The NP-Completeness of Some Edge-Partition Problems

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Abstract. We show that for each fixed $n \geq 3$ it is NP-complete to determine whether an arbitrary graph can be edge-partitioned into subgraphs isomorphic to the complete graph K_n. The NP-completeness of a number of other edge-partition problems follows immediately.

Key words. computational complexity, NP-complete problems, edge-partition problems

1. Introduction. Many graph theory problems have been shown to be NP-complete and so are believed not to have polynomial time algorithms. Garey and Johnson [1] give an account of the theory of NP-completeness, a list of known NP-complete problems and a bibliography of the subject. In particular, they list several NP-complete vertex-partition problems [1, p. 193] including vertex-partition into cliques [2] and vertex-partition into isomorphic subgraphs [3].

In this paper, we consider some similar problems for edge-partitions. We define the edge-partition problem EP$_n$ as follows. Given a graph $G = (V, E)$, the problem is to determine whether the edge-set E can be partitioned into subsets E_1, E_2, $\ldots$ in such a way that each E_i generates a subgraph of G isomorphic to the complete graph K_n on n vertices. Our main result is that the problem EP$_n$ is NP-complete for each $n \geq 3$. From this we deduce that a number of other edge-partition problems are NP-complete.

In order to show that EP$_n$ is NP-complete, we will exhibit a polynomial reduction from the known NP-complete problem 3SAT which is defined as follows. A set of clauses $C = \{C_1, C_2, \ldots, C_r\}$ in variables u_1,u_2, $\ldots$,u_s is given, each clause C_i consisting of three literals $l_{i,1}, l_{i,2}, l_{i,3}$ where a literal $l_{i,j}$ is either a variable u_k or its negation $\overline{u}_k$. The problem is to determine whether C is satisfiable, that is, whether there is a truth assignment to the variables which simultaneously satisfies all the clauses in C. A clause is satisfied if one or more of its literals has value “true”.

2. The main theorem. Our first task is to find a graph which can be edge-partitioned into K_n’s in exactly two distinct ways. Such a graph can be used as a “switch” to represent the two possible values “true” and “false” of a variable in an instance of 3SAT.

For each $n \geq 3$ and $p \geq 3$ we define a graph $H_{n,p} = (V_{n,p}, E_{n,p})$ by

$$V_{n,p} = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{Z}_p^n : \sum_{i=1}^{n} x_i \equiv 0 \right\},$$

$$\begin{aligned} E_{n,p} = \{\mathbf{xy} : & \text{ there exist } i, j \text{ such that } y_k \equiv x_k \\ & \text{for } k \neq i, j \text{ and } y_i \equiv x_i + 1, y_i \equiv x_j - 1\} \end{aligned}$$

where the equivalences are modulo p. Note that $H_{n,p}$ can be regarded as embedded in the $(n-1)$-dimensional torus $T^{n-1} = S^1 \times S^1 \times \ldots \times S^1$, and that the local structure of $H_{n,p}$ is the same for each p (see Fig. 1). The properties of $H_{n,p}$ are given in the following lemma.

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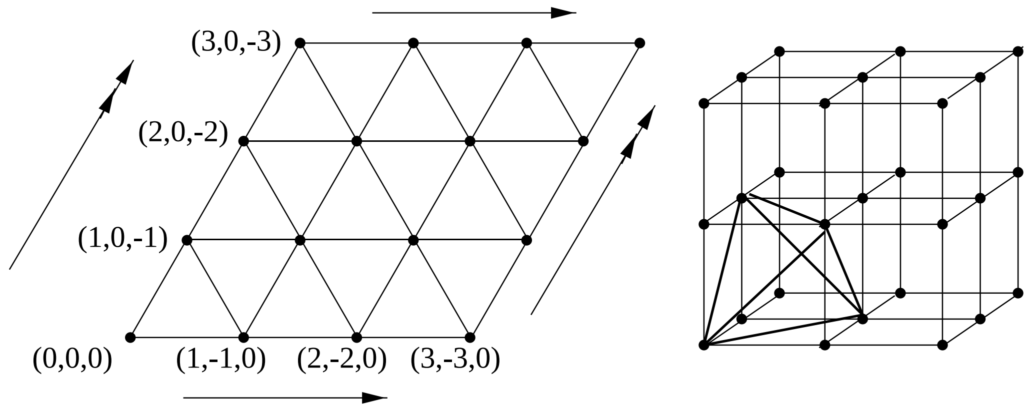

(i) (ii)

Figure 1: *(i) $H_{3,3}$ embedded in the (2-dimensional) torus. Opposite sides are identified as shown. (ii) The local structure of $H_{4,p}$. The edges of a single K_4 are shown.*

LEMMA. *The graph $H_{n,p}$ has the following properties:*

(i) *The degree of each vertex is $2\binom{n}{2}$.*

(ii) *The largest complete subgraph is K_n, and any K_3 is contained in a unique K_n.*

(iii) *The number of K_n's containing a particular vertex is $2n$.*

(iv) *Each edge occurs in just two K_n's.*

(v) *Each two distinct K_n's are either edge-disjoint or have just one edge in common.*

(vi) *There are just two distinct edge-partitions of $H_{n,p}$ into K_n's.*

Proof. (i) By translational symmetry we need only consider $\mathbf{0} = (0, \ldots, 0)$. This is adjacent to $(1, -1, 0, \ldots, 0)$ and the distinct points obtained from it by permuting its coordinates ($0, 1, -1$ are distinct modulo p as $p \geq 3$). There are clearly $2\binom{n}{2}$ of these.

(ii) By translation and coordinate permutation we may assume that a largest complete subgraph contains the vertices $\mathbf{0} = (0, \ldots, 0)$, $(1, -1, 0, \ldots, 0)$ and $(1, 0, -1, 0, \ldots, 0)$. It is then forced to be the *standard* K_n, which we call K and whose vertices are:

$$\begin{array}{c} (0, 0, 0, \ \ldots, 0) \\ (1, -1, 0, \ldots, 0) \\ (1, 0, -1, \ldots, 0) \\ \ldots \\ (1, 0, 0 \ldots, -1) \end{array}$$

(iii) The K_n's containing $\mathbf{0}$ are obtained from K and its inverse $-K$ by cyclic permutation of the coordinates. Thus there are $2n$ of them.

(iv) We need only consider a particular edge containing the vertex $\mathbf{0}$ and check that it is contained in just two of the K_n's given in (iii).

(v) If two K_n's are not disjoint, we may assume that they have vertex $\mathbf{0}$ in common. We may then use (iii) to check that they have just one more vertex in common.

(vi) The edges containing $\mathbf{0}$ can be partitioned in at most two ways, and these extend to the whole of $H_{n,p}$. All the K_n's are obtained from K or $-K$ by translation. One edge-partition consists of the translates of K, and the other consists of the translates of $-K$.

We now make the following definitions. The *T-partition* of $H_{n,p}$ (corresponding to logical value "true") consists of the translates of K, and the *F-partition* (corresponding to "false") consists of the translates of $-K$. Two K_n's in $H_{n,p}$ are called *neighbors* if they have a common edge. A *patch* is a subgraph of $H_{n,p}$ consisting of the vertices and edges of a particular K_n and of its neighbors. It is a *T-patch* if the central K_n belongs to the T-partition, and it is an *F-patch* otherwise. Two patches P_1, P_2 in $H_{n,p}$ are called *noninterfering* if the distance $d(\mathbf{x}, \mathbf{y})$ in $H_{n,p}$ between vertices $\mathbf{x} \in V(P_1)$ and $\mathbf{y} \in V(P_2)$ is always at least 10, say.

THEOREM. *The edge-partition problem* EP_n *is* NP-*complete for each* $n \geq 3$.

Proof. The problem EP_n is clearly in NP. Suppose we have an instance $C = \{C_1, C_2, \ldots, C_r\}$ of 3SAT in s variables $u_1, u_2, \ldots, u_s$ where each C_i consists of literals $l_{i,1}$, $l_{i,2}$, $l_{i,3}$. We reduce this instance of 3SAT to an instance $G_n = (V_n, E_n)$ of EP_n as follows.

Choose p sufficiently large so that up to $3r$ noninterfering patches can be chosen in $H_{n,p}$ say $p = 100r$. Take a copy U_i of $H_{n,p}$ to represent each variable u_i and copies $C_{i,1}$, $C_{i,2}$ and $C_{i,3}$ of $H_{n,p}$ to represent each clause C_i.

Join these copies of $H_{n,p}$ together as follows. If literal $l_{i,j}$ is u_k, then identify an *F*-patch of $C_{i,j}$ with an *F*-patch of U_k. If $l_{i,j}$ is $\overline{u}_k$, then identify an *F*-patch of $C_{i,j}$ with a *T*-patch of U_k as indicated for $n = 3$ in Fig. 2.

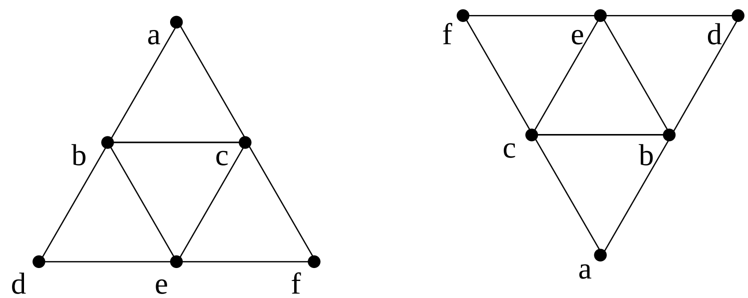

Figure 2: *The identification of an F-patch with a T-patch when* $n = 3$. *Similarly labelled vertices (and the edges between them) are identified.*

Also join $C_{i,1}$, $C_{i,2}$ and $C_{i,3}$ for each i by identifying one *F*-patch from each and then removing the edges of the central K_n (see Fig. 3).

Choose all those patches which occur in a single copy of $H_{n,p}$ to be noninterfering.

Denote by $G_n = (V_n, E_n)$ the graph obtained in this way. We now show that there is an edge-partition of G_n into K_n's if and only if the instance C of 3SAT is satisfiable.

Suppose that there is an edge-partition of G_n into a set S of K_n's, and consider a particular copy H of $H_{n,p}$ involved in the construction of G_n. Take a K_n in S, say A, which is in H, but not near any join. Using the properties in the lemma, we see that the neighbors of A do not belong to S, the neighbors of the neighbors of A do belong to S, and so on. Continuing in this way, we deduce that all the edges of H, except perhaps those involved in joins, are *T*-partitioned, or all *F*-partitioned. Thus we may say that H is *T*-partitioned or *F*-partitioned.

Now suppose $l_{i,j}$ is u_k and consider the join between $C_{i,j}$ and U_k. We claim that the edges in the vicinity of this join can be edge-partitioned into K_n's if and only if at least one of $C_{i,j}$, U_k is *T*-partitioned. If (say) $C_{i,j}$ is *T*-partitioned, this accounts for all the edges of $C_{i,j}$ near the joining patch except for those of the patch itself. The patch can then be regarded as belonging to U_k which can then be locally partitioned in either way. If on the other hand both $C_{i,j}$ and U_k are *F*-partitioned, the argument of the previous paragraph shows that the edges of the patch not belonging to the central K_n are forced to belong to the *F*-partitions of both $C_{i,j}$ and U_k, which is a contradiction.

Similarly, if $l_{i,j}$ is $\overline{u}_k$, then either $C_{i,j}$ is *F*-partitioned or U_k is *T*-partitioned.

Now consider the join between $C_{i,1}$, $C_{i,2}$ and $C_{i,3}$. We claim that the edges in the vicinity of this join can be edge-partitioned into K_n's if and only if exactly one of $C_{i,1}$, $C_{i,2}$, $C_{i,3}$ is *F*-partitioned. The argument is the same as above, except that now, as the central K_n is missing, the remaining edges of the patch must be claimed by an *F*-partition in exactly one of $C_{i,1}$, $C_{i,2}$, $C_{i,3}$.

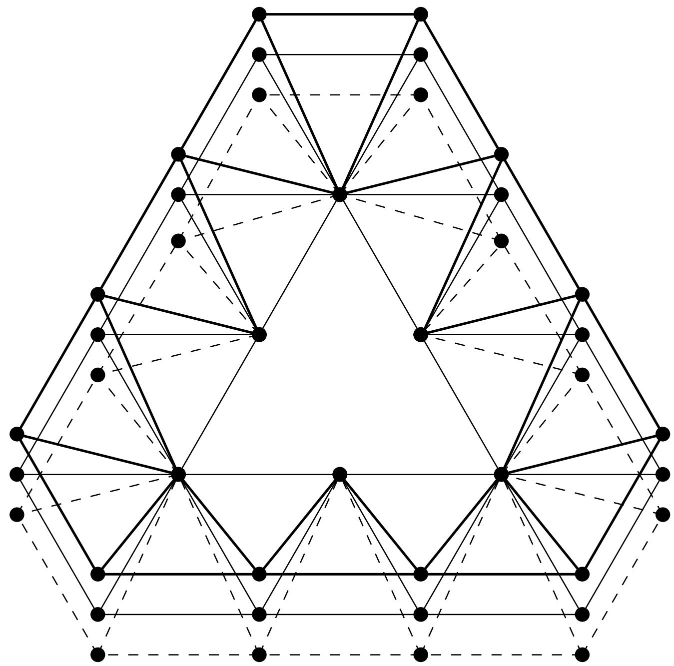

Figure 3: *The join between $C_{i,1}$, $C_{i,2}$ and $C_{i,3}$ when $n = 3$.*

Thus if G_n can be edge-partitioned into K_n's, then there is a truth assignment to $u_1, \ldots, u_s$ which satisfies C, namely u_k has value "true" if and only if U_k is *T*-partitioned.

If C is satisfiable, we partition G_n by partitioning U_k according to the truth of u_k in a satisfying assignment, choosing one "true" literal $l_{i,j}$ for each i, and *F*-partitioning the corresponding $C_{i,j}$.

It should be clear that the above reduction from 3SAT to EP$_n$ can be carried out using a polynomial time algorithm, and so the proof of the theorem is complete. □

3. Deductions. The following problems are now easily seen to be NP-complete.

(i) Find the maximum number of edge-disjoint K_n's in a graph ($n \geq 3$).

(ii) Find the maximum number of edge-disjoint maximal cliques in a graph.

(iii) Edge-partition a graph into the minimum number of complete subgraphs.

(iv) Edge-partition a graph into maximal cliques.

(v) Edge-partition a graph into cycles C_m of length m.

For (i) we use the same construction as for EP$_n$. For (ii), (iii) and (iv) we use the same construction as for EP$_3$. Note that G_3 contains no K_4's, and every edge K_2 is in a K_3, so the maximal cliques coincide with the K_3's.

For (v) we alter the construction for EP$_3$ in the following way. Note that the edges in $H_{3,p}$ occur in three distinct directions, say **a**, **b** and **c**, and that the joins in the construction of G_3 are made so that edges which are identified have the same direction. In G_3, replace each edge with direction **a** (say) by a path of $m - 2$ edges.

We conjecture that the problem of edge-partitioning a graph into subgraphs isomorphic to a fixed graph H is NP-complete for all graphs H with at least 3 edges. The problem is polynomial if H has at most 2 edges, and it is easy to show that the problem is NP-complete for a number of particular small, connected graphs H. The NP-completeness of the problem seems difficult to prove if H is disconnected, e.g., if $H = 3K_2$, that is, H has 6 vertices and 3 independent edges.

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