

$P=NP$

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*We're greatly indebted to Michael Zenzen for many valuable discussions about the $P=?NP$ problem and digital physics. Though the two arguments herein establishing $P=NP$ are for weal or woe Bringsjord's, Taylor's astute objections catalyzed crucial refinements.

The Clay Mathematics Institute offers a \$1 million prize for a solution to the $\mathbf{P}=?\mathbf{NP}$ problem.¹ We look forward to receiving our award — but concede that the expected format of a solution is an *object-level* proof, not a meta-level argument like what we provide. On the other hand, certainly the winner needn't provide a *constructive* proof that $\mathbf{P}=\mathbf{NP}$.² Despite Gödel's recently discovered position on the matter,³ the general consensus has certainly been that $\mathbf{P}\neq\mathbf{NP}$, and many of those brave, contrarian (and, alas, often confused) souls who have endeavored to show $\mathbf{P}=\mathbf{NP}$ have sought to take the beckoning route of exhibiting a polynomial-time algorithm for one or more of the 1000 or so currently catalogued \mathbf{NP} -complete problems. This is an exceedingly taxing (and, at least hitherto, unproductive) direction to take, and we eschew it. We happily concede that constructive success would have many practical implications, but we are more interested in the fact of the matter than, say, whether many current cryptographic schemes can be compromised. Very well; let's proceed.

In logic and related fields we often speak about problems in purely abstract terms. For example, we may declare a problem to be Turing-solvable, without giving any thought whatsoever to the *embodiment* of a Turing machine able to carry out a solution.⁴ So we may for instance say that the set \mathcal{C} of composite numbers is Turing-decidable: that there exists some TM M such that, for every $n \in \mathcal{N} = \{0, 1, 2, \dots\}$, with n given to M as input (suitably encoded on its tape), M produces (say) Y iff $n \in \mathcal{C}$, and N otherwise. Such facts are routinely confirmed in the absence of even a stray, evanescent thought about how M might or might not be embodied.

However, it's well-known that TMs (and other purely abstract computers) *can* be built. In fact, one such physical machine is processing the letters in the present sentence, as I (Selmer) type them. We may not know for sure that every abstract TM M^i from the countably infinite set of such devices can be physicalized to produce M_p^i , but certainly we *do* know that for every physical TM M_p^i able to accomplish some computation, there exists a corresponding purely mathematical TM that carries out the same computation (in the mathematical universe). This fact will prove convenient below.

Another well-known fact, one we also find rather helpful, is that there are simple physical processes *not reflective of the mathematical structure of TMs and the like*, which nonetheless solve some problems that are overwhelmingly difficult for TMs and their digital relatives. For example, the Steiner Tree problem (STP) is known to be \mathbf{NP} -complete (see e.g. pp. 208–209 of Garey & Johnson 1979).⁵ Nonetheless, a simple physical process (termed an **analog computation**⁶) can

¹See <http://www.claymath.org/millennium>. There are six other “millennium” problems; each of these is also associated with a \$1M prize.

²As many readers know, the history of the problem is littered with failed attempts to provide non-constructive substantiation of the received view that $\mathbf{P}\neq\mathbf{NP}$.

³His position is communicated in a stunningly prescient letter he wrote to von Neumann in 1950; this letter is reproduced, in English, in (Sipser 1992). Gödel, writing of course before the modern $\mathbf{P}=?\mathbf{NP}$ framework, inquires as to von Neumann's thoughts about what is today known as the k -symbol provability problem. Let ϕ be a formula of \mathcal{L}_I (a formula of first-order logic, or just FOL). We write $\vdash_k \phi$ provided there is a first-order proof of ϕ of $\leq k$ symbols. Gödel apparently believed that it might well be possible to answer questions of the form “ $\vdash_k \phi$?” in linear or quadratic time. When the set here is made explicit and configured so as to allow for encoding on a Turing machine tape, it's patent that it's \mathbf{NP} -complete. Gödel was quite at home with the idea that as logic and mathematics progress, machines would increasingly take over the “Yes-No” part of the enterprise. Any notion that Gödel would have embraced an argument by analogy from the undecidability of FOL to the perpetual intractability of the k -symbol provability problem is utterly misguided: He writes: “[I]t would obviously mean that in spite of the undecidability of the *Entscheidungsproblem*, the mental work of a mathematician concerning Yes-or-No questions could be completely replaced by a machine.”

⁴An exactly parallel point obviously holds of all those incorporeal models known to be equivalent to TMs: register machines, the λ -calculus, abaci, etc.

⁵STP is \mathbf{NP} -hard when the metric is non-discretized, and \mathbf{NP} -complete when the metric is discrete.

⁶Analog computers are nothing new, though they don't get much air time these days. An elegant example is

solve it quickly. STP is the problem of connecting n points on a plane with a graph of minimal overall length, using junction points if necessary. The physical process in question runs as follows. Make two parallel glass plates, and insert n pins between the plates to represent the points. Then dip the structure into a soap solution, and remove it. The soap film will connect the n pins in the minimum Steiner-tree graph (Iwamura, Akazawa & Amemiya 1998). Building the structure and the solution (and the container for the solution) requires steps linear in the size of n , and dipping and withdrawing make two steps, so despite the fact that STP is **NP**-complete, the physical process just described — let’s call it A^s — is apparently carried out well within $O(n^k)$, for some constant k .

Before starting to read this short paper, you were probably positive that $\mathbf{P} \neq \mathbf{NP}$. If you’ve now heard about it for the first time, does the soap process change your mind? We didn’t think so. But please reason further with us.

First, some simple notation. Let’s refer to the physical version of the STP problem as $B(STP)$, and the abstract version as STP . In addition, following usage above, we use M with or without superscripts to refer to Turing machines, and M_p to refer to physicalized TMs. We refer to analog processes with variable A . Now here is a naive proof, functioning as precursor to the more sophisticated successor given later, formalizable in sorted⁷ first-order logic (FOL), that $\mathbf{P} = \mathbf{NP}$ (where the predicate letter N is explained later):

The Preliminary Proof

1	$\exists M (M \text{ solves } STP \text{ in polynomial time}) \rightarrow \mathbf{P} = \mathbf{NP}$	definition of NP -completeness
2	$\exists A (A \text{ solves } B(STP) \text{ in polynomial time} \wedge N(A))$	derivable, e.g., by existential introduction from soapfilm process, i.e., A^s
3	$\exists A (A \text{ solves } B(STP) \text{ in polynomial time} \wedge N(A)) \rightarrow$ $\exists M_p (M_p \text{ solves } B(STP) \text{ in polynomial time})$	digital physics; see below
4	$\exists M_p (M_p \text{ solves } B(STP) \text{ in polynomial time}) \rightarrow$ $\exists M (M \text{ solves } STP \text{ in polynomial time})$	unassailable; see justification 3rd ¶
5	P=NP	1–4 (full FOL derivation trivial)

There is no question that the reasoning here can be certified as formally valid (e.g., using an automated proof checker). The only question is whether the premises are true. If they are, the problem is at long last solved in the affirmative. Are the premises true?

Line 2, note, isn’t a premise, but rather an intermediate conclusion; however, there are two routes to this conclusion. As noted in the justification column, one possible inference to line 2 is from A^s , where this constant is replaced by the variable A , and existential introduction is used (we assume for certification a natural deduction calculus, with rules for introducing and eliminating truth-functional connectives and quantifiers; a nice system of this sort is \mathcal{F} , from Barwise & Etchemendy 1999). The second route takes account of what we regard to be self-evident: Surely A^s is just the tip of the iceberg, with myriad analog processes out there in our physical universe waiting to be discovered and harnessed (though presumably most will remain undetected for eternity). This view can be derived from a sampling assumption, according to which a finding like A^s must be a random sampling from some small proper subset of the (probably infinite) set of all candidate processes available in the cosmos. We don’t pursue this derivation herein. Interested readers should consult a parallel form of argument explored in theoretical physics (see e.g. Bostrom 2002).

Claude Shannon’s famous differential analyzer, which solves ordinary differential equations. A nice discussion of the analyzer can be found in (Earman 1986).

⁷E.g., 1 is short for $\exists x(A(x) \wedge M \text{ solves } \dots)$.

We anticipate that some will be uncomfortable with the view that there exists a process A that accommodates ever greater values for n . In light of this, we move now to the more sophisticated of our two proofs: First, note that full specification of our proof in FOL does include universal quantification over \mathcal{N} , and a corresponding index for STP and $B(STP)$, as for example in what line 1, unpacked, becomes:

$$\exists A \forall n (\dots B(STP)_n \dots) \dots$$

But to us, this simply calls for a natural variant of induction on \mathcal{N} . The base clause is trivial. Given the induction hypothesis, it's exceedingly hard to see how A^s cannot succeed on $n + 1$ if the minimal graph has been found for n physicalized points. Whatever underlying principles of physics generate the graph in the case of n surely can be employed to generate it for $n + 1$. Even if the physical laws governing *our* universe are such that there is some point $n + 1$ at which A^s fails, surely it's physically *possible* that this failure *not* occur. This implies that there is a more formidable second proof that employs modal logic (Chellas 1980, Hughes & Cresswell 1968). If we let \diamond_p refer to physical possibility in a manner that parallels the straight \diamond of logical possibility from modal logic,⁸ then the modal version of line 3 is

$$\begin{aligned} &\diamond_p \exists A \forall n (A \text{ solves } B(STP)_n \text{ in polynomial time} \wedge N(A)) \rightarrow \\ &\quad \diamond_p \exists M_p \forall n (M_p \text{ solves } B(STP)_n \text{ in polynomial time}) \end{aligned}$$

and this technique can be easily propagated through the original proof to produce the more circumspect one. In this modal proof, line 4 becomes the key principle that if it's physically possible that a physical TM solve $B(STP)_n$ in polynomial time, then there exists (in the mathematical universe) a TM that solves STP_n in polynomial time. This principle would appear to be invulnerable. Summing up, we have:

The Modalized Proof

1'	$\exists M \forall n (M \text{ solves } STP_n \text{ in polynomial time}) \rightarrow \mathbf{P}=\mathbf{NP}$	definition of NP -completeness
2'	$\diamond_p \exists A \forall n (A \text{ solves } B(STP)_n \text{ in polynomial time} \wedge N(A))$	derivable, e.g., by induction and existential introduction from soapfilm process, i.e., A^s
3'	$\diamond_p \exists A \forall n (A \text{ solves } B(STP)_n \text{ in polynomial time} \wedge N(A)) \rightarrow$ $\diamond_p \exists M_p \forall n (M_p \text{ solves } B(STP)_n \text{ in polynomial time})$	digital physics; see below
4'	$\diamond_p \exists M_p \forall n (M_p \text{ solves } B(STP)_n \text{ in polynomial time}) \rightarrow$ $\exists M \forall n (M \text{ solves } STP_n \text{ in polynomial time})$	unassailable; see justification 3rd ¶
5'	P=NP	1'–4' (full FOL derivation trivial)

Please note that the modal version of our argument provides complete immunity from an objection that the physical universe is finite, and that therefore no analog process can scale up through all natural numbers as inputs for the minimal graph to be generated. The dominant view among theoretical physicists appears to be that the theory of inflation (Vilenkin 1983, Guth 2000) holds (which renders it likely that the universe is infinite). But we need not take a stand on the issue. All we need is what follows immediately from the fact that the theory of inflation, whether or not true, is certainly coherent: namely, that it's physically *possible* that the universe is infinite.

By “digital physics” for the justification of premise 3/3', we have in mind the position that the physical universe is fundamentally a vast physical computer — or, if you like, a computer composed of computers, which are in turn composed of computers, and so on. This view has been recently

⁸We assume a normal S5 version of the \diamond operator.

affirmed by Wolfram (2002), but Fredkin (1990) advanced the view long ago (and continues to energetically defend it now), and Feynman (1982) seems to have embraced the view as well. Even Einstein can be read as having affirmed the digital physics position. Though premise 3/3' refers to physicalized Turing machines, most digital models in physics are based on cellular automata, but this is of no matter: it's well-known that every cellular automaton can be recast as a TM.⁹

Our argument shows that if $\mathbf{P} \neq \mathbf{NP}$, digital physics is incorrect. Since it must be true that all physical phenomena can in principle be modeled in information-processing terms of *some* kind, $\mathbf{P} \neq \mathbf{NP}$ thus immediately implies, courtesy of our arguments, that hypercomputational processes exist in the physical universe.¹⁰ If you believe, as many do, that hypercomputational processes are always merely mathematical, and never physically real, you can't be rational and at the same time refuse to accept our case for $\mathbf{P} = \mathbf{NP}$.

Perhaps you do indeed refuse to accept the digital physics view, and have no qualms about physical hypercomputation. It was with skeptics like you in mind that the predicate $N(\)$ (for “normal”) was included in our proofs. While many are perhaps right to point out, *contra* Wolfram and company, that some physical phenomena (e.g., those associated with quantum mechanics) are so bizarre and complicated that they resist formalization in TM-level computational models, the fact of the matter is that the analog process we exploit is a painfully simple macroscopic phenomenon — as we say, a “normal” physical process. The burden of proof is surely on those who would maintain that the formal machinery of digital physics is insufficient to model something as straightforward as submerging nails in, and retrieving them from, a bucket of soapy water.

⁹The transformation preserves polynomial-time processing, as cognoscenti know. For others, a sketch: Use an n -dimension TM in which n is high enough to sufficiently represent the CA which is the universe. Let each cell of the TM's tapes represent a cell of the CA. The alphabet of the TM will contain some representation of all states for the CA's cells. The computation performed by the CA is finite, so the TM's states are as well. It is known that the transformation from multidimensional Turing machines to standard Turing machines is a polynomial transformation. If the computation performed in each cell of the CA is in class \mathbf{P} , the equivalent TM will be in class \mathbf{P} .

¹⁰Physical phenomena that can be rigorously modeled only via information processing above the **Turing Limit** (Siegelmann 1999, Bringsjord & Zenzen 2003) would be phenomena calling for hypercomputational machines. Just as the class of mathematical devices equivalent to TMs is infinite, so also there are an infinite number of hypercomputational machines. Examples include **analog chaotic neural nets** (Siegelmann & Sontag 1994) and **infinite time Turing machines** (Hamkins & Lewis 2000). Other examples include **analog “knob” TMs** (Bringsjord 2001) and **accelerated TMs** (Copeland 1998).

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