{(b)} > 0$ if $\bar{x}_j^* > 0$, hence $wa_j = c_j$ if $\bar{x}_j^* > 0$. To verify (4.14) note that if $x \geq 0$ and $\sum a_j x_j = \sum a_j \bar{x}_j^*$, then $cx \geq \sum (wa_j) x_j = w(\sum a_j x_j) = w(\sum a_j \bar{x}_j^*) = c\bar{x}^*$.

Theorem 4.6. *For any c, v such that $I_v \neq \emptyset$ and $G_c(v) > -\infty$ there is an $x^* \in I_v$ satisfying* (4.13), (4.14) *and*

$$G_c(v) = cx^*. \tag{4.15}$$

Proof. Any $x \in I_v$ can be decomposed as $\bar{x}$ plus a vector x' all of whose components are between 0 and K_2. Since there are only finitely many x' and at most one $x \in I_v$ satisfying (4.13) for each choice of x' and linearly independent set, there are only finitely many $x \in I_v$ satisfying (4.13) and (4.14). Let x^* be an x with cx minimal.

Corollary 4.7. *For every $c \in R^n$ there is a K_3 such that if $I_v \neq \emptyset$ and $R_c(v) > -\infty$, then*

$$R_c(v) \leq G_c(v) \leq R_c(v) + K_3. \tag{4.16}$$

Proof. $R_c(v) \leq G_c(v)$ is immediate. Parametric linear programming theory[1] implies that there is an M_1 such that $|R_c(v) - R_c(w)| \leq M_1 \|v - w\|$. Let $M_2 = \max\{|cx| \mid 0 \leq x \leq K_2 e\}$. Let $M_3 = \max\{\|\sum_1^n a_j x_j\|, 0 \leq x \leq K_2 e\}$. From theorem 4.6 we know there is an $x^* \in I_v$ such that $cx^* = G_c(v)$ and $c\bar{x}^* = R_c(\sum a_j \bar{x}_j^*)$. Since $0 \leq x^* - \bar{x}^* \leq K_2 e$ we have

$$G_c(v) = cx^* \leq c\bar{x}^* + M_2 = R_c\left(\sum a_j \bar{x}_j^*\right) + M_2 \leq R_c(v) + M_1 M_3 + M_2,$$

so we may take $K_3 = M_1 M_3 + M_2$.

5. Value functions are Gomory functions

We will use notation (especially (4.3)–(4.6)) and results from Section 4. Let $a_1, \ldots, a_n \in Q^m$ and $c \in Q^n$ be fixed. The two main results of this section are:

Theorem 5.1. *There is a Gomory function $f : R^m \to R$ such that, for every v, $f(c) \leq 0$ if and only if $I_v \neq \emptyset$.*

Theorem 5.2. *There is a Gomory function g such that, for every v such that $I_v \neq \emptyset$, $g(v) = G_c(v)$.*

The function f is a 'consistency tester' for the integer program. g is a function

[1] This follows from the fact that $R_c(v) = \max \lambda_i v$, where the λ_i are the extreme points of the polyhedron $\{\lambda \mid \lambda a_j \leq c_j, 1 \leq j \leq n\}$. We take $M_1 = \max \|\lambda_j\|$.

which is equal to the value function of the given integer program, whenever it is consistent. Our proof of Theorem 5.2 uses Theorem 5.1, which requires several preliminary results. The proofs are constructive.

Lemma 5.3. *Let*

$$S = \{v \mid v = \sum a_j x_j,\ x_j \text{ integer (positive or negative)}\}. \tag{5.1}$$

There is a linearly independent $U \subset Q^m$ such that

$$S = \left\{v \mid v = \sum_{u \in U} a_u u,\ \alpha_u \text{ integer}\right\}.$$

Proof. Let $H = \{j \mid$ some member of S has first $(j-1)$ components zero and jth component positive$\}$. For each $j \in H$, $u_j \in U$ is a vector where the first $(j-1)$ components are zero and whose jth component is the smallest possible positive number. Set $U = \{u_j \mid j \in H\}$. It is easy to show that if $v \in S$ and the first $(j-1)$ components of v are zero, then $v - \alpha u_j$ will have the first j components zero for some integer α. This process can be continued to yield a representation of v as an integer linear combination of the u_j.

Remark. The proof of Lemma 5.3 consists essentially of taking the Smith normal form of A.

Corollary 5.4. *Let S be as in Lemma 5.3. There is a Gomory function f_1 such that $v \in S$ if and only if $f_1(v) \leq 0$.*

Proof. Let d be the dimension of $L(S)$, the linear span of S. There are $w_1, \ldots, w_{m-d} \in Q^m$ such that $v \in L(S)$ if and only if $w_i v = 0$, $1 \leq i \leq m-d$. There are $z_u \in Q^m$ such that if $v \in L(S)$, then $v = \sum_{u \in U} (z_u v)u$. Hence $v \in S$ if $w_i v = 0$ for all i and $z_u v$ is integer for all $u \in U$. Hence we may take

$$f_1(v) = \max\{w_i v, -w_i v; 1 \leq i \leq m-d; \ulcorner z_u v \urcorner - z_u v; u \in U\}. \tag{5.2}$$

Theorem 4.2 says that if $I_v = \emptyset$, then either S above is empty or else inserting lower bounds of K_1 on the variables produces an inconsistent linear program. Corollary 5.4 shows that the first situation can be detected by a Gomory function. Our next lemma is a fact from parametric linear programming. It states that if lower bounds produce an inconsistent LP, this inconsistency can be detected in a uniform manner over all v.

Lemma 5.5. *There exist $\lambda_1, \ldots, \lambda_M \in Q^m$ such that*

$$\lambda_i a_j \leq 0 \quad \text{for all } 1 \leq i \leq M,\ 1 \leq j \leq n; \tag{5.3}$$

for every $v \in R^m$, $k \geq 0$, *if there is no* $x \in L_v$ *with* $x \geq ke$, *then, for some* i,

$$\lambda_i v + k \sum_{j=1}^{n} (-\lambda_i a_j) > 0. \tag{5.4}$$

Proof. $\{w \mid a_j w \leq 0,\ 1 \leq j \leq n\}$ is a cone. We apply the finite basis theorem to obtain $\lambda_1, \ldots, \lambda_M$ such that every member of the cone is a non-negative linear combination of the λ_i.

Standard results of linear programming (e.g., Farkas' lemma) show that if there is no $x \in L_v$ with $x \geq ke$, then there exists $w \in R^m$ and scalars $s_1, \ldots, s_n \geq 0$ such that $wa_j + s_j = 0$, $1 \leq j \leq n$, and $wv + k(\sum s_j) > 0$. There are non-negative α_i such that $w = \sum \alpha_i \lambda_i$. If $s_{ij} = -\lambda_i a_j$, then $s_j = \sum_{i=1}^{M} \alpha_i s_{ij}$. Since

$$\sum_{i=1}^{M} \alpha_i \left(\lambda_i v + k \sum_{j=1}^{n} s_{ij} \right) = wv + k \sum s_j > 0,$$

there must be at least one i such that $\lambda_i v + k \sum_{j=1}^{n} (-\lambda_i a_j) > 0$.

Our next result is motivated by the fact that Gomory functions, being sub-additive, generate valid inequalities. Suppose A and v are such that every $x \in I_v$ satisfies $x_1 \geq p$, for some integer p. Suppose we also know that every $x \in I_{v-pa_1}$ satisfies $\sum \beta_j x_j \geq \gamma$. Then we can conclude that every $x \in I_v$ satisfies $\sum \beta_j x_j \geq \gamma + \beta_1 p$. We will show that if there are Gomory functions generating the first two inequalities, then one can construct a Gomory function that generates the third one.

Lemma 5.6. *Let p be a Gomory function such that $p(a_1) = 1$; $p(a_j) \leq 0$, $2 < j \leq n$ [i.e., p generates an inequality of the form $x_1 \geq$ something]. Let h be any Gomory function. Then there is a Gomory function s such that:*

$$s(a_j) \leq h(a_j), \quad 2 \leq j \leq n, \tag{5.5}$$

$$s(a_1) = h(a_1), \tag{5.6}$$

for any v, if $p(v)$ is integer, then

$$p(v)s(a_1) + h(v - p(v)a_1) = s(v). \tag{5.7}$$

Proof. Our argument proceeds by induction on the formation of h. If h is linear we take $s(v) = h(v)$. If $h(v) = \lceil h_1(v) \rceil$ where h_1 is a Gomory function, then by induction hypothesis there is an s_1 such that (5.5)–(5.7) hold for s_1, h_1. We define

$$s(v) = \lceil s_1(v) + (\lceil s_1(a_1) \rceil - s_1(a_1)) p(v) \rceil. \tag{5.8}$$

For $2 \leq j \leq n$, $s(a_j) \leq \lceil s_1(a_j) \rceil \leq \lceil h_1(a_j) \rceil = h(a_j)$, hence (5.5) holds. $s(a_1) =$

$\lceil s_1(a_1) \rceil = \lceil h_1(a_1) \rceil = h(a_1)$ so (5.6) holds. If $p(v)$ is integer,

$$\begin{aligned} s(v) &= \lceil s_1(v) - s_1(a_1)p(v) \rceil + \lceil s_1(a_1) \rceil p(v) \\ &= \lceil h_1(v - p(v)a_1) \rceil + s(a_1)p(v) \\ &= h(v - p(v)a_1) + p(v)s(a_1) \end{aligned}$$

so (5.7) holds.

If $h(v) = \alpha h_1(v)$ where $\alpha \geq 0$, we take $s(v) = \alpha s_1(v)$. If $h(v) = h_1(v) + h_2(v)$, we take $s(v) = s_1(v) + s_2(v)$. If $h(v) = \max\{h_1(v), h_2(v)\}$ and $h_1(a_1) \geq h_2(a_1)$, we take

$$s(v) = \max\{s_1(v), s_2(v) + (h_1(a_1) - h_2(a_1))p(v)\}. \tag{5.9}$$

For $2 \leq j \leq n$,

$$s(a_j) \leq \max\{s_1(a_j), s_2(a_j)\} \leq \max\{h_1(a_j), h_2(a_j)\} = h(a_j).$$

Also $s(a_1) = s_1(a_1) = h(a_1)$. If $p(v)$ is integer,

$$\begin{aligned} s(v) &= \max\{p(v)s_1(a_1) + h_1(v - p(v)a_1), h_2(v - p(v)a_1) + h_1(a_1)p(v)\} \\ &= p(v)s_1(a_1) + h(v - p(v)a_1) \end{aligned}$$

so (5.5)–(5.7) hold in this case and the induction is complete.

Remark. The construction of s is based on the idea that a Gomory function represents a method of obtaining valid inequalities, with each step in the formation of the function corresponding to the generation of a new valid inequality from those previously obtained. The function s represents the same sequence of operations on inequalities as the function h, except (see (5.8) and (5.9)) that whenever h uses the inequality $x_1 \geq 0$, s uses the inequality $x_1 \geq p(v)$ generated by p.

Our next task is to show how we can use information about the consistency of an integer program with $n-1$ columns to obtain valid inequalities for an integer program with n columns. Let

$$LI_v = \left\{(x_2, \ldots x_n) \mid \sum_2^n a_j x_j = v,\ x_j \geq 0 \text{ and integer}\right\}. \tag{5.10}$$

Suppose we know that $x_1 \geq p$ (p non-negative integer) if $x \in I_v$ and that $LI_{v-pa_1} = \emptyset$. Then we may conclude that $x_1 \geq p+1$ if $x \in I_v$. Our next result uses this idea in the context Gomory functions.

Lemma 5.7. *Suppose there is a Gomory function h such that $h(v) \leq 0$ if and only if $LI_v \neq \emptyset$. Then for any k there is a Gomory function p_k such that*

$$p_k(a_1) \leq 1, \tag{5.10}$$

$$p_k(a_j) \leq 0, \quad 2 \leq j \leq n, \tag{5.11}$$

$$\textit{for any } v, \quad p_k(v) \geq k+1 \textit{ if } I_v = \emptyset. \tag{5.12}$$

Proof. We argue by induction on k. For $k=0$ we may take $p_0(v)=\lceil \alpha h(v)\rceil$ for some $\alpha>0$. If $p_k(v)$ satisfies (5.10)–(5.12) and $p_k(a_1)<1$ we may take $p_{k+1}(v)=\lceil (1/\alpha)p_k(v)\rceil$ where $\alpha=\max\{p_k(a_1),\frac{1}{2}\}$.[2] If $h(a_1)\leq 0$ we may take $p_{k+1}(v)=(k+2)\lceil h(v)\rceil$.

The interesting case is $p_k(a_1)=1$, $h(a_1)>0$. By scaling we may assume $h(a_1)=1$. We apply Lemma 5.6 with $p=p_k$ to obtain s such that $s(a_j)\leq h(a_j)\leq 0$ and $s(a_1)=h(a_1)=1$. We define $p_{k+1}(v)=\lceil \max\{p_k(v),s(v)\}\rceil$. If $I_v=\emptyset$, then by (5.12) either $p_k(v)>k+1$ or $p_k(v)=k+1$. Since $p_{k+1}(v)\geq\lceil p_k(v)\rceil$ we are done in the first case. If $p_k(v)=k+1$, then (5.7) implies $s(v)=p_k(v)+h(v-(k+1)a_1)>k+1$, hence $p_{k+1}(v)\geq\lceil s(v)\rceil\geq k+2$.

We are ready to carry out:

Proof of Theorem 5.1. Our proof proceeds by induction on n. First we deal with the case $n=1$. There are $\lambda_1,\ldots,\lambda_{m-1}$ such that ν is a scalar multiple of a_1 if $\lambda_j v=0$, $1\leq j\leq m-1$. There is w such that if $v=\alpha a_1$, then $\alpha=wv$ (e.g. we may take $w=a_1/\|a_1\|$. We may take

$$f(v)=\max\{\lambda_i v,-\lambda_i v,-wv,\lceil wv\rceil-wv\}.$$

Now we deal with the induction step. We are assuming for every $n-1$ rational vectors there is a Gomory function f such that $f(v)>0$ if and only if v is not a non-negative integer combination of the $n-1$ vectors. In particular we are assuming there are Gomory functions h_j, $1\leq j\leq n$, such that $h_j(v)>0$ if and only if there is no $x\in I_v$ with $x_j=0$. We apply Lemma 5.7 with $k=K_1$ to obtain functions T_j such that

$$T_j(a_j)\leq 1, \tag{5.13}$$

$$T_j(a_i)\leq 0,\quad i\neq j, \tag{5.14}$$

$$T_j(v)\geq K_1 \tag{5.15}$$

if there are no $x\in I_v$ with $x_j=0$.

Let $\lambda_1,\ldots,\lambda_M$ be as in Lemma 5.5. Define

$$f_2(v)=\max_{1\leq i\leq M}\left\{\lambda_i v+\sum_{j=1}^{n}(-\lambda_i a_j)T_j(v)\right\}, \tag{5.16}$$

$$f(v)=\max\{f_1(v),f_2(v)\}, \tag{5.17}$$

where $f_1(v)$ was constructed in Lemma 5.4.

If $I_v=\emptyset$, then by Lemma 4.2, either $S=\emptyset$ (hence $f_1(v)>0$ by Lemma 5.4) or there is no $x\in L_v$ with $x\geq K_1 e$. In this last case Lemma 5.5 implies that, for

[2] The original manuscript used an incorrect choice of α, as was remarked to us by P. Carstensen.

some i, $\lambda_i v + K_1 \sum_{j=1}^n (-\lambda_i a_j) > 0$. Using (5.15) and (5.3),

$$f_2(v) \geq \lambda_i v + \sum_{j=1}^{n} (-\lambda_i a_j) T_j(v) > 0,$$

hence $f(v) > 0$ if $I_v = \emptyset$.

To show $f(v) \leq 0$ if $I_v \neq \emptyset$ it suffices to show $f(a_j) \leq 0$ and use the subadditivity of f. $f_1(a_j) \leq 0$ by Corollary 5.4. By (5.16), (5.3), (5.13) and (5.14), $f_2(a_j) \leq 0$, hence $f(a_j) \leq 0$.

Our next task is the proof of Theorem 5.2. Let $N > 0$ be such that

$$Nc_j \text{ is integer}, \quad 1 \leq j \leq n; \qquad N \text{ integer}. \tag{5.18}$$

Our method of proof is to deduce valid inequalities by making use of information about I_v together with information about $I_v \cap \{x \mid cx = p\}$. Suppose we know that if $x \in I_v$, then $cx \geq p$, where Np is integer; and that if $x \in I_v$ and $cx = p$, then $\alpha x \geq \beta$. There should be some way of combining these two inequalities into a single inequality $(\alpha + Lc)x \geq \beta + Lp$ for some $L \geq 0$. The next result shows that this does happen when the inequalities are generated by Gomory functions.

Lemma 5.8. *Let $p : R^m \to R$ be a Gomory function such that $p(a_j) \leq c_j$ for all j. Let $f : R^{m+1} \to R$ be any Gomory function. Let $p'(v) = (1/N)\lceil Np(v) \rceil$ ($p'(a_j) \leq c_j$ by (5.18)). There is a Gomory function $h : R^m \to R$ and an $L \geq 0$ such that*

$$h(a_j) \leq f(c_j, a_j) + Lc_j, \quad 1 \leq j \leq n, \tag{5.19}$$

$$\textit{for every } v, \quad h(v) \geq f(p'(v), v) + Lp'(v). \tag{5.20}$$

Proof. We construct h by induction on the formation of f. If f is a linear functional $f(r, v) = ar + wv$ with $\alpha \leq 0$, we take $h(v) = wu$, $L = -\alpha$ and (5.19) and (5.20) hold as equations.

If $f(r, v) = \alpha r + wv$ with $\alpha > 0$, take $h(v) = \alpha p'(v) + wv$, $L = 0$. (5.20) is an equation, (5.19) follows because $p'(a_j) \leq c_j$.

If $f(r, v) = \lceil f_1(r, v) \rceil$, then by induction hypothesis there are h_1, L_1 such that (5.19) and (5.20) hold. Take $L = N \lceil L_1 \rceil$ and define

$$h(v) = \lceil h_1(v) + (L - L_1)p'(v) \rceil = \lceil h_1(v) - L_1 p'(v) \rceil + Lp'(v). \tag{5.21}$$

We have

$$h(a_j) \leq \lceil h_1(a_j) + (L - L_1)c_j \rceil = \lceil h_1(a_j) - L_1 c_j \rceil + Lc_j \leq f(c_j, a_j) + Lc_j,$$

so (5.19) holds. Also

$$h(v) \geq \lceil f_1(p'(v), v) \rceil + Lp'(v) = f(p'(v), v) + Lp'(v)$$

so (5.20) holds.

If $f(r, v)=\alpha f_1(r, v)$, $\alpha \geq 0$, we take $h=\alpha h_1$, $L=\alpha L_1$. If $f(r, v)=f_1(r, v)+f_2(r, v)$, we take $h=h_1+h_2$, $L=L_1+L_2$. If $f(r, v)=\max\{f_1(r, v); f_2(r, v)\}$ with $L_1 \geq L_2$ take $L=L_1$ and

$$h(v)=\max\{h_1(v); h_2(v)+(L_1-L_2)p'(v)\}. \tag{5.22}$$

We have

$$h(a_j) \leq \max\{f_1(c_j, a_j)+L_1c_j; f_2(c_j, a_j)+L_1c_j\}=f(c_j, a_j)+Lc_j.$$

Also

$$\begin{aligned} h(v) &\geq \max\{f_1(p'(v), v)+L_1p'(v); f_2(p'(v), v)+L_1p'(v)\} \\ &= f(p'(v), v)+Lp'(v). \end{aligned}$$

Thus (5.19) and (5.20) hold in this case and the induction is complete.

Remark. The idea behind this construction is similar to that for Lemma 5.6. The function h represents the same sequence of operations as the function f, except that at every step at which f uses the equation $cx=p$, h uses the inequality $cx \geq p'$ generated by the Gomory function p'.

Corollary 5.9. *For every $k \geq 0$ there is a Gomory function $T_k : R^m \to R$ such that*

$$T_k(a_j) \leq c_j, \quad 1 \leq j \leq n, \tag{5.23}$$

$$\text{for all } v, \quad NT_k(v) \text{ is integer,} \tag{5.24}$$

$$\text{for all } v, \quad \text{if } I_v \neq \emptyset, \quad T_k(v) \geq \min\left\{G_c(v), \frac{1}{N}\ulcorner NR_c(v)\urcorner+\frac{k}{N}\right\}. \tag{5.25}$$

Recall N defined by (5.18), R_c by (4.5). Note that (5.23) and sub-additivity imply $T_k(v) \leq G_c(v)$.

Proof. We argue by induction on k. We take $T_0(v)=(1/N)\ulcorner NR_c(v)\urcorner$.[3] $R_c(a_j) \leq c_j$ is immediate and (5.23) follows because Nc_j is integer. (5.24) and (5.25) are also easy.

Suppose we have constructed $T_k(v)$. By Theorem 5.1 applied to $a_j'=(c_j, a_j)$ there is an $f: R^{m+1} \to R$ such that $f(r, v) \leq 0$ if and only if there is an $x \in I_v$ such that $cx=r$. We apply Lemma 5.8 with $p=T_k, f$ as described. By (5.24), $p'=p=T_k$. Define

$$T_{k+1}(v)=\max\left\{T_k(v), \frac{1}{N}\ulcorner Nh(v)/L\urcorner\right\} \quad \text{if } L>0, \tag{5.26}$$

$$T_{k+1}(v)=\max\left\{T_k(v), T_k(v)+\frac{1}{N}\ulcorner Nh(v)\urcorner\right\} \quad \text{if } L=0. \tag{5.27}$$

[3] R_c is a Gomory function by the remark at the end of Corollary 4.7.

(5.23) holds for T_{k+1} because $T_k(a_j) \leq c_j$ and $f(c_j, a_j) \leq 0$ in (5.19). (5.24) is immediate. If $T_k(v) = G_c(v)$, we see that (5.25) holds for T_{k+1}, because $T_{k+1}(v) \geq T_k(v)$. If $T_k(v) < G_c(v)$, then $f(T_k(v), v) = f(p'(v), v) > 0$. (5.20) implies $T_{k+1}(v) > T_k(v)$, which implies that $T_{k+1}(v)$ is a rational with denominator N and hence

$$T_{k+1}(v) \geq \frac{1}{N} \lceil NR_c(v) \rceil + \frac{(k+1)}{N}.$$

Since $NG_c(v)$ is integer, $T_k(v) < G_c(v)$ implies

$$G_c(v) \geq \frac{1}{N} \lceil NR_c(v) \rceil + \frac{(k+1)}{N}$$

hence (5.25) is established for T_{k+1}.

Now we can return to:

Proof of Theorem 5.2. With K_3 constructed by Corollary 4.7,[4] we let $f = T_{NK_3}$. Using (4.16) condition (5.25) becomes $T_{NK_3}(v) \geq G_c(v)$. As remarked above, (5.23) implies the opposite inequality, hence $f(v) = G_c(v)$.

The referee suggests an alternate proof of Theorem 5.2, sketched as follows. In Wolsey (1979) it is shown that there is a finite set of subadditive functions which, independent of the r.h.s. b, generate a set of inequalities of the form (2.16) which define the convex hull of feasible solutions. From Schrijver (1979) one can deduce that each of these subadditive functions is a Chvátal function. By combining these two facts with standard results on the value function of linear programs, our sketch is complete. We have not checked the details of this approach.

Next we consider the dependence of the optimal solution to (IP) on the right-hand-side v. Consider the one-row problem

$$\begin{aligned} \min \quad & x + y, \\ & 3x + y = v, \\ & x, y \geq 0 \text{ and integer.} \end{aligned}$$

The optimal solutions for $v = 4, 5, 9$, have $x = 1, 1, 3$, respectively. The optimal solution value for x is not a subadditive function of v, hence cannot be a Gomory function. However, our next result shows that the optimal solution can be obtained by using unrestricted Gomory functions (defined in Proposition 2.7).

To deal with cases involving more than one optimal solution, we define the lexicographically smallest optimal solution to be that optimal solution which makes x_1^* as small as possible. If there is more than one such x we make x_2^* as small as possible, given the specified value of x_1^*, etc.

[4] The assumption $R_c(v) > -\infty$ needed to invoke Corollary 4.7 is not restrictive. It is easy to show that if $R_c(v) = -\infty$ for any v, then for all v, either $I_v = \emptyset$ or $G_c(v) = -\infty$.

Corollary 5.10. *Assume $R_c(v) > -\infty$ for all right-hand sides v. If $I_v \neq \emptyset$, let x_v^* be the lexicographically smallest member of I_v such that $cx_v^* = G_c(v)$ [i.e. x_v^* is an optimal solution]. Then there are unrestricted Gomory functions $f_j : R^m \to R$ such that if $I_v \neq \emptyset$, then the* jth *component of x_v^* is $f_j(v)$.*

Proof. The first component of x_v^* is the value of the optimal solution to the integer program

$$\begin{aligned} \min \quad & x_1, \\ \text{subject to} \quad & cx = \alpha_0, \\ & \textstyle\sum a_j x_j = v, \\ & x \geq \mathbf{0},\ x \text{ integer}, \end{aligned} \tag{5.28}$$

when we set $\alpha_0 = G_c(v)$. By Theorem 5.2, there are Gomory functions $g_0(v)$, $g_1(\alpha, v)$ such that $g_0(v) = G_c(v)$ and $g_1(\alpha_0, v)$ is the optimal value of (5.28). Then the first component of x_v^* is $g_1(g_0(v), v)$, which is an unrestricted Gomory function of v. Similarly, the second component of x_v^* is the value of the optimal solution to

$$\begin{aligned} \min \quad & x_2, \\ \text{subject to} \quad & cx = \alpha_0, \\ & x_1 = \alpha_1, \\ & \textstyle\sum a_j x_j = v, \\ & x_j \geq \mathbf{0}, \quad x \text{ integer}, \end{aligned} \tag{5.29}$$

where $\alpha_0 = g_0(v)$, $\alpha_1 = g_1(g_0(v), v)$. By Theorem 5.2, there is a Gomory function $g_2(\alpha_0, \alpha_1, v)$ which is the optimal value of (5.29). Hence the second component of x_v^* is $g_2(g_0(v), g_1(g_0(v), v), v)$. The other components of x_v^* are developed similarly.

We next present the analogues of Theorems 5.1 and 5.2 for an integer program in inequality format:

$$\begin{aligned} \min \quad & c_1 x_1 + \cdots + c_n x_n, \\ \text{subject to} \quad & a_1 x_1 + \cdots + a_n x_n \geq v, \\ & x_1, \ldots, x_n \geq 0 \text{ and integer}. \end{aligned} \tag{5.30}$$

We will assume that the vectors a_j have all components integer (the extension to the rational case is straightforward). Then (5.30) is equivalent to the integer program in equation form:

$$\begin{aligned} \min \quad & c_1 x_1 + \cdots + c_n x_n, \\ \text{subject to} \quad & a_1 x_1 + \cdots + a_n x_n - e_1 y_1 - \cdots - e_m y_m = \lceil v \rceil, \\ & x, y \geq 0 \text{ and integer}, \end{aligned} \tag{5.31}$$

where $e_i \in R^m$ has one in ith component, zero in other components, and v is taken componentwise. Application of theorems 5.1, 5.2 yields:

Corollary 5.11. *There is a Gomory function f such that* (5.30) *is consistent if and only if* $f(\lceil v \rceil) \leq 0$.

Corollary 5.12. *There is a Gomory function g such that* $g(\lceil v \rceil)$ *is the value of* (5.30) *for any v for which* $f(\lceil v \rceil) \leq 0$.

We can extract further information about f, g. A Gomory function h is specified by a definition giving the precise order in which the various operations (sums, round-ups, etc.) are carried out. A Gomory function can have several different definitions, e.g. $\frac{3}{2}x$ defines the same function as $x + \frac{1}{2}x$. We will use $\hat{h}$ to denote a definition of h.

Definition 5.13. For a given $\hat{h}$ we associate $T(\hat{h}) \subseteq Q^m$, the set of all λ occurring in linear functionals used in $\hat{h}$. Formally $T(\hat{h})$ is defined by

(i) if $\hat{h}(v) = \lambda v$, then $T(\hat{h}) = \{\lambda\}$;

(ii) if $\hat{h} = \alpha\hat{h}$, or $\hat{h} = \lceil \hat{h}_1 \rceil$, then $T(\hat{h}) = T(\hat{h}_1)$;

(iii) if $\hat{h} = \hat{h}_1 + \hat{h}_2$ or $\hat{h} = \max\{\hat{h}_1, \hat{h}_2\}$, then $T(\hat{h}) = T(\hat{h}_1) \cup T(\hat{h}_2)$.

The class $\mathcal{MG}$ consists of those Gomory functions h for which there is $\hat{h}$ such that every $\lambda \in T(\hat{h})$ has non-negative components.

Every $h \in \mathcal{MG}$ is a monotone non-decreasing Gomory function (the converse is also true, but non-trivial).[5] $\mathcal{MG}$ is closed under composition in the sense that if $f: R^Q \to R$, and $g_i: R^m \to R$, $1 \leq i \leq Q$, are in $\mathcal{MG}$, then so is $h(v) = f(g_1(v), \ldots, g_Q(v))$. In particular, if $f \in \mathcal{MG}$, then $h(v) = f(\lceil v \rceil) \in \mathcal{MG}$.

Lemma 5.14. *Let* $1 \leq j \leq n$. *Let* h_0 *be a Chvátal function defined by* $\hat{h}_0$ *such that* $h_0(-e_j) \leq 0$. *Then there is a Chvátal function* h_1 *defined by* $\hat{h}_1$ *such that*:

(i) $h_1(v) = h_0(v)$ *for all v with integer components*;

(ii) *if* $\lambda \in T(\hat{h}_1)$, *then* $\lambda e_j \geq 0$;

(iii) *if* $\lambda \in T(\hat{h}_1)$, $\lambda = \lambda' + ke_j$, *where* λ' *is a non-negative linear combination of members of* $T(h_0)$.

Proof. We construct $\hat{h}_1$ by moving integer quantities through the round-up operations $\lceil\ \rceil$ which occur in $\hat{h}_0$. For example, if $h_0(v) = 4\frac{1}{3}e_1v + \lceil -2\frac{1}{2}e_1v \rceil$ we could take $h_1(v) = 1\frac{1}{3}e_1v + \lceil \frac{1}{2}e_1v \rceil$.

Formally, we proceed by induction on the number of round-up operations used in h_0. For any $\hat{h}$ define $n(\hat{h})$ by

(i) if $\hat{h}(v) = \lambda v$, $n(\hat{h}) = 0$;

(ii) if $\hat{h} = \alpha\hat{f}$, $n(\hat{h}) = n(\hat{f})$;

(iii) if $\hat{h} = \hat{f} + \hat{g}$, $n(\hat{h}) = n(\hat{f}) + h(\hat{g})$;

(iv) if $\hat{h} = \lceil \hat{f} \rceil$, $n(\hat{h}) = n(\hat{f}) + 1$.

[5] The proof is by induction on the formation of h. The key step is that if $h = f + g$ is a monotone Gomory function, then for some linear function λ. $f + \lambda$ and $g - \lambda$ are monotone Gomory functions.

If $n(\hat{h}_0)=0$, then we may take $h_1(v)=\lambda v$, since h_0 is linear. If $n(\hat{h}_0)>0$, then there is $\lambda \in Q^{m}$' $\alpha_1,\dots,\alpha_k\geq 0$; $f_1,\dots,f_k$ such that

$$\hat{h}_0(v)=\lambda v+\alpha_1\ulcorner \hat{f}_1(v)\urcorner+\alpha_2\ulcorner \hat{f}_2(v)\urcorner+\cdots+\alpha_k\ulcorner \hat{f}_k(v)\urcorner$$

where $n(f_i)<n(\hat{h}_0)$. Since $h_0(-e_j)\leq 0$ there are integers $m_0, m_1,\dots,m_k$ such that

(i) $\lambda m_0+\sum_1^k \alpha_i m_i=0$;

(ii) $\lambda e_j+m_0\leq 0$;

(iii) $\ulcorner f_i(-e_j)\urcorner+m_i=0$.

Define $h_1(v)=(\lambda-m_0e_j)v+\sum \alpha_i\ulcorner g_i(v)\urcorner$ where $g_i(v)=f_i(v)-(m_ie_j)v$. $h_1(v)=h_0(v)$ for integer v by (i), (ii) and (iii) mean we may apply the induction hypothesis to produce suitable $\hat{g}_i$.

Corollary 5.15. *If h_0 is a Gomory function and $h_0(-e_j)\leq o$ for all j, then there is an $h\in \mathcal{MG}$ such that $h(v)=h_0(v)$ for all integer v.*

Proof. By Proposition 2.16, h_0 is a maximum of Chvátal functions. Use Lemma 5.13 each Chvátal function for each $1\leq j\leq n$ to get the desired representation.

Now the strengthening of corollaries 5.4 and 5.5 is immediate.

Theorem 5.16. *There is an $f\in\mathcal{MG}$ such that* (5.30) *is consistent if and only if $f(v)\leq 0$.*

Theorem 5.17. *There is a $g\in\mathcal{MG}$ such that $g(v)=$ optimum value of* (5.30) *if $f(v)\leq 0$.*

The next result was first proven by Wolsey (1981) by an analysis of Gomory's method of integer forms (Gomory, 1963). However, Gomory assumes that the initial linear programming relaxation has a tableau of lexicographically positive columns (Gomory, 1963, bottom page 286; also pp. 287 and 289). Hence the method of proof in Wolsey (1981) cannot be used for all integer programs.

Theorem 5.18. *If* (1.1) *is consistent and has finite value, there is an optimal solution f to the subadditive dual problem* (2.19) *which is a Chvátal function.*

Proof. By Theorem 2.15, the value function G of (1.1) optimally solves (2.19); hence by Theorem 5.2, there is a Gomory function g which is an optimum in (2.19). By Proposition 2.18, $g=\max\{f_1,\dots,f_1\}$ for certain Chvátal functions f_1, $1\leq i\leq t$. If f_j is such that $g(b)=f_j(b)$, then f_j is an optimum for (2.19).

Remark. Several alternative proofs of Theorem 5.18 are possible. Schrijver (1979) building on work of Edmonds and Giles (1977) has recently established that finitely many applications of Chvátal's operation (as in (2.6)) yields the convex hull of integer points for any integer program (without the restriction in

Gomory (1963)). This can be used to construct the appropriate f in Theorem 5.18. Another proof is based on (non-trivial) modifications of the method of integer forms so that it will work for all integer programs.

An interesting 'separation principle' follows from Theorem 5.18 which we give next.

Corollary 5.19. *If b is not an element of a finitely generated integer monoid M, there is a Chvátal function f such that*

(i) $f(m) \leq 0$ *for all* $m \in M$; *and*

(ii) $f(b) > 0$.

Proof. Let the generators of M be $a_1, \dots, a_n$. Then the following integer program is consistent and has finite value one:

$$\begin{aligned} \text{minimize} \quad & x_{n+1}, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_j x_j + b x_{n+1} = b, \\ & x_j \geq 0 \text{ and integer.} \end{aligned} \tag{5.32}$$

The subadditive dual of (5.32) is the program:

$$\begin{aligned} \max \quad & F(b), \\ \text{subject to} \quad & F(a_j) \leq 0, \quad j = 1, \dots, n, \\ & F(b) \leq 1, \end{aligned} \tag{5.33}$$

and by Theorem 5.18, the optimum value of this dual is achieved by a Chvátal function f; hence $f(b) = 1$. From $f(a_j) \leq 0$ for $j = 1, \dots, n$ one easily derives $f(m) \leq 0$ for all $m = \sum_{j=1}^{n} a_j x_j$ ($x_j \geq 0$ and integer) by induction on $\alpha = \sum_{j=1}^{n} x_j$.

We conclude this section with a result which relates the value function G_c of (1.1) to that of the linear relaxation.

Theorem 5.20. *Let g be any Gomory function such that*

$$g(v) = G_c(v) \quad \textit{whenever } I_v \neq \emptyset, \tag{5.34}$$

and let $\tilde{g}$ be the carrier of g. Then

$$\tilde{g}(v) = R_c(v) \quad \textit{whenever } R_c(v) < +\infty. \tag{5.35}$$

Proof. Suppose that there is a v_0 with $R_c(v_0) < +\infty$ and $\tilde{g}(v_0) \neq R_c(v_0)$. Then for suitably large integral $D \geq 1$, $I_{Dv_0} \neq \emptyset$, and as $\tilde{g}$ and R_c are homogeneous functions, $\tilde{g}(Dv_0) \neq R_c(Dv_0)$. Then without loss of generality, $D = 1$ and $I_{v_0} \neq \emptyset$.

We established in Proposition 2.10 that there exists $k_1 \geq 0$ with

$$0 \leq g(v) - \tilde{g}(v) \leq k_1 \quad \text{for all } v \in Q_m. \tag{5.36}$$

By Corollary 4.7, there exists $k_2 \geq 0$ such that

$$0 \leq G_c(v) - R_c(v) \leq k_2 \quad \text{whenever } I_v \neq \emptyset, \tag{5.37}$$

and hence

$$0 \leq g(v) - R_c(v) \leq k_2 \quad \text{whenever } I_v \neq \emptyset. \tag{5.38}$$

Starting from (5.36) and (5.38), we may apply the kind of reasoning as in the proof of Corollary 2.11 (particularly as in the display (2.14)) to the homogeneous functions $\tilde{g}$ and R_c, and we obtain a contradiction from our supposition that $\tilde{g}(v_0) \neq R_c(v_0)$.

Theorem 5.20 has this interpretation; if we start with a closed-form Gomory expression g for the optimal value of (1.1) and simply go through the expression erasing all round-up symbols, we obtain a closed-form expression for the optimal value of the linear relaxation of (1.1).

6. The structure of $G_c(v)$ as c varies

Throughout this section $a_1, \ldots, a_n \in Q^m$ will be fixed. In Section 5 we determined the parametric form of the value of (1.1) in its right-hand side; now we seek a simultaneous uniformity in the criterion vector c.

We begin with a result which says that there is a finite set F such that, if x is any feasible but not optimal solution to an integer program, there is a better feasible solution obtained by adding some member of F to x. The set F is independent of the criterion vector c. This type of result was first established by Graver (1975); we give an alternate proof (and a somewhat different statement of the result) via monoid basis results.

Lemma 6.1. *There is a finite $F \subset Z^n$ such that, for any $v \in Q^m$, $c \in R^n$, $x \in I_v$, either:*

(i) *$cx = G_c(v)$; or*

(ii) *for some $y \in F$, $x + y \in I_v$ and $c(x+y) < cx$.*

Proof. Define $M \subseteq Z^{2n}$ by

$$M = \{(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \mid \sum a_j\alpha_j = \sum a_j\beta_j,\ \alpha_j,\ \beta_j \geq 0 \text{ and integer}\}. \tag{6.1}$$

M is a monoid defined by rational polyhedral constraints. Theorem 7 of Jeroslow (1978b) (indeed, Hilbert's result) implies that there is a finite $W \subset M$ such that every member of M is a non-negative integer combination of members of W. Define $F \subseteq Z^n$ by

$$F = \{y \mid y = \alpha - \beta \text{ where } (\alpha, \beta) \in W\}. \tag{6.2}$$

If $v \in Q^m$, $x \in I_v$ and (i) fails, there is a $z \in I_v$ with $cz < cx$. Since $(z, x) \in M$ there are non-negative integers n_w such that

$$\sum_{w \in W} n_w w = (z, x). \tag{6.3}$$

Since $cz < cx$ there is at least one $w = (\alpha, \beta) \in W$ such that $n_w \geq 1$ and $c\alpha < c\beta$. As $(\sum n_w w) - w + (\alpha, \alpha) \in M$, $x - \beta + \alpha \in I_v$. Then $(\alpha - \beta) \in F$ and $c(\alpha - \beta) < 0$.

Theorem 6.2. *There is a finite set $T = \{d_1, \ldots, d_N\} \subset Q^n$ such that, for any $c \in R^n$, $v \in R^m$, if $I_v \neq \emptyset$ and $G_c(v) > -\infty$, there are $\alpha_d \geq 0$, $d \in T$, such that:*
(i) $\sum_{d \in T} \alpha_d d = c$, *and*
(ii) $\sum_{d \in T} \alpha_d G_d(v) = G_c(v)$.

The algebraic content of Theorem 6.2 is that any inequality $cx \geq G_c(v)$ valid for I_v can be obtained by taking non-negative linear combinations of the inequalities $dx \geq G_d(v)$, $d \in T$. Geometrically, this means that for every v the finitely many inequalities $dx \geq G_d(v)$, $d \in T$ include the facets of I_v, uniformly in v.

Essentially the same result has been stated by Wolsey (1979) as Theorem 2′.

Proof. Define

$$C = \{c \mid \text{for some } w \in R^m,\ wa_j \leq c_j,\ 1 \leq j \leq n\} \subset R^n. \tag{6.4}$$

C is a polyhedral cone. If $I_v \neq \emptyset$, $G_c(v) > -\infty$ if and only if $c \in C$, since $c \in C$ if and only if $R_c(v) > -\infty$. (The 'if' part is easy. The 'only if' follows from the remark at the end of Theorem 5.2.)

Let F be as in Lemma 6.1. For each $H \subseteq F$ define the polyhedral cone

$$B_H = \{c \mid c \in C \text{ and } cy \geq 0 \text{ for every } y \in H\} \subset R^n. \tag{6.5}$$

By the Finite Basis Theorem there is a finite $A_H \subset B_H$ such that the cone generated by A_H is B_H. We define

$$T = \bigcup_{H \subseteq F} A_H. \tag{6.6}$$

We must establish that T has the desired properties. Let $v \in Q^m$, $c \in R^n$ satisfy our hypotheses. By Theorem 4.6 (or Meyer (1974)) there is an $\boldsymbol{x} \in I_v$ with $G_c(v) = c\boldsymbol{x}$. Let $\boldsymbol{H} = \{y \in F \mid \boldsymbol{x} + y \in I_v\}$. Clearly $c \in B_{\boldsymbol{H}}$. Hence (i) holds for some $\alpha_d \geq 0$ where we may further specify $\alpha_d = 0$ if $d \notin A_{\boldsymbol{H}}$. By Lemma 6.1, $G_d(v) = d\boldsymbol{x}$ for every $d \in B_{\boldsymbol{H}}$. Hence

$$\sum_{d \in A_H} \alpha_d G_d(v) = \Big(\sum_{d \in A_H} \alpha_d d\Big)\boldsymbol{x} = c\boldsymbol{x} = G_c(v),$$

and (ii) holds.

By Theorem 5.2, there is, for each $d \in T$, a Gomory function g_d such that $g_d(v) = G_d(v)$ if $I_v \neq \emptyset$. Also, it follows from the definition of $G_d(v)$ that $\sum \alpha_d G_d(v) \leq G_c(v)$ for all non-negative α_d such that $\sum \alpha_d d = c$.

Thus we can strengthen Theorem 6.2:

Theorem 6.3. *There are finitely many Gomory functions g_d, $d \in T$, such that, if $I_v \neq \emptyset$ and $G_c(v) > -\infty$, then $G_c(v)$ is the value of the optimal solution to this programming problem with linear constraints*:

$$\begin{aligned} \text{maximize} \quad & \sum_{d \in T} \alpha_d g_d(v), \\ \text{subject to} \quad & \sum_{d \in T} \alpha_d d = c, \\ & \alpha_d \geq 0. \end{aligned} \tag{6.7}$$

Remark. If c is fixed and v varies, only finitely many optimal solutions $\boldsymbol{\alpha}$ to (6.7) arise, as each optimal $\boldsymbol{\alpha}$ is an extreme point to the linear constraints. Each of the optimal solutions gives a Gomory function $\sum \boldsymbol{\alpha}_d g_d(v) \leq G_c(v)$, where, for all $v, G_c(v)$ is the maximum of this finite family of Gomory functions. Thus we have extended Theorem 5.2 to $c \in R^n$.

7. Examples of valid inequalities generated by Chvátal functions

In Gomory (1969) we find tabulated the facets of the group problem

$$\begin{aligned} & t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 \equiv 0 \pmod 6, \\ & t_i \geq 0, \quad \text{integer}, \quad \text{not all } t_i = 0. \end{aligned} \tag{7.1}$$

This is equivalent to an integer programming problem with a single constraint

$$\begin{aligned} & x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 - 6x_6 = 6, \\ & x_i \geq 0, \quad x_i \quad \text{integer}. \end{aligned} \tag{7.2}$$

One facet given in Gomory (1968) is

$$5t_1 + 4t_2 + 3t_3 + 2t_4 + t_5 \geq 6. \tag{7.3}$$

This is generated by the Chvátal function

$$f(\alpha) = 6\lceil \tfrac{1}{5}\alpha \rceil - \alpha. \tag{7.4}$$

More generally, the inequality

$$kt_1 + (k-1)t_2 + \cdots + t_k \geq k+1 \tag{7.5}$$

is valid for the group program constraint

$$t_1 + 2t_2 + 3t_3 + \cdots + kt_k \equiv 0 \pmod{k+1}. \tag{7.6}$$

(7.5) is generated by the Chvátal function $f(\alpha) = (k+1)\lceil \alpha/k \rceil - \alpha$.

Theorem 5.11 guarantees that any valid inequality for an integer program with fixed right-hand side is generated by a Chvátal function. However, it seems too much to expect that the facets will be generated by particularly simple functions.

Another facet of (7.1) is (see Gomory (1969))

$$4x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5 \geq 6. \tag{7.7}$$

One function that generates (7.7) is

$$f(\alpha) = 3\lceil -\tfrac{2}{3}\alpha + \tfrac{2}{3}\lceil (\tfrac{3}{4}\lceil -\tfrac{2}{5}\alpha \rceil - \tfrac{1}{2}\alpha)\rceil\rceil + 4\alpha. \tag{7.8}$$

(7.8) was obtained by using the method of integer forms (see Gomory (1963)) to solve (7.2) with objective function $4x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5$. (7.7) can probably be generated by a simpler function, but it can be shown that (7.7) cannot be generated by a function of the form $f(\alpha) = \lambda_1\alpha + \lambda_2\lceil \lambda_3\alpha \rceil$. There is room for further investigation.

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