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Strengthening Chvátal–Gomory cuts and Gomory fractional cuts ☆

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Abstract

Chvátal–Gomory and *Gomory fractional* cuts are well-known cutting planes for pure integer programming problems. Various methods for strengthening them are known, for example based on subadditive functions or disjunctive techniques. We present a new and surprisingly simple strengthening procedure, discuss its properties, and present some computational results. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The use of valid linear inequalities as cutting planes is now a well-established, and powerful, tool in integer programming and combinatorial optimization (see for example [16,1]). Broadly speaking, valid inequalities are of two types. There are those which are tailored to particular problems, typically derived via polyhedral techniques; and there are those which are (more or less) general purpose, derived by algorithmic or algebraic techniques. In this second class we find, for example, the *fractional* and *mixed-integer* cuts of Gomory [10,11], the *Chvátal–Gomory cuts* of Chvátal [7], the *disjunctive cuts* of Balas [2] and others, and the *lift-and-project cuts* of Lovász and Schrijver [15] and Balas et al. [4].

This paper is concerned with a new and surprisingly simple technique for *strengthening* the Chvátal–Gomory cuts (and the closely-related Gomory fractional cuts). In Section 2 we define these cuts and review some of the known procedures for strengthening them. In Section 3, we give the new procedure. Computational results given in Section 4 illustrate the improvements which can be gained by using the new inequalities in place of the original ones.

Throughout the paper the following notation is used. An *Integer Linear Program* (ILP) with n variables and m constraints is a problem of the form

$$\min\{c^{\mathrm{T}}x\colon Ax \leqslant b,\ x \in Z_+^n\}, \tag{1}$$

where $A \in Z^{m\times n}$, $b \in Z^m$ and $c \in \mathbb{R}^n$. Associated with any ILP are the two polyhedra

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- $P := \{x \in R^n_+ : Ax \leqslant b\}$ (the feasible region of the linear programming relaxation of the problem), and
- $P_I := \text{conv}\{x \in Z^n_+ : Ax \leqslant b\}$ (the convex hull of feasible integer solutions).

For any $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the *lower integer part* of r, that is, the largest integer not exceeding r. Also, $f(r)$ denotes $r - \lfloor r \rfloor$, the so-called *fractional part* of r. We also apply the same operators to vectors (component-wise).

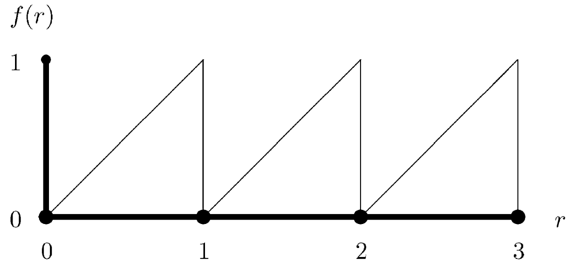

Fig. 1. The subadditive function $f(r)$.

2. Known results

2.1. Gomory fractional cuts and variants

The original cutting plane algorithm, due to Gomory [10], works as follows. Insert slack variables into the ILP (1) to obtain a system of the form

$$\min\{c^T x : (A, I)x = b,\ x \in Z^{n+m}_+\} \tag{2}$$

and find a basic optimal solution x^* to the LP relaxation by the simplex method. If x^* is integral, it is also optimal. If not, pick a variable x_j such that x^*_j is fractional and consider the corresponding row of the simplex tableau:

$$x_j + \sum_{i \in Q} \alpha_i x_i = x^*_j, \tag{3}$$

where Q is the set of indices associated with the non-basic variables.

Proposition 1 (Gomory [10]). *The inequality*

$$\sum_{i \in Q} f(\alpha_i) x_i \geqslant f(x^*_j) \tag{4}$$

is violated by x^*, *yet satisfied by all non-negative integer solutions to* (3), *and therefore by all solutions to* (2).

The cutting plane (4), which has come to be known as a *Gomory fractional cut* [16], can then be added to the LP relaxation.

In practice, Gomory fractional cuts tend to be rather weak. However, there are several methods in the literature for *strengthening* them. The first two are in fact due to Gomory himself:

Proposition 2 (Gomory [10]). *For any integer* t, *the cut*

$$\sum_{i \in Q} f(t\alpha_i) x_i \geqslant f(t x^*_j) \tag{5}$$

is satisfied by all non-negative integer solutions to (3), *and therefore by all solutions to* (2). *Moreover, if* $f(x^*_j) < \frac{1}{2}$, t *is positive and* $\frac{1}{2} \leqslant t f(x^*_j) < 1$, *then the cut* (5) *dominates the original fractional cut* (4).

Proposition 3 (Gomory [11]). *The cut*

$$\sum_{i \in Q} \min\left\{ f(\alpha_i), f(x^*_j)\frac{(1 - f(\alpha_i))}{(1 - f(x^*_j))} \right\} x_i \geqslant f(x^*_j) \tag{6}$$

is valid and stronger than the original cut (4).

The inequality (6) is a special case of the *Gomory mixed-integer cut*. As the name suggests, it can be generalized to mixed-integer linear programs.

Note that both of these procedures can be applied consecutively. That is, for any t, the cut:

$$\sum_{i \in Q} \min\left\{ f(t\alpha_i), f(t x^*_j)\frac{(1 - f(t\alpha_i))}{(1 - f(t x^*_j))} \right\} x_i \geqslant f(t x^*_j) \tag{7}$$

is valid and stronger than the cut (5).

In 1972, Gomory and Johnson [12,13] made an interesting series of observations. Consider the function $f(\cdot)$, displayed in Fig. 1, mapping the coefficients in (3) onto the coefficients in (4). This function is *subadditive*, i.e., for any $r_1, r_2 \in \mathbb{R}$, $f(r_1 + r_2) \leqslant f(r_1) + f(r_2)$. Moreover, the more complicated function mapping the coefficients in (3) onto the coefficients in (6), displayed in Fig. 2 for the case $x^*_j = \frac{2}{3}$, is also

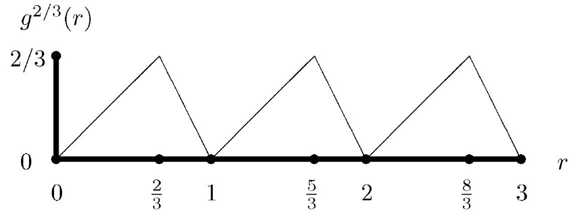

Fig. 2. The subadditive function $g^{2/3}(r) := \min\{f(r), 2(1-f(r))\}$, obtained when $x_j^* = \frac{2}{3}$.

subadditive. This led Gomory and Johnson to derive the following result:

Proposition 4 (Gomory and Johnson [12,13]). *Let $g(\cdot)$ be any subadditive function with domain $[0,1)$ such that*:

- $g(0) = 0$,
- $g(x_j^*) = f(x_j^*)$, *and*
- $g(r+1) = g(r)$ *for all* $r \in \mathbb{R}$.

Then the cut

$$\sum_{i \in Q} g(x_i^*) x_i \geqslant f(x_j^*) \tag{8}$$

is satisfied by all non-negative integer solutions to (3). *Moreover, the non-dominated cuts of this type are obtained when $g(r) + g(1-r) = 1$ for all r.*

In fact, Gomory and Johnson showed the following much stronger result:

Proposition 5 (Gomory and Johnson [12,13]). *Any non-dominated valid inequality which is implied only by the Eq.* (3) *and the fact that the variables are non-negative integers can be derived by a subadditive function of the type mentioned in Proposition* 4.

The catch of course is that constructing the required subadditive function is difficult in general. However, procedures for constructing some useful *continuous* subadditive functions of this type can be found in [12,13].

Stronger cuts can of course be obtained by using more of the information in the ILP. Burdett and Johnson [5] obtain such cuts by using subadditive functions of more than one variable. See also Balas [2] and Balas and Jeroslow [3], where stronger cuts are obtained using disjunctive methods.

2.2. *Chvátal–Gomory cuts and variants*

In the previous subsection, we relied on the addition of slack variables to the ILP. In fact the slack variables are not needed and it is possible to work with the system $Ax \leqslant b$ directly. This was first noticed by Chvátal [7] in 1973.

Proposition 6 (Chvátal [7]). *Given an ILP of the form* (1) *and any vector $\lambda \in \mathbb{R}_+^m$, the inequality*

$$\lfloor \lambda^{\mathrm{T}} A \rfloor x \leqslant \lfloor \lambda^{\mathrm{T}} b \rfloor \tag{9}$$

is valid for P_{I} (*though in general not for P*).

It is well-known (see, e.g. [16]) that any fractional cut of the form (4) is equivalent to a cut of the form (9). The reverse is also true (with some qualifications—see Cornuéjols and Li [8]).

For this reason the cuts (9) have come to be known as *Chvátal–Gomory cuts*, or *CG cuts* for short.

The components of λ are called *CG multipliers*. As noted in [7], one can assume without loss of generality that all CG multipliers are rational and less than 1. Moreover, it is easy to show that any CG cut can be generated using a λ which has at most $\min\{m,n\}$ strictly positive coefficients. This implies that the coefficients in a CG cut will be 'reasonably small' integers whenever the coefficients in A and b are 'reasonably small'.

CG cuts may induce facets of P_{I} in certain cases (see for example Caprara et al. [6]). However, just as with the fractional cuts, in general they are weak and it is desirable to strengthen them. The known strengthening procedures are essentially analogues of the above-mentioned procedures for strengthening the fractional cuts. For example, the analogue of Proposition 2 is:

Proposition 7 (Chvátal [7]). *If $f(\lambda^{\mathrm{T}} b) < \frac{1}{2}$ and t is any positive integer such that $\frac{1}{2} \leqslant t\, f(\lambda^{\mathrm{T}} b) < 1$, then replacing λ by $f(t\lambda)$ yields a stronger* (*or equivalent*) *CG cut.*

To state the equivalent of Proposition 3, it is helpful to introduce some further notation. Let $N = \{1, \ldots, n\}$

and $a_0 = \lambda^{\mathrm{T}} b$ and, for $i \in N$, let a_i equal the ith component of the row vector $\lambda^{\mathrm{T}} A$. Then, the obvious valid inequality $\sum_{i \in N} (\lambda^{\mathrm{T}} A) x_i \leqslant \lambda^{\mathrm{T}} b$ can be written as

$$\sum_{i \in N} a_i x_i \leqslant a_0 \tag{10}$$

and the CG cut (9) can be written as

$$\sum_{i \in N} \lfloor a_i \rfloor x_i \leqslant \lfloor a_0 \rfloor. \tag{11}$$

Using this notation, the analogue of Proposition 3 is:

Proposition 8 (Burdett and Johnson [5]). *The inequality*

$$\sum_{i \in N} \left(\lfloor a_i \rfloor + \max\left\{0, \frac{f(a_i) - f(a_0)}{1 - f(a_0)}\right\}\right) x_i \leqslant \lfloor a_0 \rfloor \tag{12}$$

is valid and dominates the CG cut (11).

Again, both of the procedures can be applied together; that is, one may replace λ by $f(t\lambda)$ and then generate the strengthened inequality of the form (12).

Finally, note that the floor function $\lfloor \cdot \rfloor$ is *superadditive*. That is, for any $r_1, r_2 \in \mathbb{R}$, $\lfloor r_1 \rfloor + \lfloor r_2 \rfloor \leqslant \lfloor r_1 + r_2 \rfloor$. The same is true for the function mapping coefficients in (10) onto coefficients in (12). The equivalent of Propositions 4 and 5 are, respectively:

Proposition 9 (Burdett and Johnson [5]). *Let $g(\cdot)$ be any non-decreasing superadditive function such that $g(0) = 0$. The cut*

$$\sum_{i \in Q} g(a_i) x_i \leqslant g(a_0) \tag{13}$$

is satisfied by all non-negative integer solutions to (10).

Proposition 10 (Burdett and Johnson [5]). *Any non-dominated valid inequality which is implied only by the inequality* (10) *and the fact that the variables are non-negative integers can be derived by a superadditive function of the type mentioned in Proposition* 9.

3. A new strengthening procedure

In this section, we present a new strengthening procedure. First we express it in terms of CG cuts.

Theorem 1. *Consider the inequality* (10) *once more. Suppose that $f(a_0) > 0$ and let $k \geqslant 1$ be the unique integer such that $1/(k+1) \leqslant f(a_0) < 1/k$. Partition N into classes $N_0, \ldots, N_k$ as follows. Let $N_0 = \{i \in N\colon f(a_i) \leqslant f(a_0)\}$ and, for $p = 1, \ldots k$, let $N_p = \{i \in N\colon f(a_0) + (p-1)(1 - f(a_0))/k < f(a_i) \leqslant f(a_0) + p(1 - f(a_0))/k\}$. The inequality*

$$\sum_{i \in N_0} (k+1) \lfloor a_i \rfloor x_i + \sum_{p=1}^{k} \sum_{i \in N_p} ((k+1) \lfloor a_i \rfloor + p) x_i \leqslant (k+1) \lfloor a_0 \rfloor \tag{14}$$

is valid for P_{I} and dominates the CG cut (11).

Proof. To show dominance, simply divide (14) by $k+1$ and compare it with (11). To prove validity, multiply the valid inequality (10) by k and apply integer rounding to obtain the CG cut

$$\sum_{p=0}^{k-1} \sum_{i \in N: p/k \leqslant f(a_i) < (p+1)/k} (k \lfloor a_i \rfloor + p) x_i \leqslant k \lfloor a_0 \rfloor. \tag{15}$$

Then let $\varepsilon > 0$ be a small real number such that $1 - \varepsilon \geqslant f(a_0)/f(a_i)$ for all $i \in N \setminus N_0$. It is easy to show that $(1-\varepsilon)/f(a_0)$ and $1 + (1 - (1-\varepsilon)/f(a_0))/k$ are non-negative. Thus, we can multiply (10) by $(1-\varepsilon)/f(a_0)$ and multiply (15) by $1 + (1 - (1-\varepsilon)/f(a_0))/k$ and sum the two resulting inequalities together to obtain the valid inequality

$$\sum_{p=0}^{k-1} \sum_{i \in N: p/k \leqslant f(a_i) < (p+1)/k} \left((k+1) \lfloor a_i \rfloor + p + \frac{(1-\varepsilon)(f(a_i) - p/k)}{f(a_0)} + p/k \right) x_i \leqslant (k+1) \lfloor a_0 \rfloor + (1 - \varepsilon). \tag{16}$$

Now, it is easy to check that the term $((1-\varepsilon)(f(a_i) - p/k))/f(a_0) + p/k$ on the left-hand side of (16) is non-negative but less than 2 and that it exceeds 1 if and only if $f(a_i) > f(a_0) + p(1 - f(a_0))/k$. Applying integer rounding to (16) (i.e., replacing each coefficient by its lower integer part) and rearranging gives (14). □

We will call an inequality of the form (14) a *strong* CG cut.

Example 1. Let $P=\{x\in\mathbb{R}^2_+\colon 6x_1+4x_2\leqslant 9\}$ and $\lambda=\{1/6\}$. The CG cut for $P_{\rm I}$ is $x_1\leqslant 1$ and the cut (12) is $x_1+\frac{1}{3}x_2\leqslant 1$. However, in Theorem 1 we have $k=1$, $N_0=\{1\}$ and $N_1=\{2\}$. Thus we obtain the strong CG cut $2x_1+x_2\leqslant 2$.

Note that, in this example, the strong CG cut dominates the cut (12) as well as the ordinary CG cut. In general, however, there is no dominance relationship between the cuts (12) and the strong CG cuts.

One advantage of strong CG cuts over cuts of the form (12) is that the coefficients are integral. Moreover, provided that k is not 'too large', the coefficients will be 'reasonably small' integers. If k *is* too large (because $f(a_0)$ is close to 0), the coefficients in (14) may be very large. In such a case, it is better to strengthen (11) in two steps: in step (i), apply the strengthening procedure mentioned in Proposition 7 to produce a CG cut with $f(a_0)\geqslant\frac{1}{2}$. Then, in step (ii), produce the strong version of this second CG cut. Now k will be equal to 1, so that the strong CG cut will have small coefficients.

This 'two step' procedure is illustrated in the following example.

Example 2. Let $P=\{x\in\mathbb{R}^2_+\colon 6x_1+4x_2\leqslant 9\}$ as before, but let $\lambda=\{4/7\}$. The CG cut for $P_{\rm I}$ is $3x_1+2x_2\leqslant 5$, the cut (12) is $3\frac{2}{21}x_1+2\frac{5}{42}x_2\leqslant 5$ and the strong CG cut is $19x_1+13x_2\leqslant 35$. The strong CG cut has large coefficients because $f(a_0)=\frac{1}{7}$ and $k=6$. A suitable scaling parameter t is 4, because $f(ta_0)=\frac{4}{7}\geqslant\frac{1}{2}$. Thus, we change λ to $f(4\times\frac{4}{7})=f(\frac{16}{7})=\frac{2}{7}$ and step (i) of the strengthening procedure yields the CG cut $x_1+x_2\leqslant 2$. In step (ii) we generate the corresponding strong CG cut (with $k=1$), $3x_1+2x_2\leqslant 4$. This has small integer coefficients and is stronger than any of the other cuts mentioned in this example.

Next, we make a comment about *Chvátal rank*. The proof of Theorem 1 shows that strong CG cuts can be obtained from the original formulation by applying the integer rounding procedure at most twice. In the terminology of [7], this shows that the *Chvátal rank* of any given strong CG cut is at most 2. In fact, we now show that the Chvátal rank of the strong CG cut obtained in Example 1 is *exactly* 2.

Proposition. *In Example* 1, *the strong CG cut* $2x_1+x_2\leqslant 2$ *has Chvátal rank* 2.

Proof. Hartmann et al. [14] showed that an inequality with integral coefficients, valid for $P_{\rm I}$, has Chvátal rank >1 if the following conditions hold:

- $P_{\rm I}$ is full-dimensional,
- The inequality induces a facet of $P_{\rm I}$,
- The left-hand side coefficients in the inequality have no common factors,
- The optimal objective value of a certain linear programming problem exceeds the right-hand side of the inequality by at least 1.

It is easy to check that the first three conditions are satisfied in the example. As for the fourth condition, the LP in question is

$$\max\{2x_1+x_2\colon 6x_1+4x_2\leqslant 9,\ x_1\geqslant 0,\ x_2\geqslant 0\}.$$

The optimal solution is $x_1=\frac{3}{2}$, $x_2=0$, with an objective value of 3. Since the right-hand side of the inequality is 2, the fourth condition is also satisfied. Therefore the Chvátal rank of the inequality is >1. But we already know that the Chvátal rank of the inequality is not >2. □

When the conditions mentioned in the proof of the above proposition do not hold, one can no longer determine the Chvátal rank by solving a linear program. Indeed, it was recently shown by Eisenbrand [9] that testing if an inequality has Chvátal rank one is strongly $\mathcal{NP}$-complete.

We can also apply the strengthened integer rounding argument of Theorem 1 to derive a new strengthened version of the *fractional cut* (4). This is shown explicitly in the following theorem.

Theorem 2. *Suppose that* $f(x_j^*)>0$ *and let* $k\geqslant 1$ *be the unique integer such that* $1/(k+1)\leqslant f(x_j^*)<1/k$. *Partition* Q *into classes* $Q_0,\dots,Q_k$ *as follows. Let* $Q_0=\{i\in N\colon f(\alpha_i)\leqslant f(x_j^*)\}$ *and, for* $p=1,\dots,k$, *let* $Q_p=\{i\in Q\colon f(x_j^*)+(p-1)(1-x_j^*)/k<f(\alpha_i)\leqslant x_j^*+p(1-f(x_j^*))/k\}$. *The following* strong fractional cut

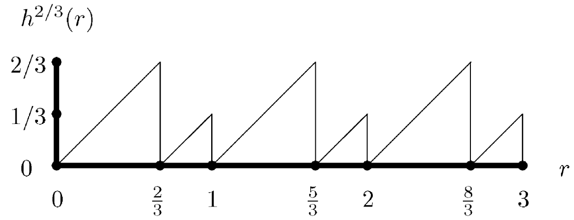

Fig. 3. The subadditive function $h^{2/3}(r)$ leading to the strong fractional cut in the case where $x_j^* = \frac{2}{3}$.

is valid for $\mathscr{P}_{\mathscr{I}}$:

$$\sum_{i \in Q_0} f(\alpha_i)x_i + \sum_{p=1}^{k} \sum_{i \in Q_p} (f(\alpha_i) - p/(k+1))x_i \geqslant f(x_j^*). \quad (17)$$

Proof. Relax the Eq. (3) to an inequality of '$\leqslant$' form and apply the strengthened integer rounding procedure of Theorem 1, to obtain

$$(k+1)x_j + \sum_{p=1}^{k} \sum_{i \in Q_p} ((k+1)\lfloor \alpha_i \rfloor + p)x_i \leqslant (k+1)\lfloor x_j^* \rfloor. \quad (18)$$

Divide (18) by $k+1$, and subtract it from (3) to yield (17). □

It is easy to show that the strong fractional cut (17) dominates the ordinary fractional cut (4) but that there is no dominance relation in general between the strong fractional cut and the mixed-integer cut (6).

Moreover, consider the function mapping the coefficients of (3) onto the coefficients in the strong fractional cut (17). This function, displayed in Fig. 3 for the case where $x_j^* = \frac{2}{3}$, can be shown to be subadditive. It also meets the other conditions of Proposition 4. However, it differs from the subadditive functions given in [12,13] in that it is *discontinuous*.

4. Computational experiments

In this section, we report the results of some computational experiments to see whether the above theoretical results might also be useful in practice. We implemented a cutting plane algorithm with Gomory cuts, using the CPLEX 7.0 callable library of ILOG. To avoid problems with rounding errors, we converted all Gomory cuts into CG cuts with integer coefficients, and a similar conversion was carried out with the other cuts. As mentioned in Section 3, the strong CG cuts can have large coefficients if k is large. Hence, we applied the following procedure: (i) the row of the simplex tableau corresponding to a fractional variable is multiplied by -1 so that the fractional part of the right-hand side becomes $> \frac{1}{2}$ by construction; then, (ii) the new strengthening operation can be applied with $k = 1$.

It is easy to see that step (i) corresponds to using $t = -1$ in Propositions 2 and 7, but note that there is no dominance relation between the cuts obtained from the original row and the modified one since t is *negative*. In other words, the above procedure is a heuristic version of the strengthening operation described in those propositions, and we illustrate its own effectiveness in Tables 1–3 by referring to it as 'strengthening 1'. Moreover, as mentioned, this is also used as step (i) of our new strengthening procedure, called 'strengthening 2' in the tables, and both procedure are compared within the same cutting plane framework with the original CG cuts ('original' in tables).

In particular, we tested the strengthening procedures on a set of 50 *multi-dimensional 0–1 knapsack problems*, i.e., problems of the form $\max\{c^{\mathrm{T}}x\colon Ax \leqslant b,\ x \in \{0,1\}^n\}$, where $c \in Z_+^n$, $A \in Z_+^{m\times n}$ and $b \in Z_+^m$, which were randomly generated as follows. For any pair (n,m) with $n \in \{5,10,15,20,25\}$ and $m \in \{5,10\}$, we constructed 5 random instances whose objective function coefficients are integers generated uniformly between 1 and 10. Moreover,

- For the instances with $m = 5$, the left-hand side coefficients are also integers generated uniformly between 1 and 10;
- For the instances with $m = 10$, the left-hand side coefficients have a 50% chance of being an integer generated uniformly between 1 and 10, but also have a 50% chance of being zero. That is, these instances are sparse.

In all cases the right-hand side of each constraint was set to half the sum of the left-hand side coefficients.

Table 1
Average %Gap of the cutting plane algorithms

m	n	Original			Strengthening 1			Strengthening 2		
		1 R.	10 R.s	25 R.s	1 R.	10 R.s	25 R.s	1 R.	10 R.s	25 R.s
		%Gap	%Gap	%Gap	%Gap	%Gap	%Gap	%Gap	%Gap	%Gap
	5	42.85	95.26	100.00	54.08	100.00	100.00	77.24	100.00	100.00
	10	28.12	79.54	91.86	26.66	80.34	95.66	35.72	86.71	96.15
5	15	16.04	70.20	81.04	21.20	75.25	82.98	25.15	80.52	85.84
	20	24.22	58.29	63.41	25.93	73.53	88.77	36.34	76.37	86.66
	25	13.47	62.64	73.80	20.39	75.68	81.82	32.33	79.35	83.12
	5	29.19	100.00	100.00	31.56	99.34	99.34	37.84	100.00	100.00
	10	15.95	55.82	67.33	15.04	60.84	76.88	18.08	67.14	81.01
10	15	8.13	26.76	36.36	14.28	32.61	35.18	17.41	38.54	42.59
	20	11.77	30.56	32.15	11.94	32.14	38.58	14.84	37.61	44.07
	25	4.02	20.54	21.81	3.94	24.61	29.30	6.59	29.62	32.07

Table 2
Number of optimal solutions and branch-and-bound nodes

m	n	Original						Strengthening 1						Strengthening 2					
		1 R.		10 R.s		25 R.s		1 R.		10 R.s		25 R.s		1 R.		10 R.s		25 R.s	
		Opt	N	Opt	N	Opt	N	Opt	N	Opt	N	Opt	N	Opt	N	Opt	N	Opt	N
	5	0	0	4	0	5	0	1	0	5	0	5	0	1	0	5	0	5	0
	10	0	9	3	1	4	9	0	9	3	5	4	4	1	13	2	12	4	0
5	15	0	21	1	7	3	18	0	25	2	12	3	4	0	30	3	10	3	2
	20	0	30	1	64	2	59	0	34	2	32	3	35	0	27	3	27	3	31
	25	0	167	2	133	2	122	0	118	2	82	2	78	0	118	3	40	3	57
	5	0	0	5	0	5	0	0	0	4	0	4	0	0	0	5	0	5	0
	10	0	0	2	0	2	0	0	0	2	0	3	1	0	3	2	1	3	1
10	15	0	28	0	14	0	38	0	34	0	17	0	29	0	47	1	30	1	12
	20	0	19	0	11	0	22	0	33	0	13	0	29	0	38	0	31	0	33
	25	0	266	0	206	0	212	0	213	0	211	0	216	0	197	0	215	0	203
Total		0	540	18	440	23	485	1	467	20	377	24	401	2	474	24	371	27	344

Table 3
Cutting plane with just one cut at a time

m	n	Original			Strengthening 1			Strengthening 2		
		Opt	# cuts[a]	%Gap	Opt	# cuts[a]	%Gap	Opt	# cuts[a]	%Gap
5	5	20	674	88.86	21	148	93.89	21	153	95.52
10	5	10	836	57.93	18	389	77.80	18	294	82.08

[a]This is the overall number of cuts needed by the algorithms to solve the 20 (resp. 10) instances which can be solved in the case 'original'.

(This is well-known to lead to non-trivial instances of the multi-dimensional 0–1 knapsack problem.)

In Table 1 we report the average (for each pair (n,m)) percentage gap closed (%Gap) by three versions of a standard cutting plane algorithm exploiting the original, strengthened and strong CG cuts. In particular, these gaps are computed as

$$(first\_lp\_value - last\_lp\_value)/$$

$$(first\_lp\_value - opt\_value) \cdot 100,$$

where *last_lp_value* is the LP value after the addition of 1, 10 and 25 *rounds* of the corresponding CG cuts (columns 1 R., 10 R.s and 25 R.s, respectively). By a *round* of cuts, we mean (as in [4]) that one cutting plane is generated for each fractional variable, the cuts are added to the LP, and the LP is re-optimized.

The results shown in Table 1 are satisfactory: both strengthening operations prove to be effective in closing the integrality gap, and in particular the strong fractional cuts close a larger amount of this gap.

Table 2 completes the results given in Table 1 by reporting for each cutting plane version (i.e., type of cut and number of rounds) and each pair (n,m), the number of instances solved to optimality over 5 (Opt), and the overall number of nodes (N) required by the standard branch-and-bound of CPLEX 7.0 to obtain the optimal solution (when the cutting plane was not able to prove optimality). The last line of the table indicates the total number of both optimal solutions and branch-and-bound nodes.

In general, the use of strong fractional cuts results in fewer branch-and-bound nodes being needed, and the number of problems that are solved to optimality is always consistently larger. However, the number of required nodes can largely vary for the same problem (or pair (n,m)), and strange situations occur: a larger amount of nodes is needed when passing from 10 to 25 rounds in both 'original' and 'strengthening 1' cases.

The last set of experiments was performed by generating 20 additional instances with $n=5$, and testing the resulting 50 instances (25 with $n=m=5$ and 25 with $n=5,\ m=10$) by using the original cutting plane algorithm of Gomory, i.e., a cutting plane in which *just one* cut at a time is added. In particular, we select in all cases the CG cut with *smallest* absolute value of the right-hand side to keep control of the numerical problems.

In Table 3 we report for each cutting plane version (i.e., type of cuts) the number of instances solved to optimality (Opt) within a time limit of 15 CPU seconds on a Digital Alpha 533 MHz, the number of cuts needed (# cuts), and the average percentage gap (%Gap).

It can be clearly seen that the cutting plane algorithm using strong CG cuts can solve more instances with a smaller number of cuts and that, even when it fails to solve an instance, it closes a larger proportion of the integrality gap.

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