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Critical Independent Sets and König–Egerváry Graphs

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Abstract A set S of vertices is *independent* or *stable* in a graph G, and we write $S \in \text{Ind}(G)$, if no two vertices from S are adjacent, and $\alpha(G)$ is the cardinality of an independent set of maximum size, while $\text{core}(G)$ denotes the intersection of all maximum independent sets. G is called a *König–Egerváry graph* if its order equals $\alpha(G) + \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching. The number $def(G) = |V(G)| - 2\mu(G)$ is the *deficiency* of G. The number $d(G) = \max\{|S| - |N(S)| : S \in \text{Ind}(G)\}$ is the *critical difference* of G. An independent set A is *critical* if $|A| - |N(A)| = d(G)$, where $N(S)$ is the neighborhood of S, and $\alpha_c(G)$ denotes the maximum size of a critical independent set. Larson (Eur J Comb 32:294–300, 2011) demonstrated that G is a König–Egerváry graph if and only if there exists a maximum independent set that is also critical, i.e., $\alpha_c(G) = \alpha(G)$. In this paper we prove that: (i) $d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G)$ holds for every König–Egerváry graph G; (ii) G is König–Egerváry graph if and only if each maximum independent set of G is critical.

Keywords Maximum independent set · Maximum matching · Deficiency · Critical difference · Critical independent set · Core

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1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, simple (i.e., loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F, and we use $G - e$, if $W = \{e\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while $N(A) = \cup\{N(v) : v \in A\}$ and $N[A] = A \cup N(A)$ for $A \subset V$.

A set $S \subseteq V(G)$ is *independent* or *stable* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the set of all the independent sets of G. An independent set of maximum size will be referred to as a *maximum independent set* of G, and the *independence number* of G is $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Let us denote the set $\{S : S \text{ is a maximum independent set of } G\}$ by $\Omega(G)$, and let $\mathrm{core}(G) = \cap\{S : S \in \Omega(G)\}$ [11]. A set $A \subseteq V(G)$ is a *local maximum independent set* of G if $A \in \Omega(G[N[A]])$ [10].

Theorem 1 ([16]) *Every local maximum independent set of a graph is a subset of a maximum independent set.*

A *matching* in a graph G is a set M of edges such that no two edges of M share a common vertex. A matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G.

The *deficiency* of G, denoted by $def(G)$, is the number of exposed vertices relative to a maximum matching [15]. In other words, $def(G) = |V(G)| - 2\mu(G)$.

It is well-known that $\lfloor |V|/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq |V|$ holds for any graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then G is called a *König–Egerváry graph*. We attribute this definition to Deming [6], and Sterboul [18]. These graphs were studied in [3,12,13,15], and generalized in [2,17].

It is well-known that every bipartite graph is a König–Egerváry graph. This class includes non-bipartite graphs as well (see, for instance, the graphs H_1 and H_2 from Fig. 1).

It is easy to see that if G is a König–Egerváry graph, then $\alpha(G) \geq \mu(G)$, and that a graph G having a perfect matching is a König–Egerváry graph if and only if $\alpha(G) = \mu(G)$.

The number $d(G) = \max\{|S| - |N(S)| : S \in \mathrm{Ind}(G)\}$ is called the *critical difference* of G. An independent set A is *critical* if $|A| - |N(A)| = d(G)$, and the *critical independence number* $\alpha_c(G)$ is the cardinality of a maximum critical independent set

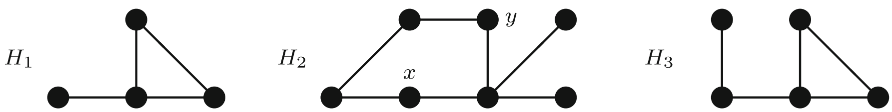

Fig. 1 $\lfloor |V(H_3)|/2 \rfloor + 1 < \alpha(H_3) + \mu(H_3) < |V(H_3)|$

[19]. Clearly, $\alpha_c(G) \leq \alpha(G)$ holds for any graph G. The problem of finding a critical independent set is polynomially solvable [1,19].

Proposition 1 ([8]) *If S is a critical independent set, then there exists a matching M from $N(S)$ into S, i.e., M matches each vertex belonging to $N(S)$ with some vertex of S.*

If S is an independent set of a graph G and $H = G - S$, then we write $G = S * H$. Evidently, every graph admits such representations.

Proposition 2 ([12]) *G is a König–Egerváry graph if and only if for every set $S \in \Omega(G)$, there is a decomposition $G = S * H$, such that $|S| \geq \mu(G) = |V(H)|$.*

Various characterizations of König–Egerváry graphs could be found in the following papers [6,7,14,18]. In [9] it was shown that G is a König–Egerváry graph if and only if $\alpha_c(G) = \alpha(G)$, thus providing a positive answer to the Graffiti.pc 329 conjecture [5].

In this paper we prove that the critical difference for a König–Egerváry graph G is given by

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G),$$

and using this finding, we show that G is a König–Egerváry graph if and only if each of its maximum independent sets is critical.

2 Results

Proposition 3 *Every critical independent set of a graph G is a local maximum independent set.*

Proof Suppose, to the contrary, that some critical independent set S of G is not a local maximum independent set. It means that there exists an independent set $A \subseteq N[S]$, larger than S. It follows that $|A \cap N(S)| > |S - S \cap A|$, which contradicts the fact that, according to Proposition 1, there is a matching from $A \cap N(S)$ into S, in fact, from $A \cap N(S)$ into $S - S \cap A$. □

Remark 1 The converse of Proposition 3 is not true; e.g., the set $\{x, y\}$ is a local maximum independent set of the graph H_2 from Fig. 1, but it is not critical.

Using Theorem 1 and Proposition 3, we easily deduce the following.

Corollary 1 ([4]) *Every critical independent set of a graph is contained in some maximum independent set.*

The following results will be used in the sequel.

Theorem 2 *If G is a König–Egerváry graph, then*

(i) [12] $N(\text{core}(G)) = \cap\{V(G) - S : S \in \Omega(G)\}$;
(ii) [13] $\alpha(G) + |\cap\{V(G) - S : S \in \Omega(G)\}| = \mu(G) + |\cap\{S : S \in \Omega(G)\}|$;
(iii) [13] $G - N[core(G)]$ *is a König–Egerváry graph with a perfect matching.*

Remark 2 Let us notice that for non-König–Egerváry graphs every relation between

$$\alpha(G) - \mu(G) \text{ and } |\text{core}(G)| - |N(\text{core}(G))|$$

is possible. For example, the graphs from Fig. 2 satisfy:

$$\alpha(G_1) - \mu(G_1) = 1 = |\text{core}(G_1)| - |N(\text{core}(G_1))|$$

and

$$\alpha(G_2) - \mu(G_2) = 1 < 2 = |\text{core}(G_2)| - |N(\text{core}(G_2))| .$$

The opposite direction of the above inequality may be seen in $G_3 = K_{2n} - e, n \geq 3$:

$$\alpha(G_3) - \mu(G_3) = 2 - n > 2 - (2n - 2) = |\text{core}(G_3)| - |N(\text{core}(G_3))| .$$

Theorem 3 *If G is a König–Egerváry graph, then the following equalities hold*

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G).$$

Proof Firstly, let us prove that

$$\alpha(G) - \mu(G) \geq |S| - |N(S)|$$

is true for every $S \in \text{Ind}(G)$, i.e., $d(G) \leq \alpha(G) - \mu(G)$. If $\alpha(G) = \mu(G)$, then G has a perfect matching, and

$$|S| - |N(S)| \leq 0 = \alpha(G) - \mu(G)$$

holds for every $S \in \text{Ind}(G)$.

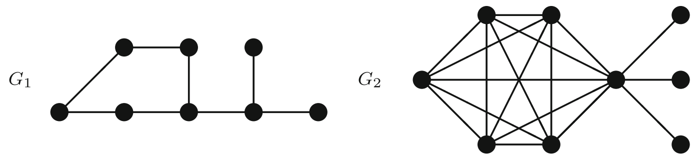

Fig. 2 Non-König–Egerváry graphs

Suppose that $\alpha(G) > \mu(G)$. Let $S_0 \in \Omega(G)$ and let M be a maximum matching in G. By Proposition 2, it follows that $|M| = |V(G) - S_0| = \mu(G)$. Assume that $S \in \mathrm{Ind}(G)$ satisfies $|S| - |N(S)| > 0$. Then one can write $S = S_1 \cup S_2 \cup S_3$, where

$$S_3 \subseteq V(G) - S_0, S_1 \cup S_2 \subset S_0, S_1 \cap S_2 = \emptyset,$$

and S_2 contains every $v \in S$ matched by M with some vertex of $V(G) - S_0$. Since M is a maximum matching, we obtain that $|S_2| - |N(S_2)| \leq 0$ and $|S_3| - |N(S_3)| \leq 0$. Consequently, we infer that

$$\alpha(G) - \mu(G) = |S_0| - |V(G) - S_0| \geq |S_1| \geq |S| - |N(S)|,$$

as required.

In Fig. 3 are illustrated various examples of $S = S_1 \cup S_2 \cup S_3$ with respect to the matching $M = \{y_1x_4, y_2x_5, y_3x_6, y_4x_7, y_5x_8\}$ and $S_0 = \{x_i : 1 \leq i \leq 8\}$: $S_2 = \{x_5\}$, $S_3 = \{y_4, y_5\}$, while S_1 belongs to $\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_3\}\}$.

The fact that $\mathrm{core}(G)$ is an independent set of G ensures that

$$\alpha(G) - \mu(G) \geq |\mathrm{core}(G)| - |N(\mathrm{core}(G))|.$$

Since G is a König–Egerváry graph, we get that

$$\alpha(G) + \mu(G) = |V(G)| = |\mathrm{core}(G)| + |N(\mathrm{core}(G))| + |V(G - N[\mathrm{core}(G)])|.$$

Assuming that $\alpha(G) - \mu(G) > |\mathrm{core}(G)| - |N(\mathrm{core}(G))|$, we obtain the following contradiction

$$\begin{aligned} 2\alpha(G) &> 2\,|\mathrm{core}(G)| + |V(G - N[\mathrm{core}(G)])| = \\ &= 2\,|\mathrm{core}(G)| + 2\alpha(G - N[\mathrm{core}(G)]) = 2\alpha(G), \end{aligned}$$

because $|V(G - N[\mathrm{core}(G)])| = 2\alpha\,(G - N[\mathrm{core}(G)])$, by Theorem 2(iii).

Therefore, we get that

$$\alpha(G) - \mu(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))|\,.$$

Actually, this equality immediately follows from Theorem 2(i), (ii), but the current way of proof exploits different aspects of $\mathrm{Ind}(G)$.

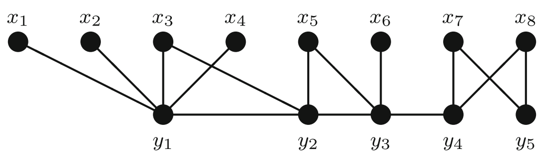

Fig. 3 Proof of Theorem 3

Further, using the inequality $d(G) \leq \alpha(G) - \mu(G)$ and the equality

$$\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|,$$

we finally deduce that

$$|\text{core}(G)| - |N(\text{core}(G))| \leq \max\{|S| - |N(S)| : S \in \text{Ind}(G)\} = \\ = d(G) \leq \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|,$$

i.e., $\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))| = d\,(G)$.

Since G is a König–Egerváry graph, we infer that

$$\alpha(G) - \mu(G) = \alpha(G) + \mu(G) - 2\mu(G) = |V\,(G)| - 2\mu(G) = \mathit{def}(G),$$

and this completes the proof. □

Remark 3 There exist non-König–Egerváry graphs satisfying the equalities

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G),$$

see, for instance, the graph G_1 from Fig. 2.

Corollary 2 *If G is a König–Egerváry graph, then $d(G) = 0$ if and only if G has a perfect matching.*

Remark 4 The graph $G = G_1 - h$, where G_1 is depicted in Fig. 2, is an example of a non-König–Egerváry graph G satisfying $d(G) = 0$ but without perfect matchings.

Theorem 4 *For a graph G, the following assertions are equivalent:*

(i) *G is a König–Egerváry graph;*
(ii) *there is some $S \in \Omega(G)$, such that S is critical, i.e., $\alpha_c(G) = \alpha(G)$;*
(iii) *every $S \in \Omega(G)$ is critical.*

Proof (i) $\Longrightarrow$ (iii) Let $S \in \Omega(G)$, and let us denote

$$A = S - \text{core}(G) \text{ and } B = V\,(G) - S - N(\text{core}(G)).$$

By Theorem 2(iii), we infer that $|A| = |B|$, since $G - N[\text{core}(G)]$ has a perfect matching. Hence, we obtain that

$$|S| - |N(S)| = |A| + |\text{core}(G)| - (|B| + |N(\text{core}(G))| \\ = |\text{core}(G)| - |N(\text{core}(G))|.$$

In other words, according to Theorem 3, the equality $|S| - |N(S)| = d(G)$ is true for every $S \in \Omega(G)$.

(iii) $\Longrightarrow$ (ii) It is clear.

(ii) $\Longrightarrow$ (i) This was done in [9]. For the sake of completeness we add the proof.

There is a critical independent set S such that $|S| = \alpha_c(G) = \alpha(G)$. In accordance with Proposition 1, there exists a matching M from $N(S)$ into S, and clearly, $|M| = |N(S)| = \mu(G)$. Hence, we finally obtain that

$$|V(G)| = |S| + |N(S)| = \alpha(G) + \mu(G),$$

i.e., G is a König–Egerváry graph. $\square$

3 Conclusions

In this paper we have given a new characterization of König–Egerváry graphs. On the one hand, it is similar in form to Sterboul's theorem [18]. On the other hand, it extends Larson's finding [9]. We found that the critical difference of a König–Egerváry graph G is given by

$$d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G) = def(G).$$

Clearly, $\alpha(G) - \mu(G) = def(G)$ is equivalent to $\alpha(G) + \mu(G) = |V(G)|$, i.e., G is a König–Egerváry graph. It seems interesting to find other families of graphs satisfying $d(G) = |\mathrm{core}(G)| - |N(\mathrm{core}(G))| = \alpha(G) - \mu(G)$.

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