Notes

Steinberg's Conjecture is false

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A B S T R A C T

Steinberg conjectured in 1976 that every planar graph with no cycles of length four or five is 3-colorable. We disprove this conjecture.

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1. Introduction

One of major open problems on colorings of planar graphs is Steinberg's Conjecture from 1976. The conjecture asserts that every planar graph with no cycles of length four

or five is 3-colorable. This problem has been attracting a substantial amount of attention among graph theorists, see [3, Section 7]. It is also one of the six graph theory problems ranked with the 4-star (highest) importance in the Open Problem Garden [10]. As a possible approach towards proving the conjecture, Erdős in 1991 (see [12]) suggested to determine the smallest k such that every planar graph with no cycles of length $4, \ldots, k$ is 3-colorable. Borodin et al. [7], improving up on [1,11], showed that $k \leq 7$, i.e., every planar graph with no cycles of length four, five, six or seven is 3-colorable. Many other relaxations of the conjecture have been established, e.g. [2,5,6,8,14]. On the other hand, the conjecture is known to be false in the list coloring setting [13]. We refer the reader for further results on the conjecture and other problems related to colorings of planar graphs to [3,9].

We disprove Steinberg's Conjecture by constructing a planar graph with no cycles of length four or five that is not 3-colorable. This also disproves Strong Bordeaux Conjecture from [8] and Novosibirsk 3-Color Conjecture from [4].

Conjecture 1 *(Strong Bordeaux Conjecture). Every planar graph with no pair of cycles of length three sharing an edge and with no cycle of length five is* 3-*colorable.*

Conjecture 2 *(Novosibirsk* 3-*Color Conjecture). Every planar graph without a cycle of length three sharing an edge with a cycle of length three or five is* 3-*colorable.*

We would like to remark that we have not tried to minimize the number of vertices in our construction; we are actually aware of the existence of smaller counterexamples (the smallest one with 85 vertices) but we decided to present the one that is the simplest to analyze among those that we have found.

2. Counterexample

Our construction has three steps. In each of these steps, we construct a particular planar graph with no cycles of length four or five that has some additional properties. The next two lemmas present the first two steps of the construction.

Lemma 1. *The graph G_1 that is depicted in Fig. 1 has no cycles of length four or five and there is no* 3-*coloring that would assign the same color to all the three vertices a, b and c. In addition, the distance between a and b is three, between a and c is three, and between b and c is four.*

Proof. It is easy to check that G_1 contains no cycle of length four or five, the distance between a and b is three, between a and c is three, and between b and c is four. Suppose that G_1 has a 3-coloring such that all the three vertices a, b and c have the same color, say γ. The other two colors must be used in an alternating way to color the vertices d, e, f, g, h and i. This implies that the vertices j, k and l have the color γ. Hence, the two
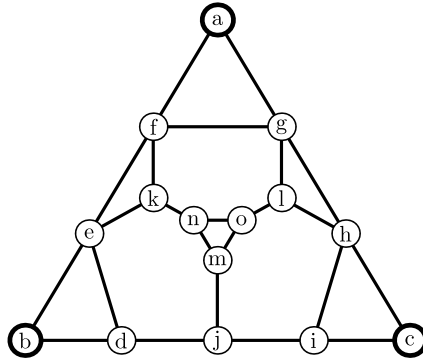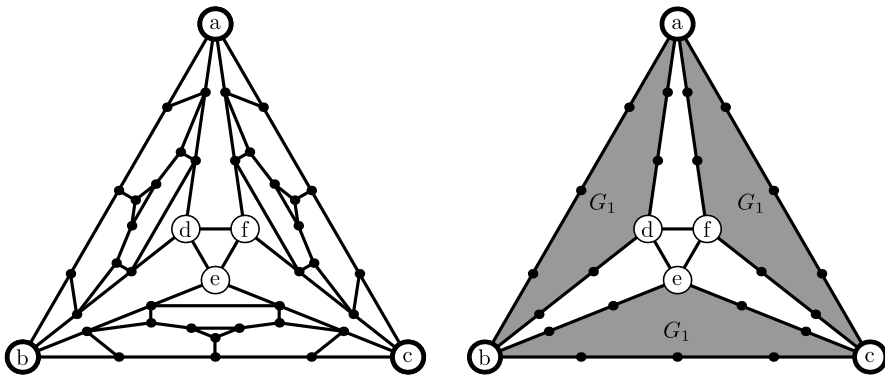
Fig. 1. The graph G_1 analyzed in Lemma 1.

Fig. 2. The graph G_2 analyzed in Lemma 2. The actual graph is drawn on the left; its abstract structure using the graph G_1 is given on the right. The three contact vertices a, b and c are drawn with bold circles.

colors different from γ must be used to color the vertices m, n and o, which is impossible since these three vertices form a triangle. □

Lemma 2. *The graph G_2 that is depicted in Fig. 2 has no cycles of length four or five and there is no 3-coloring that would assign the same color to all the three vertices a, b and c. In addition, the distance between any pairs of the vertices a, b and c is four.*

Proof. Note that the graph G_2 consists of three copies of the graph G_1 pasted together as depicted in Fig. 2. We start by checking that the distance between the vertices a and b is four (the cases of the other two pairs are symmetric). Since a and b are joined by a path of length four, their distance is at most four. Lemma 1 implies that there is no path of length less than four inside the copy of G_1 containing a and b. On the other hand, any path passing through more than one copy of G_1 must have length at least three inside each traversed copy of G_1, so its total length cannot be less than six. We conclude that the distance between a and b is four.
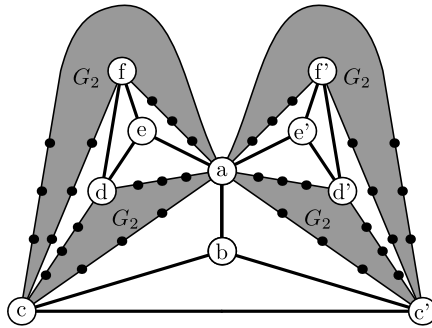
Fig. 3. The graph G constructed in the proof of Theorem 3. The gray areas represent the graph from Fig. 2.

We next check that G_2 has no cycle of length four or five. By Lemma 1, no cycle inside one of the copies of G_1 has length four or five. Since the distance between any two of the contact vertices of the copies of G_1 (the vertices used to glue the copies together) is at least three by Lemma 1, the length of any cycle passing through two or three of the copies is at least six. The only cycle that does not contain an edge inside a copy of G_1 is the cycle def and its length is three. Therefore, G_2 has no cycle of length four or five.

It remains to check that G_2 has no 3-coloring such that the vertices a, b and c get the same color. Suppose that G_2 has a 3-coloring such that the vertices a, b and c receive the same color, say γ. By Lemma 1, the color of any of the vertices d, e and f is different from γ. However, this is impossible since the vertices d, e and f form a triangle and there are only three colors in total. Hence, G_2 has no 3-coloring such that the vertices a, b and c would get the same color. □

We are now ready to provide the counterexample construction.

Theorem 3. *There exists a planar graph with no cycles of length four or five that is not 3-colorable.*

Proof. Consider the graph G obtained by taking four copies of the graph G_2 from Lemma 2 and pasting them together with additional vertices b, e and e' in the way depicted in Fig. 3. We first argue that G has no cycle of length four or five. By Lemma 2, there is no cycle of length four or five inside a single copy of G_2. Any cycle containing an edge inside a copy of G_2 that is not fully contained inside the copy must contain at least four edges inside the considered copy of G_2, since the distance between any two of the contact vertices of G_2 is four by Lemma 2. Since such a cycle must contain at least two edges outside the copy of G_2, its length must be at least six. Hence, there is no cycle with length four or five that contains an edge inside a copy of G_2. Observe that there are exactly three cycles not containing an edge inside any copy of G_2 and these are triangles bcc', def and $d'e'f'$. Therefore, G has no cycle of length four or five.

Suppose that G has a 3-coloring and let γ be the color assigned to the vertex a. Since the vertices of the triangle bcc' must be assigned all three colors and the color of b is

not γ, either c or c' is colored with γ. By symmetry, we can assume that the color of c is γ. Since the vertices a, c and d are contact vertices of a copy of G_2, the color of d is not γ by Lemma 2. Similarly, since the vertices a, c and f are contact vertices of a copy of G_2, the color of f is also not γ. Hence, none of the vertices d, e and f (note the vertex e is adjacent to a) has the color γ, which is impossible since the vertices d, e and f form a triangle. We conclude that the graph G has no 3-coloring.   □

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