Perfect Digraphs

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Abstract: The *clique number* $\omega(D)$ of a digraph D is the size of the largest bidirectionally complete subdigraph of D. D is *perfect* if, for any induced subdigraph H of D, the dichromatic number $\chi(H)$ defined by Neumann-Lara (The dichromatic number of a digraph, J. Combin. Theory Ser. B 33 (1982), 265–270) equals the clique number $\omega(H)$. Using the Strong Perfect Graph Theorem (M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Ann. Math. 164 (2006), 51–229) we give a characterization of perfect digraphs by a set of forbidden induced subdigraphs. Modifying a recent proof of Bang-Jensen et al. (Finding an induced subdivision of a digraph, Theoret. Comput. Sci. 443 (2012), 10–24) we show that the recognition of perfect digraphs is co-$\mathcal{NP}$-complete. It turns out that perfect digraphs are exactly the complements of clique-acyclic superorientations of perfect graphs. Thus, we obtain as a corollary that complements of perfect digraphs have a kernel, using a result of Boros and Gurvich (Perfect graphs are kernel solvable, Discrete Math. 159 (1996), 35–55). Finally, we prove that it is $\mathcal{NP}$-complete to decide whether a perfect digraph has a kernel. © 2014 Wiley Periodicals, Inc. J. Graph Theory 79: 21–29, 2015

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1. INTRODUCTION

A conjecture that kept mathematicians busy with for a long time was Berge's Conjecture (cf. [3,4]) which says that a graph is perfect if and only if it is a Berge graph, i.e., it does neither contain odd holes nor odd antiholes as induced subgraphs. After many partial results, the most famous being Lovasz' proof of the (Weak) Perfect Graph Theorem [16] stating that a graph is perfect if and only if its complement is perfect, a proof of Berge's Conjecture was published in 2006 by Chudnovsky et al. [9], so that Berge's Conjecture is now known as the Strong Perfect Graph Theorem (SPGT).

The importance of perfect graphs lies in computer science. Many problems that are $\mathcal{NP}$-complete for graphs in general are polynomially solvable for perfect graphs, e.g. the maximum clique problem, the maximum stable set problem, the graph coloring problem and the minimum clique covering problem (see [15] resp. [13]). This is applicable, since the members of many important classes of graphs are known to be perfect, e.g. bipartite graphs and their line graphs, split graphs, chordal graphs, and comparability graphs.

In this note we make a first step to generalize the theory of perfect graphs to digraphs. For that purpose we replace the underlying coloring parameter, the chromatic number, by the dichromatic number introduced by Neumann-Lara [17] and independently by Jacob and Meyniel [14]. As main result we obtain that a digraph is perfect if and only if it does not contain induced directed cycles of length at least 3 and its symmetric part is a perfect graph. Hence, using the SPGT, we derive a characterization of perfect digraphs by means of forbidden induced subdigraphs. In this article we describe some further consequences of the main result in complexity theory and kernel theory.

Bokal et al. [5] proved that 2-coloring a digraph feasibly is an $\mathcal{NP}$-complete problem. By our results, k-coloring of perfect digraphs is in $\mathcal{P}$ for any k. We also show that it is possible to determine a maximum induced acyclic subdigraph of a perfect digraph in polynomial time. Note that for symmetric digraphs this is equivalent to computing a stable set.

By a result of Chudnovsky et al. [8] the recognition of Berge graphs is in $\mathcal{P}$, and so, by the SPGT [9], the same holds for the recognition of perfect graphs. In contrast to this, the recognition of induced directed cycles of length at least 3, which are a main obstruction for perfect digraphs, is $\mathcal{NP}$-complete by a result of Bang-Jensen et al. [2]. By a small modification of their proof we, unfortunately, must conclude that the recognition of perfect digraphs is co-$\mathcal{NP}$-complete.

Our belief that our definition of perfection in digraphs is a natural concept is supported by the observation that perfect digraphs are exactly the complements of clique-acyclic superorientations of perfect graphs. Using a result of Boros and Gurvich [6] we conclude that complements of perfect digraphs have a kernel, whereas it is $\mathcal{NP}$-complete to decide whether a perfect digraph has a kernel.

2. NOTATION

We start with some definitions. For basic terminology we refer to Bang-Jensen and Gutin [1]. For the rest of the article, we only consider digraphs without loops. Let $D = (V, A)$ be a digraph. The *dichromatic number* $\chi(D)$ of D is the smallest cardinality $|C|$ of a color set C, so that it is possible to assign a color from C to each vertex of D such that for every color $c \in C$ the subdigraph induced by the vertices colored with c is acyclic, i.e. it does not contain a directed cycle. A *clique* in a digraph D is a subdigraph in which for any two distinct vertices v and w both arcs (v, w) and (w, v) exist. The *clique number* $\omega(D)$ of D is the size of the largest clique in D. The clique number is an obvious lower bound for the dichromatic number. D is called *perfect* if, for any induced subdigraph H of D, $\chi(H) = \omega(H)$.

An (undirected) graph $G = (V, E)$ can be considered as the symmetric digraph $D_G = (V, A)$ with $A = \{(v, w), (w, v) \mid vw \in E\}$. In the following, we will not distinguish between G and D_G. In this way, the dichromatic number of a graph G is its chromatic number $\chi(G)$, the clique number of G is its usual clique number $\omega(G)$, and G is perfect as a digraph if and only if G is perfect as a graph. For us, an *edge* vw in a digraph $D = (V, A)$ is the set $\{(v, w), (w, v)\} \subseteq A$ of two antiparallel arcs, and a *single arc* in D is an arc $(v, w) \in A$ with $(w, v) \notin A$. The *oriented part* $O(D)$ of a digraph $D = (V, A)$ is the digraph (V, A_1) where A_1 is the set of all single arcs of D, and the *symmetric part* $S(D)$ of D is the digraph (V, A_2) where A_2 is the union of all edges of D. Obviously, $S(D)$ is a graph, and by definition we have

Observation 1. *For any digraph D, $\omega(D) = \omega(S(D))$.*

The *(loopless) complement* $\overline{D}$ of a digraph $D = (V, A)$ is the digraph

$$\overline{D} = (V, ((V \times V) \setminus \{(v, v) \mid v \in V\}) \setminus A).$$

For any digraph $D = (V, A)$, the *underlying graph* $G(D) = (V, E)$ is a graph that has an edge $vw \in E$ if and only if $(v, w) \in A$ or $(w, v) \in A$ (possibly both). By definition we have

Observation 2. *For any digraph D, $\overline{S(D)} = G(\overline{D})$.*

A digraph D is a *superorientation* of a graph G, if $G = G(D)$. A superorientation D of G is *clique-acyclic* if there does not exist a clique of G which is induced by the vertex set of a directed cycle of $O(D)$. Whenever a set $S \subseteq V$ induces a subdigraph $H = (S, \emptyset)$ of a digraph $D = (V, A)$, the set S is called *stable*. A *kernel* in a digraph $D = (V, A)$ is a stable set $K \subseteq V$ that is *absorbing*, i.e. for any $v \in V \setminus K$ there is an arc $(v, w) \in A$ with $w \in K$.

In the formulation of the SPGT and in our directed generalization some special types of graphs, respectively, digraphs are needed. An *odd hole* is an undirected cycle C_n with an odd number $n \geq 5$ of vertices. An *odd antihole* is the complement of an odd hole (without loops). A *filled odd hole/antihole* is a digraph H, so that $S(H)$ is an odd hole/antihole. For $n \geq 3$, the directed cycle on n vertices is denoted by $\vec{C}_n$. Furthermore, for a digraph $D = (V, A)$ and $V' \subseteq V$, by $D[V']$ we denote the subdigraph of D induced by the vertices of V'.

3. A STRONG PERFECT DIGRAPH THEOREM

The following main result is the basis of all results of this article.

Theorem 3. *A digraph $D = (V, A)$ is perfect if and only if $S(D)$ is perfect and D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

Proof. Assume $S(D)$ is not perfect. Then there is an induced subgraph $H = (V', E')$ of $S(D)$ with $\omega(H) < \chi(H)$. Since $S(D[V']) = H$, we conclude by Observation 1,

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi(H) = \chi(S(D[V'])) \leq \chi(D[V']),$$

therefore D is not perfect. If D contains a directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph, then D is obviously not perfect, since $\omega(\vec{C}_n) = 1 < 2 = \chi(\vec{C}_n)$.

Now assume that $S(D)$ is perfect but D is not perfect. It suffices to show that D contains an induced directed cycle of length at least 3. Let $H = (V', A')$ be an induced subdigraph of D such that $\omega(H) < \chi(H)$. As $S(H)$ is perfect, there is a proper coloring of $S(H) = S(D)[V']$ with $\omega(S(H))$ colors, i.e., by Observation 1, with $\omega(H)$ colors. This cannot be a feasible coloring for the digraph H. Hence there is a (not necessarily induced) monochromatic directed cycle $\vec{C}_n$ with $n \geq 3$ in $O(H)$. Let C be such a cycle of minimal length. C cannot have a chord that is an edge vw, since both terminal vertices v and w of any such edge vw are colored in the same color contradicting the fact that the coloring is a proper coloring of $S(H)$. By minimality, C does not have a chord that is a single arc. Therefore, C is an induced directed cycle (of length at least 3) in H, and thus in D. ■

We actually have proven:

Remark 4. If D is a perfect digraph, then any feasible coloring of $S(D)$ is also a feasible coloring for D.

Corollary 5. *A digraph $D = (V, A)$ is perfect if and only if it does neither contain a filled odd hole, nor a filled odd antihole, nor a directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

Proof. If D contains any configuration of the three forbidden types, D is obviously not perfect, since each of these configurations is not perfect.

Assume, D does not contain any of these configurations. Then $S(D)$ does neither contain odd holes nor odd antiholes, therefore, by the Strong Perfect Graph Theorem [9], $S(D)$ is perfect. Using Theorem 3, we conclude that D is perfect. ■

4. SOME COMPLEXITY RESULTS

Using some well-known complexity results, in this section we describe several immediate consequences of Theorem 3.

Proposition 6. *There is a polynomial time algorithm to determine an induced acyclic subdigraph of maximum cardinality of a perfect digraph D.*

Proof. Let D be a perfect digraph. By Theorem 3, $S(D)$ is a perfect graph. By a result of Grötschel, Lovász, and Schrijver [12] it is possible to find a stable set I of $S(D)$ of maximum cardinality in polynomial time. $D[I] = O(D)[I]$ is an acyclic digraph, since D is perfect and therefore does not contain induced directed cycles by Theorem 3, hence, as $S(D[I]) = (I, \emptyset)$, it does not contain any directed cycles. By the maximality of I, $D[I]$ is a maximal induced acyclic subdigraph of D. ■

Proposition 7. *k-coloring of perfect digraphs is in $\mathcal{P}$ for any $k \geq 1$.*

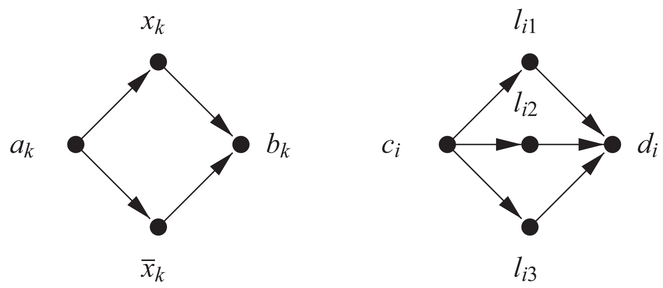

FIGURE 1. Variable gadget $VG(k)$ (left) and clause gadget $CG(i)$ (right).

Proof. By Remark 4 it follows that a coloring of a perfect digraph D with $\omega(D)$ colors can be obtained by coloring the perfect graph $S(D)$, which is possible in polynomial time (see [12]). ■

The preceding result does not depend on an efficient recognition of perfect digraphs. The good news on the tractability of the above problems is bedimed, though, by the result that the recognition problem for perfect digraphs is hard. In order to test, whether a digraph D is perfect, by Theorem 3 we have to test

1. whether $S(D)$ is perfect, and
2. whether D does not contain an induced directed cycle $\vec{C_n}$, $n \geq 3$.

The first can indeed be tested efficiently by the results of Chudnovsky et al. [8] and the SPGT [9], but the second is a co-$\mathcal{NP}$-complete problem by a recent result of Bang-Jensen et al. ([2], Theorem 11). The proof of Bang-Jensen et al. can be easily modified to prove the following.

Theorem 8. *The recognition of perfect digraphs is co-$\mathcal{NP}$-complete.*

Proof. We reduce 3-SAT to nonperfect digraph recognition. We consider an instance of 3-SAT

$$F = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3}) \quad \text{with } l_{ij} \in \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}.$$

For each variable x_k we construct a variable gadget $VG(k)$ and for each clause C_i a clause gadget $CG(i)$, as shown in Figure 1. These gadgets are very similar to those used in Theorem 11 of the article of Bang-Jensen et al. [2], only the edges (which are redundant for correctness of the reduction) are missing here. The rest of the construction is the same as in [2]: We form a chain of variable gadgets by introducing vertices b_0 and a_{n+1} and the arcs (b_k, a_{k+1}) for $k \in \{0, 1, \ldots, n\}$, and a chain of clause gadgets by introducing the vertices d_0 and c_{m+1} and the arcs (d_i, c_{i+1}) for $i \in \{0, 1, \ldots, m\}$. We close the two chains to form a ring by introducing the arcs (a_{n+1}, d_0) and (c_{m+1}, b_0). Finally, for each literal l_{ij} (which is x_k or $\bar{x}_k$) we connect the vertex l_{ij} in the clause gadget $CG(i)$ with the vertex $\overline{l_{ij}}$ in the variable gadget $VG(k)$ by an edge. This completes the construction of the digraph $D(F)$.

We remark that $S(D(F))$ is a forest of stars, thus bipartite and hence perfect, so by Theorem 3 testing whether $D(F)$ is not perfect and testing whether $D(F)$ has an induced directed cycle of length at least 3 is the same. We have to show that $D(F)$ has an induced directed cycle if and only if F is satisfiable. Let $(z_1, \dots, z_n) \in \{0, 1\}^n$ be an assignment satisfying F. Then the directed path through all y_k with

$$y_k = \begin{cases} x_k & \text{if } z_k = 1 \\ \bar{x}_k & \text{if } z_k = 0 \end{cases}$$

can be extended to an induced directed cycle through the clause gadgets by the construction, since in every clause there is a literal y_k the adjacent edge of which is only connected to the vertex $\bar{y}_k$ in $VG(k)$. On the other hand, if there is an induced directed cycle through the ring using the literals $(y_1, \dots, y_n)$ in the variable gadgets, then $(z_1, \dots, z_n)$ with $z_k = 1$ if $y_k = x_k$ and $z_k = 0$ if $y_k = \bar{x}_k$ is an assignment satisfying F, since for every clause some literal y_k that lies on the cycle is satisfied. ■

5. A WEAK PERFECT DIGRAPH THEOREM

Note that perfection of digraphs does not behave as well as perfection of graphs in a second aspect: there is no analogon to Lovasz' Weak Perfect Graph Theorem [16] in an obvious way. A digraph may be perfect but its complement may be not perfect. An easy instance of this type is the directed 4-cycle $\vec{C}_4$, which is not perfect, and its complement $\overline{\vec{C}_4}$, which is perfect.

However, the following might be considered a weak perfect digraph theorem.

Theorem 9. *A digraph D is perfect if and only if the complement $\overline{D}$ is a clique-acyclic superorientation of a perfect graph.*

Before we prove the above theorem we note

Lemma 10. *Let D be a digraph. The following are equivalent.*

(i) D contains no induced directed cycle.
(ii) $\overline{D}$ contains no induced complements of directed cycles.
(iii) $\overline{D}$ is clique-acyclic.

Proof. (iii) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (i) are obvious. For (ii) $\Longrightarrow$ (iii) assume that $\overline{D}$ is not clique-acyclic, i.e. there is a directed cycle which induces a clique in $G(\overline{D})$. Let C be such a cycle of minimal length. If there were a single arc chord there would be a directed cycle of smaller length, contradicting minimality. So every chord of C is an edge, i.e. C induces the complement of a directed cycle. ■

Proof of Theorem 9. By our main result, D is perfect if and only if $S(D)$ is perfect and D does not contain induced directed cycles. By Lemma 10 the latter is equivalent to $S(D)$ being perfect and $\overline{D}$ being clique-acyclic. By Lovasz' weak perfect graph theorem [16] and Observation 2 we have

$$\begin{aligned} S(D) \text{ perfect} &\Longleftrightarrow \overline{S(D)} \text{ perfect} \\ &\Longleftrightarrow G(\overline{D}) \text{ perfect} \\ &\Longleftrightarrow \overline{D} \text{ is superorientation of perfect graph.} \end{aligned}$$

Summarizing we obtain the proposed equivalence. ■

Our result on the complexity of perfect digraph recognition implies

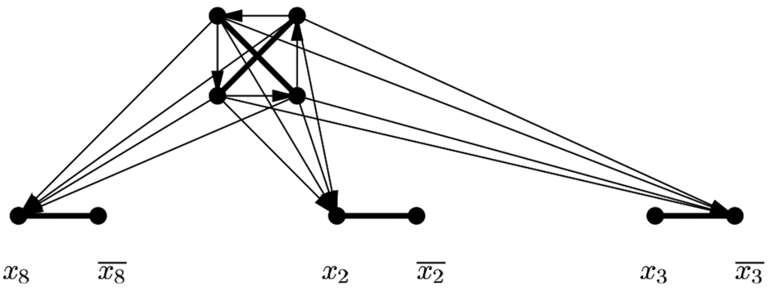

FIGURE 2. Connections for the clause $x_8 \vee x_2 \vee \overline{x_3}$ in the construction of D_f.

Corollary 11. *The recognition of clique-acyclic superorientations of perfect graphs is co-$\mathcal{NP}$-complete.*

Proof. This is immediate from Theorems 8 and 9, using the fact that the complement of a given digraph can be constructed in polynomial time. ■

Whenever one considers digraphs the question of the existence of a kernel is interesting and has quite some applications (cf. [7]). A fascinating relation between perfect graphs and kernels was given by Boros and Gurvich [6].

Theorem 12 (Boros and Gurvich [6]). *Perfect graphs are kernel solvable.*

In our terminology, Theorem 12 means that every clique-acylic superorientation of a perfect graph has a kernel.

Corollary 13. *For any perfect digraph D, the complement $\overline{D}$ has a kernel.*

Proof. Let D be perfect. By Theorem 9, $\overline{D}$ is a clique-acylic superorientation of a perfect graph. By the result of Boros and Gurvich (Theorem 12), $\overline{D}$ has a kernel. ■

The preceding result is in contrast to the following theorem.

Theorem 14. *It is $\mathcal{NP}$-complete to decide whether a perfect digraph has a kernel.*

Proof. Obviously, the problem of the existence of a kernel in perfect digraphs is in $\mathcal{NP}$. We give a reduction from 3-SAT which is a slight variation of Chvatal's classical proof (see [10]). Given a 3-SAT formula f with clauses $C_1, \ldots, C_m$ and variables $x_1, \ldots, x_n$, we construct a digraph D_f as follows. For each variable x_j, we introduce two *literal vertices* $x_j, \overline{x_j}$ joined by an edge $x_j\overline{x_j}$. For each clause $C_i = l_{i1} \vee l_{i2} \vee l_{i3}$ we add a copy H_i of $\vec{C_4}$, and add the 12 arcs connecting its four vertices to the three vertices representing the literals of the clause, see Figure 2.

D_f is a perfect digraph, since its symmetric part is a matching and the only directed cycles are the directed 4-cycles in the H_i's, which are not induced. Since each of the H_i does not admit a kernel, and any kernel must contain exactly one literal vertex for each variable, it is immediate that D_f has a kernel if and only if the formula is satisfiable. ■

6. OPEN QUESTIONS

Since there are many special classes of perfect graphs with algorithms of their own it seems natural to ask:

Open question 15. *Are there any interesting special classes of perfect digraphs with efficient algorithms for problems different from coloring and the maximum induced acyclic subdigraph problem?*

Open question 16. *Are there other problems that are $\mathcal{NP}$-complete or co-$\mathcal{NP}$-complete for digraphs in general as well as for perfect digraphs?*

Being clique-acyclic is a sufficient but not a necessary condition for a superorientation of a perfect graph to have a kernel. This raises the question:

Open question 17. *What is the complexity of recognizing superorientations of perfect graphs that have a kernel?*

ACKNOWLEDGMENTS

The work in Section 5 was motivated by a question of César Hernández Cruz at the Bordeaux Graph Workshop 2012.

NOTE ADDED IN PROOF

A digraph D is *kernel-perfect* if every induced subdigraph of D has a kernel. Tamás Király brought to our attention that Corollary 11 and Theorem 12 imply the following.

Theorem 18. *Deciding kernel-perfectness of a digraph is co-$\mathcal{NP}$-hard.*

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