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On the complexity of some subgraph problems

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ABSTRACT

We study the complexity of the problem of deciding the existence of a spanning subgraph of a given graph, and of that of finding a maximum (weight) such subgraph. We establish some general relations between these problems, and we use these relations to obtain new *NP*-completeness results for maximum (weight) spanning subgraph problems from analogous results for existence problems and from results in extremal graph theory. On the positive side, we provide a decomposition method for the maximum (weight) spanning chordal subgraph problem that can be used, e.g., to obtain a linear (or $O(n \log n)$) time algorithm for such problems in graphs with vertex degree bounded by 3.

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1. Introduction

We study the complexity of the problem of deciding the existence, in some classes of graphs, of a spanning subgraph of a given graph, and that of finding a maximum (weight) such subgraph.

We identify the set of all (undirected, simple, loopless, connected) graphs on a set V of vertices with the power set $2^{\binom{V}{2}}$ of all subsets of the set $\binom{V}{2}$ of all two-element subsets of V. The graphs on V that satisfy a certain property (e.g., chordality, connectedness, acyclicity, etc.) can thus be identified with families $\mathcal{F} \subseteq 2^{\binom{V}{2}}$.

Given a family $\mathcal{F} \subseteq 2^{\binom{V}{2}}$ of graphs and a graph $G = (V, E)$, a subset F of E is called *an $\mathcal{F}$-subgraph* of G if $F \in \mathcal{F}$.

An $\mathcal{F}$-subgraph F of G is called a *spanning* $\mathcal{F}$-subgraph of G if its edges meet all the nodes of G, i.e., if $\bigcup_{\{u,v\}\in F}\{u, v\} = V$.

Given two families $\mathcal{F}, \mathcal{G} \subseteq 2^{\binom{V}{2}}$ of graphs with vertices in V, a graph $G = (V, E)$ with weights $w_e \geq 0$ on its edges and a subgraph $F \subseteq E$, we let $w(F) = \sum_{e\in F} w_e$ denote the *weight* of F, and we consider the following decision problems:

- EXISTENCE OF A SPANNING $\mathcal{F}$-SUBGRAPH IN $\mathcal{G}$
  Given G in $\mathcal{G}$, does there exist a spanning $\mathcal{F}$-subgraph of G?
- MAXIMUM SPANNING $\mathcal{F}$-SUBGRAPH IN $\mathcal{G}$
  Given G in $\mathcal{G}$ and any positive integer k, does there exist a spanning $\mathcal{F}$-subgraph F of G such that $|F| \geq k$?
- MAXIMUM WEIGHT SPANNING $\mathcal{F}$-SUBGRAPH IN $\mathcal{G}$
  Given G in $\mathcal{G}$ and any $k \geq 0$, does there exist a spanning $\mathcal{F}$-subgraph F of G such that $w(F) \geq k$?

Note that many well-known graph problems fall into one of the three classes of problems just mentioned. Consider, e.g., the problem of the existence of – or of the maximum weight – Hamiltonian or Eulerian cycles in a graph; the maximum (weight) matching problem; or the maximum (weight) spanning tree problem.

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The maximum (weight) $\mathcal{F}$-subgraph problem is trivially equivalent to the *edge-deletion problem*: find a subset E' of E of minimum cardinality (weight) such that $E \setminus E' \in \mathcal{F}$.

The edge-deletion problem has been introduced by Yannakakis, who proved its *NP*-completeness for several classes of subgraphs (or, equivalently, of families $\mathcal{F}$) including bipartite, outerplanar, and degree-constrained graphs [23–25].

After the work of Yannakakis, the edge-deletion problem has been proved to be *NP*-complete (even to approximate) for a number of other classes of subgraphs [1–3,9,15,18]. On the other hand, not many classes of subgraphs are known for which this problem is polynomially solvable.

In Section 2 we describe some simple relations among the problem of the existence of a spanning $\mathcal{F}$-subgraph, the maximum spanning $\mathcal{F}$-subgraph problem, and the maximum weight spanning $\mathcal{F}$-subgraph problem. These relations allow one to obtain *NP*-completeness or polynomiality results for one type of problems from analogous results for another type of problems.

In particular, using also some results from extremal graph theory, we prove in Section 3 that the problem of finding a maximum chordal subgraph in a planar graph with maximum vertex degree 6 is *NP*-complete by using the *NP*-completeness of the existence of 2-trees in such graphs.

On the positive side, we provide a decomposition method for the maximum spanning chordal subgraph problem that can be used, e.g., to obtain a linear time algorithm for such problem in graphs with node degree bounded by three, and an $O(n \log n)$ time algorithm for the weighted case.

In view of the *NP*-completeness result above, this leaves open the complexity of the same problem in (planar) graphs of degree four and five.

2. Relations between subgraph problems

Given a family $\mathcal{F} \subseteq 2^{\binom{V}{2}}$ of graphs with vertices in V and a fixed graph $G = (V, E)$ with weights w_e on the edges $e \in E$, we let

$$\mathcal{F}_E = \{F \in \mathcal{F} : F \subseteq E\}$$

denote the family of all $\mathcal{F}$-subgraphs of G, and we let

$$M(\mathcal{F}) = \max_{F \in \mathcal{F}} |F|,$$

$$M(E, \mathcal{F}) = \max_{F \in \mathcal{F}_E} |F|, \quad \text{and}$$

$$M(E, \mathcal{F}, w) = \max_{F \in \mathcal{F}_E} w(F)$$

denote the maximum number of edges of an $\mathcal{F}$-subgraph, the maximum number of edges of an $\mathcal{F}$-subgraph of G, and the maximum weight of an $\mathcal{F}$-subgraph of G, respectively.

Furthermore, let $\overline{\mathcal{F}} = \{F \in \mathcal{F} : |F| = M(\mathcal{F})\}$ denote the subset of $\mathcal{F}$ that is extremal with respect to the number of edges, so that $\overline{\mathcal{F}}_E = \{F \in \overline{\mathcal{F}} : F \subseteq E\}$ is the set of $\overline{\mathcal{F}}$-subgraphs of G. For example, if $\mathcal{F}$ is the family of graphs with maximum vertex degree q, then $\overline{\mathcal{F}}_E$ is the family of q-regular subgraphs of G.

Given two families $\mathcal{G}$ and $\mathcal{G}'$ of graphs on a vertex set V, we say that $\mathcal{G}'$ *polynomially dominates* $\mathcal{G}$, if for every graph $F \in \mathcal{G}$ one can find in polynomial time a graph $F' \in \mathcal{G}'$ such that $F \subseteq F'$. In other words, we are requiring that it should always be possible to extend a graph of $\mathcal{G}$ to a graph of $\mathcal{G}'$ in polynomial time. Note that this is trivially true when $\mathcal{G}'$ consists only of the complete graph on the vertex set V.

The following lemmata provide some reductions among the three subgraph problems described in the introduction. Note that the inequality $M(E, \mathcal{F}) \leq M(\mathcal{F})$ trivially holds for all E and $\mathcal{F}$. Hence, the inequality $M(E, \mathcal{F}) \geq M(\mathcal{F})$ is equivalent to $M(E, \mathcal{F}) = M(\mathcal{F})$.

Lemma 1 (*Existence–Max Reduction*). *Let $\mathcal{F}$ and $\mathcal{G}$ be any families of graphs on the vertex set V. Then EXISTENCE OF A SPANNING $\overline{\mathcal{F}}$-SUBGRAPH IN $\mathcal{G}$ can be reduced to MAXIMUM SPANNING $\mathcal{F}$-SUBGRAPH IN $\mathcal{G}$.*

Proof. It suffices to observe that $\overline{\mathcal{F}}_E \neq \emptyset \Leftrightarrow M(E, \mathcal{F}) \geq M(\mathcal{F})$. □

Lemma 2 (*Existence–Max weight Reduction*). *Let $\mathcal{F}, \mathcal{G}, \mathcal{G}'$ be families of graphs on the vertex set V with $\mathcal{G}'$ polynomially dominating $\mathcal{G}$. Then EXISTENCE OF A SPANNING $\overline{\mathcal{F}}$-SUBGRAPH IN $\mathcal{G}$ can be reduced to MAXIMUM WEIGHT SPANNING $\mathcal{F}$-SUBGRAPH IN $\mathcal{G}'$*

Proof. Given a graph $G = (V, E)$ in $\mathcal{G}$ we can find in polynomial time a graph $G' = (V, E')$ in $\mathcal{G}'$ such that $E \subseteq E'$. We define weights $w_e = 1$ when $e \in E$ and $w_e = 0$ when $e \in E' \setminus E$. Then, for $F' \in \mathcal{F}_{E'}$ we have $w(F') \leq |F'| \leq M(\mathcal{F})$. We will now show that $\overline{\mathcal{F}}_E \neq \emptyset \Leftrightarrow (M(E', \mathcal{F}, w) \geq M(\mathcal{F}))$. Indeed, if there exists an $\overline{\mathcal{F}}$-spanning subgraph F in G, then F is also an $\overline{\mathcal{F}}$-spanning subgraph in G', so that $M(E', \mathcal{F}, w) \geq M(\mathcal{F})$. Conversely, if $M(E', \mathcal{F}, w) \geq M(\mathcal{F})$, we clearly have $M(E', \mathcal{F}, w) = M(\mathcal{F})$, so that there exists $F' \subseteq E'$, $F' \in \mathcal{F}$ with $w(F') = |F'| = M(\mathcal{F})$. Thus we must have $w_e = 1$ for all $e \in F'$ and hence $F' \subseteq E$. Therefore $\overline{\mathcal{F}}_E \neq \emptyset$. □

Note that the proof above shows that the Existence–Max weight reduction holds also when the weights are restricted to take only two nonnegative values.

In the next section we provide some applications of the reductions described in these two lemmata to obtain, in conjunction with results in extremal graph theory, some new complexity results for the maximum (weight) spanning subgraph problem from known results for the problem of the existence of a spanning subgraph.

3. Some *NP*-complete spanning subgraph problems

Let us consider the family $\mathcal{F}_{chord}$ of all connected chordal graphs on V. As customary, we call a graph *chordal* if it does not contain any chordless cycle of length greater than three as an induced subgraph.

Since spanning trees are connected chordal graphs, the problem of the existence of a spanning chordal subgraph is trivially solvable in linear time.

On the contrary, the complexity of the maximum spanning chordal subgraph problem has been uncertain for a long time. Dearing, Shier and Warner [14] have explicitly stated this as an open problem and they have described an $O(|E|\Delta)$ algorithm for finding a *maximal* chordal subgraph in a graph, where Δ is the maximum vertex degree in G. Also Erdős and Laskar [17] pointed out the interest for this problem and gave an asymptotic estimate on maximum number of edges to delete to make an n-vertex graph chordal. Yannakakis [25] has proved *NP*-completeness of the related maximum spanning $\mathcal{F}_{-C_\ell}$-subgraph problem, where $\mathcal{F}_{-C_\ell}$ is the class of connected subgraphs without cycles of specified length ℓ. The first proof of *NP*-completeness of the maximum spanning chordal subgraph is attributed to A. Ben-Dor in [18].

The interest for the maximum spanning chordal subgraph is also due to the possibility of solving several hard graph problems in polynomial time in chordal graphs. This fact has been used, e.g., by Balas and Yu [4] to find a maximum clique in a graph.

We now show that the maximum spanning chordal subgraph problem remains *NP*-complete also in planar graphs with maximum vertex degree $\Delta = 6$ by using the Existence–Max reduction Lemma 1, and an analogous *NP*-complexity result for 2-trees. On the other hand, in Section 5 we provide a linear time algorithm for the maximum spanning chordal subgraph problem in general graphs with $\Delta = 3$. This leaves still open the complexity of the problem for (planar) graphs with $\Delta = 4$ or 5.

A *q-tree* can be recursively defined as follows (see, e.g., [10,12,21] for properties and alternative definitions of q-trees):

1. A clique on q vertices (i.e., a K_q) is a q-tree.
2. From a q-tree T with $n > q$ vertices, we obtain a new q-tree with $n+1$ vertices, by adding a new vertex to T and making it adjacent to all vertices of a clique K_q of T.

It is known that q-trees are extremal chordal graphs that do not contain a K_{q+2} subgraph [8].

Theorem 1 ([12,21]). *A chordal graph with $n \geq q$ vertices that does not contain a K_{q+2} subgraph has at most $qn - q(q+1)/2$ edges, and has exactly $qn - q(q+1)/2$ edges if and only if it is a q-tree.*

In particular, a 2-tree with n vertices has exactly $2n-3$ edges and does not contain a K_4 subgraph. Bern [5] has shown that the existence problem for spanning q-trees in a graph is *NP*-complete. This result has been refined by Cai and Maffray [11] who showed that the existence problem for spanning q-trees is *NP*-complete even in split graphs or in graphs with maximum vertex degree $3q+2$. Furthermore, the existence problem for spanning 2-trees is *NP*-complete even in planar graphs with maximum vertex degree 6. In fact, a careful analysis shows that all bounded degree graphs in the family $\mathcal{G}_{CM}$ used by Cai and Maffray in the proof of this result do not contain K_{q+2} subgraphs, so that its statement can be slightly strengthened as follows.

Theorem 2 (*Cai and Maffray*). *The existence problem for spanning* 2*-trees is NP-complete even in planar graphs with maximum vertex degree* 6 *that do not contain a K_4 subgraph. The existence problem for spanning q-trees is NP-complete even in split graphs or in graphs with maximum vertex degree* $3q+2$ *that do not contain a K_{q+2} subgraph.*

We can use Theorems 1 and 2 and Lemma 1 to strengthen the *NP*-complexity result of Ben-Dor and of Natanzon, Shamir and Sharan (see [18]) for the maximum spanning chordal subgraph problem.

We first introduce some notation that will be used also in what follows. Let $\mathcal{G}_2$ denote the family of planar graphs with maximum vertex degree 6 that do not contain a K_4 subgraph, and for $q > 2$ let $\mathcal{G}_q$ denote the family of graphs with maximum vertex degree $3q+2$ that do not contain a K_{q+2} subgraph.

Theorem 3. *The MAXIMUM SPANNING CHORDAL SUBGRAPH PROBLEM IN $\mathcal{G}_q$ is NP-complete.*

Proof. Let $\mathcal{F}_{q\text{-}chord}$ denote the family of *q-chordal graphs* on V, defined as chordal graphs on V that do not contain a K_{q+2} subgraph. Trivially, a subgraph of a graph G that does not contain a K_{q+2} subgraph is chordal if and only if it is q-chordal. In particular, this holds for all graphs G in the family $\mathcal{G}_q$. Furthermore, from Theorem 1 we derive that $\overline{\mathcal{F}}_{q\text{-}chord}$ coincides with the family $\mathcal{F}_{q\text{-}trees}$ of all q-trees on V. Hence, EXISTENCE OF A SPANNING $\mathcal{F}_{q\text{-}trees}$-SUBGRAPH coincides with EXISTENCE OF A SPANNING $\overline{\mathcal{F}}_{q\text{-}chord}$-SUBGRAPH and, by Lemma 1 the latter can be reduced to MAXIMUM SPANNING $\mathcal{F}_{q\text{-}chord}$-SUBGRAPH, which coincides with MAXIMUM SPANNING $\mathcal{F}_{chord}$-SUBGRAPH in the family $\mathcal{G}_q$. Since EXISTENCE OF A SPANNING $\mathcal{F}_{q\text{-}trees}$-SUBGRAPH in $\mathcal{G}_q$ is *NP*-complete by Theorem 2, the thesis follows. □

Let us now consider the family $\mathcal{F}_{pqT}$ of partial q-trees, where a *partial q-tree* is any spanning subgraph of a q-tree (see, e.g., [6]). Note that partial 2-trees actually coincide with *series–parallel graphs* (see [8] for definitions and characterizations of series–parallel graphs).

The problem of finding a spanning partial q-tree of a graph G arises often in problems concerning the construction of reliable networks where the vertices are subject to failure (see [16] for the case $q = 2$), and in transportation networks.

From the definition of partial q-trees we have $\mathcal{F}_{q\text{-}trees} \subseteq \mathcal{F}_{pqT}$. Furthermore, q-trees are precisely the edge maximal graphs in the class of partial q-trees, that is, $\overline{\mathcal{F}}_{pqT} = \mathcal{F}_{q\text{-}trees}$. Hence, with the same argument of Theorem 3 it is straightforward to prove the following:

Theorem 4. *The MAXIMUM SPANNING SERIES–PARALLEL (or PARTIAL 2-TREE) SUBGRAPH PROBLEM is NP-complete even in planar graphs with maximum vertex degree 6 that do not contain a K_4 subgraph. The MAXIMUM SPANNING PARTIAL q-TREE SUBGRAPH PROBLEM is NP-complete in split graphs or in graphs with maximum vertex degree $3q + 2$ that do not contain a K_{q+2} subgraph.*

The first part of this theorem strengthens a result from Asano [2] who proved *NP*-completeness of the problem of finding a maximum spanning series–parallel subgraph of a planar graph.

We note that partial q-trees are closely related to the concept of *treewidth* introduced by Robertson and Seymour [20] that refers to the *width* of a tree decomposition of a graph G. There are several equivalent characterizations of the notion of treewidth, but probably the best-known characterization is the one in terms of partial q-trees. Indeed, a graph G has treewidth at most q if and only if G is a partial q-tree [8]. Treewidth plays an important role in algorithmic graph theory. Actually, many problems that are NP-hard for general graphs become polynomial or linear time solvable when restricted to graphs of bounded treewidth (see [6,7] for an overview). Since q-trees are precisely the edge maximal graphs of treewidth q [19], Theorem 4 can be rephrased by substituting treewidth at most q in place of partial q-trees.

Consider now the family $\mathcal{F}_{\Delta=q}$ of all connected graphs on V with maximum vertex degree q. When $|V|$ or q are even, the extremal elements of $\mathcal{F}_{\Delta=q}$ with the maximum number of edges are exactly the connected *q-regular graphs* $\mathcal{F}_{q\text{-reg}}$ that have all vertex degrees equal to q. Clearly, a q-regular graph has exactly $\frac{1}{2}q|V|$ edges. Cheah and Corneil [13] have proved the following *NP*-completeness result.

Theorem 5 (*Cheah and Corneil*). *The existence problem for connected q-regular spanning subgraphs is NP-complete for graphs of maximum degree $\Delta = q + 1$.*

Hence, a straightforward application of Lemma 1 implies the following:

Theorem 6. *The MAXIMUM SPANNING $\mathcal{F}_{\Delta=q}$-SUBGRAPH PROBLEM is NP-complete even for graphs of maximum degree $\Delta = q + 1$.*

Note that Theorem 6 strengthens an analogous result proved by Yannakakis [25] for general graphs.

We now show three examples of application of Lemma 2 to establish *NP*-completeness of some maximum weight spanning $\mathcal{F}$-subgraph problem.

Theorem 7. *Let $\mathcal{G}'_q$ be a family of graphs that do not contain a K_{q+2} subgraph, polynomially dominating the family $\mathcal{G}_q$. Then the MAXIMUM WEIGHT SPANNING CHORDAL SUBGRAPH PROBLEM in $\mathcal{G}'_q$ is NP-complete.*

Proof. By Theorem 1 we have that $\overline{\mathcal{F}}_{q\text{-}chord} = \mathcal{F}_{q\text{-}trees}$. Furthermore, every chordal subgraph of a graph G in $\mathcal{G}'_q$ is also q-chordal. Hence, EXISTENCE OF A SPANNING $\mathcal{F}_{q\text{-}trees}$-SUBGRAPH in $\mathcal{G}_q$ can be reduced to MAXIMUM WEIGHT SPANNING $\mathcal{F}_{chord}$-SUBGRAPH in $\mathcal{G}'_q$ by Lemma 2. The conclusion then follows from Theorem 2, which guarantees *NP*-completeness of the former problem. □

A similar argument can be used to prove the following theorem, taking into account that $\overline{\mathcal{F}}_{pqT} = \mathcal{F}_{q\text{-}trees}$.

Theorem 8. *Let $\mathcal{G}'$ be either any family of graphs polynomially dominating the family of planar graphs with maximum vertex degree 6 that do not contain a K_4 subgraph, or any family of graphs polynomially dominating the family of split graphs or of graphs with maximum vertex degree $3q+2$ that do not contain a K_{q+2} subgraph. Then the MAXIMUM WEIGHT SPANNING PARTIAL q-TREE SUBGRAPH PROBLEM in $\mathcal{G}'$ is NP-complete.*

Theorem 9. *Let $\mathcal{G}'$ be any family of graphs polynomially dominating the family $\mathcal{G}$ of graphs of maximum vertex degree $\Delta = q+1$. Then the MAXIMUM WEIGHT SPANNING $\mathcal{F}_{\Delta=q}$-SUBGRAPH PROBLEM in $\mathcal{G}'$ is NP-complete.*

Proof. The proof follows from Lemma 2 and from Theorem 5, by observing that $\overline{\mathcal{F}}_{\Delta=q} = \mathcal{F}_{q\text{-}reg}$. □

In particular, we obtain that the maximum weight spanning $\mathcal{F}_{\Delta=q}$-subgraph problem is *NP*-complete in complete graphs.

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4. A decomposition method

In this section we describe a method for solving the problem of finding a maximum weight spanning chordal subgraph in a connected graph $G = (V, E)$ by solving analogous problems in a family $G_i = (V_i, E_i)$, $i = 1, \ldots, k$, of induced subgraphs of G, where $V_1, \ldots, V_k$ is a partition of the node set V, and an additional maximum spanning tree problem in an associated graph.

For a partition $V_1, \ldots, V_k$ of the node set V we consider the following properties:

P1. the subgraph induced by V_i in G is connected for every $i = 1, \ldots, k$;
P2. each *triangle* of G (i.e., cycle of G of length 3) is contained in exactly one subgraph induced by some V_i, $i = 1, \ldots, k$;
P3. for every $i \neq j$ there is at most one edge in G joining nodes in V_i and nodes in V_j (i.e., $|(V_i \times V_j) \cap E| \leq 1$).

Let $G' = (V', E')$ be the graph obtained from G by contracting the nodes in each V_i, i.e., $V' = \{V_1, \ldots, V_k\}$, and $\{V_i, V_j\} \in E'$ whenever at least one node from V_i is joined to one node from V_j in G (in fact, under assumption P3, this is equivalent to saying that there is exactly one edge in G joining one node from V_i to one node from V_j for $i \neq j$). We consider the graph homomorphism $\varphi : G \to G'$ defined by $\varphi(v) = V_i$ whenever $v \in V_i$.

Given a subgraph $F = (V_F, E_F)$ of G, we denote by $\varphi(F) = (V_{\varphi(F)}, E_{\varphi(F)})$ the image of F in G' where

$$V_{\varphi(F)} = \varphi(V_F),$$
$$E_{\varphi(F)} = \{(\varphi(i), \varphi(j)) : (i, j) \in E_F\}.$$

In the following lemmata and theorems we always assume that properties P1, P2 and P3 hold.

Lemma 3. *Let F be a subgraph of G and assume that $\varphi(F)$ contains a chordless cycle in G' of length ℓ'. Then F contains a chordless cycle in G of length $\ell \geq \max\{4, \ell'\}$.*

Proof. Let $C' = \langle V_{i_1}, \ldots, V_{i_{\ell'}} \rangle$ be a chordless cycle in G'. Then for $h = 1, \ldots, \ell'$ there exist (not necessarily distinct) nodes $v_h, u_h \in V_{i_h}$ such that $(u_h, v_{h+1}) \in E$ for $h = 1, \ldots, \ell' - 1$, and $(u_{\ell'}, v_1) \in E$. Since each V_i is connected, there is a chordless path P_h (possibly with a single element) in V_{i_h} between v_h and u_h for $h = 1, \ldots, \ell'$. Furthermore, since C' is chordless, there is no edge joining nodes in $P_{h'}$ to nodes in $P_{h''}$ for $h' \neq h''$. Hence, the path obtained by joining $u_h \in P_h$ with $v_{h+1} \in P_{h+1}$ can be closed to form a chordless cycle C in G by joining $u_{\ell'} \in P_{\ell'}$ with $v_1 \in P_1$. Furthermore, since $|P_h| \geq 1$ for all h, we trivially have that the length of C is at least ℓ'. When $\ell' = 3$, at least one of the P_h must have more than one node, since otherwise there exists a triangle of G not contained in any of the subgraphs induced by V_i, thus contradicting property P2. Thus the length of C must be at least 4. □

Lemma 4. *Let F be a chordal subgraph of G. Then $\varphi(F)$ is acyclic in G'.*

Proof. This follows immediately from Lemma 3. □

To simplify notation, in what follows we identify a subgraph with the set of its edges. Furthermore, under assumption P3, an acyclic subgraph F' of G' is also identified with the unique minimal subgraph F of G satisfying $\varphi(F) = F'$.

Theorem 10. *A spanning subgraph F of a connected graph G is chordal if and only if*

$$F = S \cup \bigcup_{i=1}^{k} F_i, \tag{1}$$

where S is a spanning tree of G' and, for $i = 1, \ldots, k$, F_i is a spanning chordal subgraph of the subgraph induced by V_i in G.

Proof. ($\Leftarrow$) Assume that there exists a (chordless) cycle $\langle v^1, \ldots, v^\ell \rangle$ in $S \cup \bigcup_{i=1}^k F_i$ with $\ell \geq 4$. Then, there exist $V_{i_1}, \ldots, V_{i_m}$ such that $v^1, \ldots, v^{h_1} \in V_{i_1}$; $(v^{h_1}, v^{h_1+1}) \in S$; $v^{h_1+1}, \ldots, v^{h_2} \in V_{i_2}$; ...; $(v^{h_m} = v^\ell, v^1) \in S$. Thus, $V_{i_1}, \ldots, V_{i_m}$ form a cycle in (V', S) contradicting the assumption that S is a tree of G'.

($\Rightarrow$) Suppose now that F is a spanning chordal subgraph of G and let F_i denote the subgraph induced by V_i in F. Then F_i is a spanning chordal subgraph of the subgraph induced by V_i in G. Let $S = \varphi(F)$. Then, S is a spanning tree in G' by Lemma 4. Furthermore, for every edge (v', v'') in F with $v' \in V_i$ and $v'' \in V_j$, we have $(v', v'') \in F_i$ whenever $i = j$, and $(v', v'') \in S$ otherwise. □

As a straightforward consequence of Theorem 10 we can find the maximum weight spanning chordal subgraph in G by solving the maximum spanning tree problem in G' and finding the maximum weight spanning chordal subgraphs in all V_i's.

Corollary 1. *Let $\mathcal{C}_G$ denote the set of all (spanning) chordal subgraphs of G and let $\mathcal{C}_{V_i}$ denote the set of all (spanning) chordal subgraphs of the subgraph induced by V_i, $i = 1, \ldots, k$, in G. Let $\mathcal{S}_{G'}$ denote the set of all spanning trees of G'. Then, we have:*

$$\max_{F \in \mathcal{C}_G} w(F) = \max_{S \in \mathcal{S}_{G'}} w(S) + \sum_{i=1}^{k} \max_{F_i \in \mathcal{C}_{V_i}} w(F_i).$$

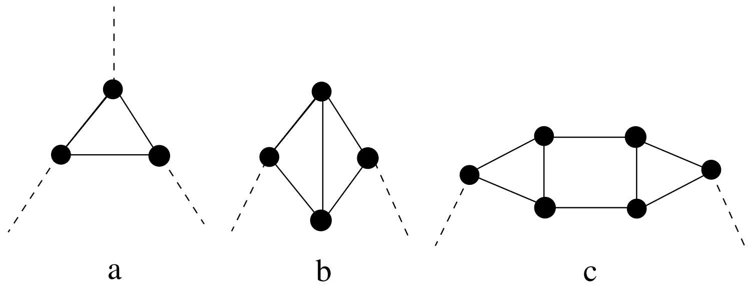

Fig. 1. (a) Triangle. (b) Diamond. (c) Hammock.

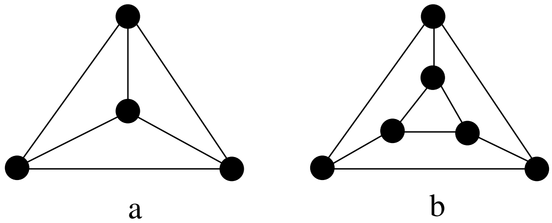

Fig. 2. (a) K_4. (b) 3-regular graph containing two triangles.

Remark 1. Let $ST(n, m)$ denote the time required to solve a maximum spanning tree problem in a graph with n nodes and m edges, β denote the time required to construct the partition $V_1, \ldots, V_k$, and α denote the time needed to solve the maximum weight spanning chordal subgraph problem in all the subsets V_i for $i = 1, \ldots, k$. Since $|V'| \leq |V|$, the time required to solve the maximum weight spanning chordal subgraph problem in a graph G with n nodes and m edges with the decomposition method is $O(ST(n, m) + \alpha + \beta)$. This is clearly polynomial in n and m whenever α and β are polynomial in n and m.

In the next section we show how to use the decomposition method in the case where G has maximum node degree equal to 3. In this case, we find that $ST(n, m) = n \log n$, $\alpha = O(n)$ and $\beta = O(n)$, so that the overall time complexity is $O(n \log n)$.

5. Spanning chordal subgraph problems on degree bounded graphs

When the graph G has (small) maximum vertex degree Δ, several – but not all – $\mathcal{F}$-subgraph problems can be solved in polynomial time.

Yannakakis [25] shows that the maximum spanning line-invertible subgraph problem can be solved in polynomial time in graphs with $\Delta = 3$, but it is NP-complete in graphs with $\Delta = 4$.

Natanzon, Shamir and Sharan [18,22] prove that the maximum spanning $\mathcal{F}$-subgraph problem can be solved in polynomial time in graphs with bounded degree when $\mathcal{F}$ is the family of chain, split, or threshold graphs (see, e.g., [18] for definitions). The proofs are based on the observation that in all these cases the search space becomes bounded when the problem is restricted to bounded degree graphs. Furthermore, the same authors provide a general polynomial approximation result for graphs with bounded degree.

Okawa, Nishitani, and Honda [26] show that the maximum bipartite subgraph problem is NP-complete even in graphs with $\Delta = 3$, or in 3-regular graphs (i.e., in *cubic graphs*).

In this section we provide a polynomial time algorithm, based on the decomposition method, for the problem of finding a maximum weight spanning chordal subgraph in a graph $G = (V, E)$ with maximum vertex degree $\Delta = 3$. In particular, we show how to obtain a suitable partition of the vertex set V into sets $V_1, \ldots, V_k$ satisfying properties P1, P2 and P3.

Let T be a triangle of G. When T has exactly two nodes connected to another single node $v \in V$, we call the subgraph D induced by T and v a *diamond*, while when T has exactly two nodes connected to two nodes of another triangle, we call *hammock* the resulting induced subgraph H (see Fig. 1).

In Fig. 1, dashed lines represent edges that may connect a given subgraph to the rest of the graph G. Consider the following procedure for finding a partition of the vertex set V:

- Visit the graph G and find all the triangles of G;
- If a triangle T has three nodes connected to another single node, then G is a complete graph on four vertices (i.e., K_4, see Fig. 2(a)), while if T has three nodes connected to the nodes of another triangle, then G is the graph in Fig. 2(b). In these two cases the maximum weight spanning chordal subgraph can trivially be found in time $O(1)$.
- If a triangle T does not have two nodes connected to another single node (i.e., it does not belong to a diamond), or to two nodes belonging to another triangle (i.e., it does not belong to a hammock), or if it does not have two nodes connected to

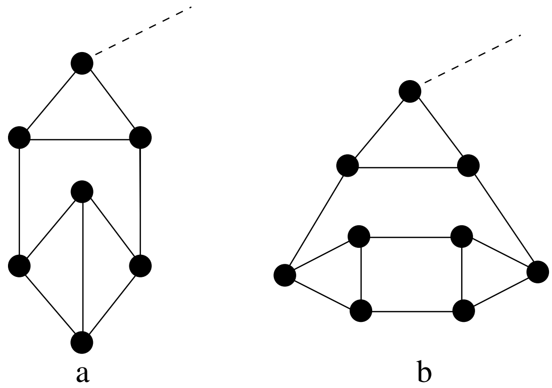

Fig. 3. (a) Triangle biconnected to a diamond. (b) Triangle biconnected to a hammock.

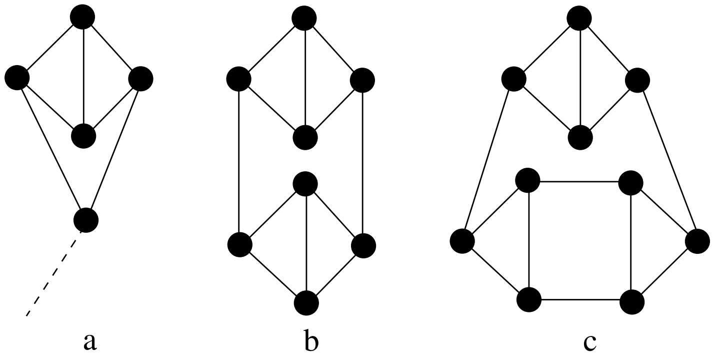

Fig. 4. (a) Diamond biconnected to a single vertex. (b) Two biconnected diamonds. (c) Diamond biconnected to a hammock.

two nodes of a diamond D or to two nodes of a hammock H (see Fig. 3), then we can take T as one of the elements V_i of the partition;

- If a triangle T is connected to two nodes of (i.e., it is *biconnected* to) a diamond D or to two nodes of a hammock H (see Fig. 3), then we can take the subgraphs induced by T and D, and by T and H as elements V_i of the partition of V.
- If a diamond D does not have two nodes connected to another single node or to two nodes belonging to a triangle (see Figs. 3(a) and 4(a)), or if it does not have two nodes connected to another diamond or to two nodes of a hammock (see Fig. 4(b) and (c)), then we can take D as one of the elements V_i of the partition;
- If a diamond D is connected to a single node v, or to two nodes of another diamond D', or to two nodes of a hammock H (see Fig. 4), we can take the subgraphs induced by D and v, by D and D', and by D and H as elements V_i of the partition of the node set V. The case of a subgraph induced by D and a triangle T was already considered (see Fig. 3(a)). Note that the subgraphs induced by a diamond D biconnected to another diamond, or to a hammock are in fact 3-regular graphs that must coincide with the whole graph G, so that a maximum weight spanning chordal subgraph problem can be solved in time $O(1)$.
- If a hammock H does not have two nodes connected to a single node or to two nodes belonging to a triangle (see Figs. 3(b) and 5(a)), or if it does not have two nodes connected to a diamond (see Fig. 4(c)), or to two nodes of another hammock (see Fig. 5(b)), then we can take H as one of the elements V_i of the partition;
- If a hammock H is biconnected to a single node v (see Fig. 5(a)), we can take the subgraph induced by H and v as an element V_i of the partition of the node set V. As before, the cases of subgraphs induced by H biconnected to a triangle T, and by H biconnected to a diamond D were already considered (see Figs. 3(b), and 4(c)). The subgraph induced by a hammock H biconnected to another hammock H' is in fact 3-regular (see Fig. 5(b)), and must thus coincide with the whole graph G, so that a maximum weight spanning chordal subgraph can be found in time $O(1)$.
- Each node v of G that has not been inserted in some subsets V_i, $i = 1, \ldots, k$, in the previous steps, can be taken to form a (singleton) element of the partition of V.

Theorem 11. *The partition $V_1, \ldots, V_k$ obtained with the above procedure satisfies properties* P1, P2 *and* P3.

Proof. Properties P1 and P2 are trivially satisfied by construction. Property P3 is trivially satisfied by all those components V_i that are connected to the rest of the graph by at most one edge. Note that each element V_i of the partition obtained with the previous procedure has at most three free edges connecting it with the rest of the graph. However, two different elements

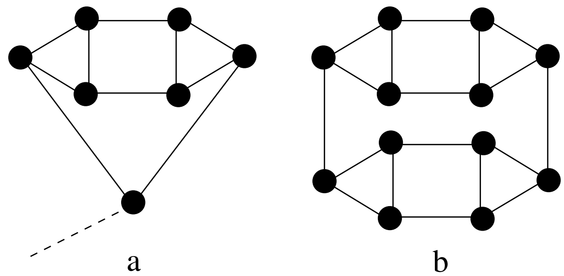

Fig. 5. (a) Hammock biconnected to a single vertex. (b) Two biconnected hammocks.

V_i and V_j, $i \neq j$, of the partition cannot be biconnected or triconnected between them, since otherwise they would have been joined in a larger element in some steps of the procedure. Hence, property P3 holds. □

Remark 2. It can be shown that the partition $V_1, \ldots, V_k$ obtained by our procedure is the finest possible among all the possible partitions that satisfy properties P1, P2 and P3. Indeed, for any other partition $U_1, \ldots, U_h$ satisfying P1, P2 and P3, we must have $h < k$, and for every i there exists j such that $V_i \subseteq U_j$.

Since the node degree of G is bounded by 3, the overall time complexity of the above procedure is $O(|V|)$ by visiting the graph G. Hence, the time β needed to construct the partition $V_1, \ldots, V_k$ is $O(|V|)$. Since each element V_i, $i = 1, \ldots, k$, of the partition has at most 12 nodes and 18 edges, the time required for finding a maximum weight spanning chordal subgraph of a subgraph induced by a V_i is $O(1)$, so that $\alpha = O(|V|)$. Hence, since a maximum spanning tree in graphs with n nodes can be found in time $O(n \log n)$ in the weighted case and $O(n)$ in the unweighted case, Remark 1 implies that for a graph G of maximum node degree $\Delta = 3$ we have the following complexity result.

Theorem 12. *The maximum weight spanning chordal subgraph problem can be solved on graphs G with maximum vertex degree* $\Delta = 3$ *in* $O(|V| \log |V|)$ *time. Furthermore, a maximum spanning chordal subgraph of G can be found in* $O(|V|)$ *time.*

Corollary 2. *The problem of the existence of a spanning* 2*-tree in a graph G with maximum vertex degree* $\Delta = 3$ *can be solved in* $O(|V|)$ *time.*

Proof. Note that a graph G with maximum vertex degree $\Delta = 3$ is either a K_4 or does not contain a K_4. Hence, a chordal subgraph of G is also a 2-chordal subgraph of G. Thus for these graphs $\overline{\mathcal{F}}_{chord} = \mathcal{F}_{2-trees}$. Then the thesis follows from Theorem 12 and from the Existence–Max reduction Lemma 1. □

6. Conclusions and further research

This paper contains two tools that might be of some help for analyzing the complexity and for solving spanning subgraph problems.

The reduction lemmata 1 and 2, in conjunction with results from extremal graph theory, allow us to obtain *NP*-completeness results for maximum (weight) spanning subgraph problems from *NP*-completeness results for the existence problem. This method has allowed, e.g., to prove *NP*-completeness of the maximum spanning chordal subgraph problem, which has been an open problem for some time that has been settled only recently in a weaker form (see [18]). Furthermore, Corollary 2 shows how to obtain a polynomiality result for the existence problem from a polynomiality result for the maximum spanning subgraph problem. However, it would be interesting to explore other possible applications of lemmata 1 and 2.

Another useful tool is the decomposition method described in Section 4, that allows us to decompose the maximum weight spanning chordal subgraph problem in a graph G into analogous subproblems on induced subgraphs of G, thereby allowing an efficient solution of the problem whenever a suitable partition of the vertex set of G can be found efficiently.

It would be interesting to investigate whether either of these two tools, or other approaches, could be applied to establish the complexity status of the problem of finding a maximum spanning chordal subgraph problem in a (planar) graph with node degree bounded by 4 or 5.

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