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Note

On maximum matchings in almost regular graphs

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ABSTRACT

In 2010, Mkrtchyan, Petrosyan, and Vardanyan proved that every graph G with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ contains a maximum matching M such that no two vertices uncovered by M share a neighbor, where $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in G, respectively. In the same paper they suggested the following conjecture: every graph G with $\Delta(G) - \delta(G) \leq 1$ contains a maximum matching M such that no two vertices uncovered by M share a neighbor. Recently, Picouleau disproved this conjecture by constructing a bipartite counterexample G with $\Delta(G) = 5$ and $\delta(G) = 4$. In this note, we show that the conjecture is false for graphs G with $\Delta(G) - \delta(G) = 1$ and $\Delta(G) \geq 4$, and for r-regular graphs when $r \geq 7$.

1. Introduction

Throughout this note all graphs are finite, undirected, and have no loops, but may contain multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a graph G, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in G, respectively. An (a, b)-*biregular* bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. The terms and concepts that we do not define can be found in [2,7].

In [3,4], Mkrtchyan, Petrosyan, and Vardanyan proved the following result.

Theorem 1. *Every graph G with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ contains a maximum matching M such that no two vertices uncovered by M share a neighbor.*

Corollary 2. *Every cubic graph G contains a maximum matching M such that no two vertices uncovered by M share a neighbor.*

In the same paper they posed the following

Conjecture 3. *Every graph G with $\Delta(G) - \delta(G) \leq 1$ contains a maximum matching M such that no two vertices uncovered by M share a neighbor.*

The authors did not even know whether the conjecture holds for r-regular graphs for any r larger than 3. In [5], Picouleau showed that the conjecture is false in the class of $(5, 4)$-biregular bipartite graphs. The question remained open for r-regular graphs with $r \geq 4$ and for the classes of graphs G such that $\Delta(G) - \delta(G) = 1$ when $\Delta(G) \geq 4$ with $\Delta(G) \neq 5$.

In this note we prove that for $r \geq 2$, there exists a $(2r, 2r - 1)$-biregular bipartite graph G such that for any maximum matching M of G, each pair of vertices uncovered by M shares a neighbor. Next we show that for $r \geq 3$, there exists a graph

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G with $\Delta(G) = 2r + 1$ and $\delta(G) = 2r$ such that for any maximum matching M of G, some two uncovered vertices have a common neighbor. We also prove that for $r \geq 3$, there exists a $(2r + 1)$-regular graph G such that for any maximum matching M of G, some two uncovered vertices have a common neighbor. Finally, we construct an 8-regular graph with the same property and prove for $r \geq 5$ that there exists a $2r$-regular graph G such that for any maximum matching M of G, each pair of vertices uncovered by M shares a neighbor.

2. Results

Before we formulate and prove our main results, we need two classical results from matching theory.

Let G be a bipartite graph with bipartition (X, Y) and $S \subseteq X$. We denote by $N_G(S)$ the set of vertices having a neighbor in S.

Theorem 4 (*Hall [1]*). *If G is a bipartite graph with bipartition (X, Y), then G has a matching that covers X if and only if $|N_G(S)| \geq |S|$ for every $S \subseteq X$.*

Let G be a graph. We denote by $o(G)$ the number of components of G that have an odd number of vertices.

Theorem 5 (*Tutte [6]*). *A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for every $S \subseteq V(G)$.*

First we consider graphs G with $\Delta(G) - \delta(G) = 1$.

Theorem 6. *For $r \geq 2$, there exists a $(2r, 2r - 1)$-biregular bipartite graph G such that for any maximum matching M of G, each pair of vertices uncovered by M shares a neighbor.*

Proof. For the proof, we construct a graph B_r for $r \geq 2$ that satisfies the specified conditions. We define a graph B_r as follows: $V(B_r) = X \cup Y$, where

$$X = \left\{x_{i,j}: 1 \leq i < j \leq 2r\right\}, \qquad Y = \left\{y_1^{(i)}, \ldots, y_r^{(i)}: 1 \leq i \leq 2r\right\}, \quad \text{and}$$
$$E(B_r) = \left\{y_s^{(i)}x_{i,j}, y_s^{(j)}x_{i,j}: 1 \leq i < j \leq 2r, 1 \leq s \leq r\right\}.$$

Clearly, B_r is a $(2r, 2r - 1)$-biregular bipartite graph with bipartition (X, Y). Moreover, $|X| = \binom{2r}{2} = 2r^2 - r$ and $|Y| = 2r^2$. Thus, B_r has no perfect matching. On the other hand, by Theorem 4, it is not hard to see that each maximum matching M covers X. This implies that for any maximum matching M of B_r, we have r vertices from Y uncovered by M. Now let $y_k^{(i)}$ and $y_{k'}^{(j)}$ be any two vertices in Y uncovered by some maximum matching M. We consider two cases.

Case 1: $i = j$. In this case, by the construction of B_r, the vertices $y_k^{(i)}$ and $y_{k'}^{(i)}$ share a neighbor $x_{i,l}$ with $i < l$ or $x_{l,i}$ with $l < i$.

Case 2: $i \neq j$. In this case, by the construction of B_r, the vertices $y_k^{(i)}$ and $y_{k'}^{(j)}$ share a neighbor $x_{i,j}$ if $i < j$ or $x_{j,i}$ if $j < i$. □

Theorem 7. *For $r \geq 3$,*

(1) *there exists a $(2r + 1)$-regular graph G such that for any maximum matching M of G, there is a pair of vertices uncovered by M that shares a neighbor,*
(2) *there exists a graph H with $\Delta(H) = 2r + 1$ and $\delta(H) = 2r$ such that for any maximum matching M of H, there is a pair of vertices uncovered by M that shares a neighbor.*

Proof. (1) For the proof, we construct a graph G_{2r+1} for $r \geq 3$ that satisfies the specified conditions. We define G_{2r+1} as follows:

1) $V(G_{2r+1}) = \{x, y, z\} \cup \left\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}: 1 \leq i \leq 2r + 1\right\}$,
2) $E(G_{2r+1})$ contains all possible pairs of vertices of the set $\left\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\right\}$, which are joined by r edges for $1 \leq i \leq 2r + 1$ and the edges $xv_1^{(i)}, yv_2^{(i)}, zv_3^{(i)}$ for $1 \leq i \leq 2r + 1$.

Clearly, G_{2r+1} is a $(2r + 1)$-regular graph with $|V(G_{2r+1})| = 6r + 6$. Let $S = \{x, y, z\}$. Since $o(G - S) = 2r + 1$ and $r \geq 3$, G_{2r+1} has no perfect matching, by Theorem 5. On the other hand, it is not hard to see that each maximum matching M covers x, y, and z. This implies that for any maximum matching M of G_{2r+1}, we have $2r - 2$ uncovered vertices from $V(G_{2r+1}) \setminus \{x, y, z\}$. Since $r \geq 3$, for any maximum matching M of G_{2r+1} there are at least four vertices in $V(G_{2r+1}) \setminus \{x, y, z\}$ uncovered by M. However, by the construction of G_{2r+1}, the vertices from $V(G_{2r+1}) \setminus \{x, y, z\}$ have only three possible subscripts; thus there are two vertices with the same subscript. Let $v_k^{(i)}$ and $v_k^{(j)}$ be these uncovered vertices from $V(G_{2r+1}) \setminus \{x, y, z\}$ with respect to some maximum matching M. By the construction of G_{2r+1}, the vertices $v_k^{(i)}$ and $v_k^{(j)}$ share a neighbor, which is x, y, or z when k is 1, 2, or 3, respectively.

(2) For the proof, it suffices to define a graph H_r for $r \geq 3$ as follows: $V(H_r) = V(G_{2r+1})$ and $E(H_r) = E(G_{2r+1}) \setminus \left\{v_1^{(i)}v_3^{(i)}: 1 \leq i \leq 2r + 1\right\}$. Clearly, H_r is a graph with $\Delta(H_r) = 2r + 1$ and $\delta(H_r) = 2r$. Similarly as in the proof of (1) it can be shown that for any maximum matching M of H_r, there is a pair of vertices uncovered by M that shares a neighbor. □

These results combined with the result of Picouleau show that Conjecture 3 is false for graphs G with $\Delta(G) - \delta(G) = 1$ and $\Delta(G) \geq 4$. Also, Conjecture 3 is false for $(2r + 1)$-regular graphs when $r \geq 3$. Next we consider regular graphs of even

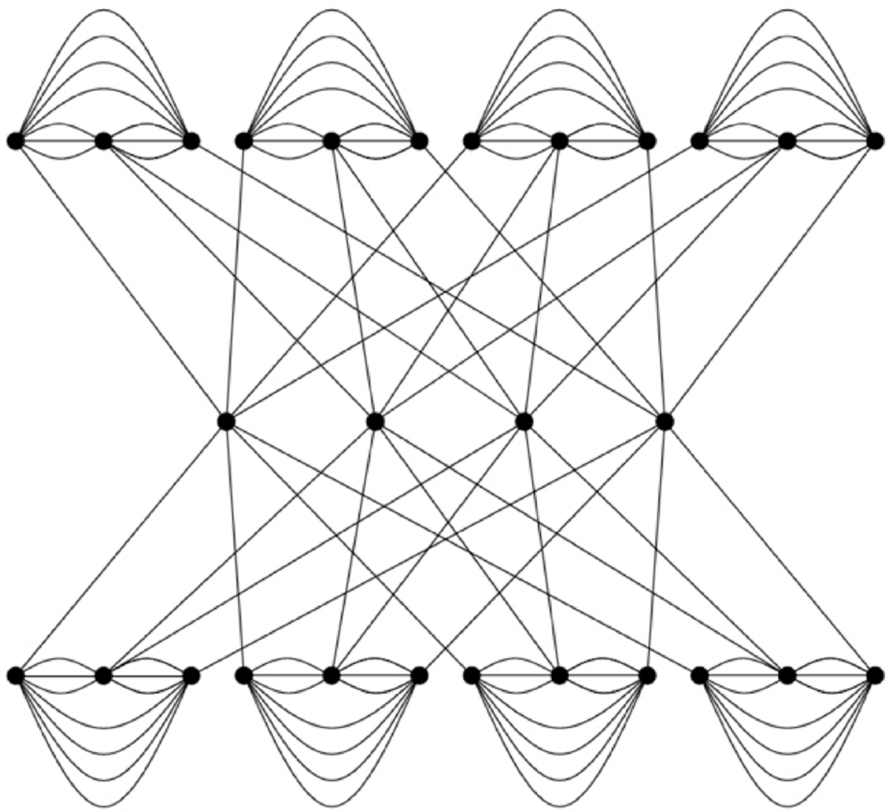

Fig. 1. The 8-regular graph G.

degree. First we consider the 8-regular graph G shown in Fig. 1. Similarly as in the proof of Theorem 7 it can be shown that for any maximum matching M of G, there is a pair of vertices uncovered by M that shares a neighbor.

Theorem 8. *For $r \geq 5$, there exists a $2r$-regular graph G such that for any maximum matching M of G, each pair of vertices uncovered by M shares a neighbor.*

Proof. For the proof, we construct a graph F_{2r} for $r \geq 5$ that satisfies the specified conditions. We define F_{2r} as follows:

1) $V(F_{2r}) = \{x, y, z\} \cup \left\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)} : 1 \leq i \leq r\right\}$,

2) $E(F_{2r})$ contains all possible pairs of vertices of the set $\left\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\right\}$, which are joined by $r-1$ edges for $1 \leq i \leq r$ and the edges $xv_1^{(i)}, xv_2^{(i)}, yv_1^{(i)}, yv_3^{(i)}, zv_2^{(i)}, zv_3^{(i)}$ for $1 \leq i \leq r$.

Clearly, F_{2r} is a $2r$-regular graph with $|V(F_{2r})| = 3r + 3$. Let $S = \{x, y, z\}$. Since $o(G - S) = r$ and $r \geq 5$, F_{2r} has no perfect matching, by Theorem 5. On the other hand, it is not hard to see that each maximum matching M covers x, y, and z. This implies that for any maximum matching M of F_{2r}, we have $r - 3$ uncovered vertices from $V(F_{2r}) \setminus \{x, y, z\}$. Since $r \geq 5$, for any maximum matching M of F_{2r} there are at least two vertices in $V(F_{2r}) \setminus \{x, y, z\}$ uncovered by M. Now let $v_k^{(i)}$ and $v_{k'}^{(j)}$ be any two vertices in $V(F_{2r}) \setminus \{x, y, z\}$ uncovered by some maximum matching M. We consider two cases.

Case 1: $k = k'$. If $k = k' = 1$ or $k = k' = 2$, then $v_k^{(i)}$ and $v_{k'}^{(j)}$ have x as a common neighbor, by the construction of F_{2r}. If $k = k' = 3$, then they have y as a common neighbor.

Case 2: $k \neq k'$. By the construction of F_{2r}, the vertices $v_k^{(i)}$ and $v_{k'}^{(j)}$ share a neighbor, which is x if $\{k, k'\} = \{1, 2\}$, is y if $\{k, k'\} = \{1, 3\}$, and is z if $\{k, k'\} = \{2, 3\}$. □

Our results show that Conjecture 3 is also false for r-regular graphs when $r \geq 7$. Thus, the question remains open only for 4-, 5-, and 6-regular graphs.

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