REGULAR PAPER

Sliding mode control of uncertain fractional-order systems: A reaching phase free approach

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Abstract
This paper proposes a sliding surface which renders the system dynamics to start directly from itself without a reaching phase. More specifically, the system dynamics is insensitive to matched disturbances/uncertainties throughout the entire system response. The controller design based on reduced-order subsystem is still preserved. It is different from integral sliding mode in which the design is based on the full order of the system to reach the same objective. The simulation results of its application to a fractional inverted pendulum system is demonstrated.

KEYWORDS
fractional-order system, reduced-order design, sliding mode control

1 | INTRODUCTION

The idea of fractional calculus was discussed for the first time over a letter from Leibniz to L'Hôpital in 1695. Fractional differential equations have been in use to model physical phenomena in the last couple of decades. The history of fractional-order calculus can be found in [1] [2]. The state space description is given in [3]. In [4], the authors instilled interest into the research community. New paths have been paved in the fractional calculus theory in [5]. Due to its wide advantages, in recent years, the study of fractional-order controllers has witnessed considerable interest [6,7]. The discussions on stability of fractional-order systems can be found in [8,9]. Some applications of fractional calculus have been given in [10–12].

The control under heavy uncertainties is one of the most challenging tasks. Sliding Mode Control (SMC) is one of the most efficient control strategies to deal with uncertainties [13]. Nowaday, it is used in control and observation of several classes of problems such as that related to power converters, vehicle motion control, etc.

The main objective of this class of controllers is to force the system states to stay in a predefined manifold (sliding surface) and maintain it there in spite of the presence of uncertainties in the system. Therefore, the sliding mode based design consists of two phases (i) Reaching Phase in which the system states are driven from the initial state to reach the sliding manifold in finite time and (ii) Sliding Phase in which the closed-loop system is induced into sliding motion. However, when the system reaches sliding phase, the consideration of robustness and order reduction come into picture which are the most important aspects of the sliding mode based design. It is worth noting that during the reaching phase, there is no guarantee of robustness [14]. In order to address robustness issue throughout the entire space, Integral Sliding Mode Control (ISMC) has been proposed in the SMC literature [14] but its design methodology has been based on full order of the system. However, the system exhibits a reduced-order dynamics.
2 | PRELIMINARIES

2.1 | Fractional-Order Calculus

Fractional-order integration and differentiation constitute the fractional calculus. They are generalization of their integer-order counterparts. The theorems and rules in fractional-order calculus are applicable to their integer-order counterparts in a more generalized representation but not always in a straightforward manner. Two of the most common definitions of fractional-order calculus are the R-L definition and Caputo definition which are inspired by the definition of Cauchy generalized $n \in \mathbb{N}−$fold integral of function by replacing the factorial function by the more generalized Gamma function $[5,22]$.

**Definition 1.** The $\alpha$th-order fractional integration of the function $f : (0, \infty) \to \mathbb{R}$ with respect to $t > 0$ and terminal value $t_0 > 0$ is given by

$$I_{t_0}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} f(\tau) (t-\tau)^{-(\alpha-1)} d\tau,$$

where $\Gamma : (0, \infty) \to \mathbb{R}$ is the Euler’s Gamma function:

$$\Gamma(\alpha) := \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx$$

**Definition 2.** The R-L definition of the $\alpha$th-order fractional derivative is given by:

$$D_{t_0}^{\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} f(\tau) (t-\tau)^{(m-\alpha-1)} d\tau,$$

where $m \in \mathbb{N}$ such that $m \geq [\alpha]$, where $[\alpha]$ is the smallest integer greater than or equal to $\alpha$ where $0 < \alpha < 1$.

**Definition 3.** The Caputo definition of the $\alpha$th-order fractional derivative of the $m$ times continuously differentiable function $f : (0, \infty) \to \mathbb{R}$ or $f \in C^m((0, \infty), \mathbb{R})$ is given by:

$$C_{t_0}^{\alpha} D_{t}^m f(t) := \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} f^{(m)}(\tau) (\tau-t)^{-(\alpha-1)} d\tau.$$
The additive index law (semigroup property)

\[ c_0^\alpha D_t^\beta D_t^\gamma f(t) = c_0^\alpha D_t^{\beta+\gamma} f(t), \]

holds for \( f(t) \in C^1[0, T] \) for some \( T > 0 \) where, \( \alpha, \beta, \gamma \in \mathbb{R}_+ \) and \( \alpha + \beta \leq 1 \) [24].

**Remark 1.** Caputo derivative and R-L are the two mostly used definitions in fractional calculus [22]. Since the initial value of fractional differential equation with Caputo derivative is the same as the initial value of integer differential equation, it is the most acceptable one. For example, the initial value of the fractional differential equation \( c_0^\alpha D_t^\gamma x(t) = f(t, x) \) with \( \alpha \in (0, 1), t > 0 \) is assumed as \( x(0) \equiv x_0 \).

However, for the same fractional differential equation with R-L in place of Caputo, \( \underline{0}_0^\alpha D_t^\gamma x(t) = f(t, x) \) with \( \alpha \in (0, 1), t > 0 \), the initial value of \( x(t) \) involves fractional integral (and/or derivative). Here, the initial condition is given as \( [\underline{0}_0^\alpha D_t^{\gamma-a} x(t)]_{t=0} = x_0 \).

On the other hand there is a limitation in case of Caputo definition. It is not able to capture the exact physical behavior of the system as illustrated in [25]. When the initial condition is non-zero, the system trajectories generated by Caputo definition differ from the actual ones. An account of physical and geometrical interpretations for initial condition value and fractional derivatives can be found in [25,26] and [27].

Given a control system, the first and the most important question is whether it is stable, because an unstable control system is typically useless and potentially dangerous. Qualitatively, a system is described as stable if by starting the system somewhere near its desired operating point, it will stay around the point ever after. The most useful and general approach for studying the stability of linear and nonlinear control systems is the theory introduced by Lyapunov. In the next subsection we are going to review the fractional extension of Lyapunov stability which has been recently proposed in [8,28].

### 2.2 Fractional Extension of Lyapunov Stability

Using Caputo definition, an \( n \)-dimensional fractional-order system can be defined as,

\[ c_0^\alpha D_t^\alpha x(t) = f(x, t); \forall t \geq t_0 \]  

where, \( \alpha \in (0, 1) \) and \( f(x, t) \) is locally bounded in \( x \) and piecewise continuous in \( t \) for all \( t \geq t_0 \) and \( x \in \mathbb{D} \), where \( \mathbb{D} \subset \mathbb{R}^n \) is a domain that contains the origin \( x = 0 \). For stability analysis of systems in (7), a fractional extension of Lyapunov’s direct method was proposed in [8] which is based on the following definition:

**Definition 4.** A continuous function \( \gamma : [0, t) \rightarrow [0, \infty) \) is a class-\( \mathcal{K} \) function if it is strictly increasing and \( \gamma(0) = 0 \).

**Theorem 1.** Let \( x = 0 \) be an equilibrium point for the non-autonomous fractional-order system i.e., \( f(x, t) = 0, \forall t \geq t_0 \). If there exists a Lyapunov function \( V(t, x(t)) : [t_0, \infty) \times \mathbb{D} \rightarrow \mathbb{R} \) and a class-\( \mathcal{K} \) function \( \gamma_i(i = 1, 2, 3) \) such that \( \gamma_1(||x||) \leq V(t, x(t)) \leq \gamma_2(||x||) \) and \( c_0^\alpha D_t^\gamma V(t, x(t)) \leq -\gamma_3(||x||) \) where, \( \alpha \in (0, 1) \) then, the system (7) is asymptotically stable.

**Theorem 2.** Let \( x \in \mathbb{R}^n \) be a continuously differentiable vector-valued function. Then, for any time instant \( t \geq t_0 \) and \( \forall \alpha \in (0, 1) \),

\[ \frac{1}{2} c_0^\alpha D_t^\gamma x^2(t) \leq x^2(t) c_0^\alpha D_t^\gamma x(t). \]  

It will be used in the later sections for the Lyapunov analysis of fractional-order systems with the proposed control input. Since this result was derived using Caputo derivatives, the same definition will be used throughout this paper unless mentioned otherwise.

### 3 Fractional-Order Sliding Mode Controller

Consider a controllable*commensurate fractional-order linear time-invariant system given by,

\[ c_0^\alpha D_t^\alpha \vec{x}(t) = \bar{A}\vec{x}(t) + \bar{B}(u(t) + d(t)) \]  

where, \( \vec{x}(\cdot) \in \mathbb{R}^n, \bar{A} \in \mathbb{R}^{nxn}, \bar{B} \in \mathbb{R}^{nxm}, u(\cdot) \in \mathbb{R}^m, \) \( d(\cdot) \in \mathbb{R}^m \) are pseudo states,\(^*\) system matrix, input matrix, control input, disturbance input respectively. It is assumed that exact evolution of disturbance with respect to time is not known but it is bounded.

There always exists an invertible non-singular matrix \( T \in \mathbb{R}^{nxn} \) such that using a linear transformation \( z(t) = T \vec{x}(t) \) (9) can be transformed into the regular form,

\(^*\)Controllability test for the commensurate fractional-order linear time-invariant system (LTI) is the same as for the integer-order LTI system [15].

\(^\dagger\)For a fractional order system, the knowledge of \( x(t_0) (t_0 \) being the initial time) is not sufficient to determine the future behavior of the system. Consequently, the collection of physical variables in a vector \( x \) does not strictly represent the state of the system. This is why the term, “Pseudo State” is coined in the literature in order to represent the physical variables of the fractional order systems [18,29]. In this manuscript same terminology has been used.
\[ c \frac{d}{dt} z_1(t) = A_{11} z_1(t) + A_{12} z_2(t) \]
\[ c \frac{d}{dt} z_2(t) = A_{21} z_1(t) + A_{22} z_2(t) + B_2 (u(t) + d(t)) \]  

(10)

where, \( z_1(t) = z(t) = z(t, z(t) \in \mathbb{R}^{n-m}, z_2(t) \in \mathbb{R}^m \).

Since the pair \((A, B)\) is controllable, the pair \((A_{11}, A_{12})\) will also be controllable. The above system of equations can be represented as,

\[ c \frac{d}{dt} z(t) = Az(t) + B(u(t) + d(t)) \]  

(11)

where, \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \).

Now, the main aim is to design SMC such that sliding takes place from the very initial time \( t \geq t_0 \) and \( z(t) \) approaches towards the origin as \( t \to \infty \) in the presence of bounded matched disturbance \( d(t) \). So, \( x(t) \) approaches asymptotically towards the origin as \( T \) is invertible. For the simplicity of presentation, it is assumed that \( u \in \mathbb{R} \) and \( B_2 \in \mathbb{R} \). However, similar results can be extended for the multi-input case in a straightforward manner. It is important to mention that the present work is based on the following assumption:

**Assumption 1.** For a non-smooth controller, the existence and uniqueness of solutions of the system are defined in the Filippov sense [15] i.e., letting \( x \) denote the pseudo states of the entire system \( c \frac{d}{dt} x(t) = f(x(t), t), a > 0 \), disturbance \( d \in \mathbb{R}^m \) and assuming \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) to be locally bounded, then the solutions are defined with the differential inclusion,

\[ c \frac{d}{dt} x(t) \in \bigcap_{\delta > 0} \bigcap_{\mu N = 0} c(\text{co}(\varepsilon_{B_0}(x) \setminus N)) \]

where, \( c \) and \( d \) denote the closure and the convex hull respectively. \( B_0(x) \) is the unit ball and the sets \( N \) are all sets of zero Lebesgue measure. Here, \( \delta \) represents the small ball around the discontinuity point on the state trajectories when the system is on the sliding surface. \( \mu N \) represents the factor by which the trajectories are scaled so that the deviation of the sliding variable from \( s = 0 \) is minimized.

In [15], the sliding surface has been designed using fractional reaching law and integer reaching law. The sliding surface for (10) using integer reaching law is,

\[ s(z, t) = t_i \frac{d^{\alpha-1}}{dt^{\alpha-1}} (c_1 z_1(t) + z_2(t)) \]  

where, \( s : \mathbb{R}^n \times (t_0, \infty) \to \mathbb{R}, c_1 \in \mathbb{R}^{1 \times (n-1)} \) and for,

\[ u(t) = B_2^{-1} (v - c_1 (A_{11} - A_{12} c_1) z_1(t)) + A_{12} t_i \frac{d^{\alpha-1}}{dt^{\alpha-1}} s \]
\[ - A_{21} z_1(t) - A_{22} z_2(t) \]

where, \( v = -k_1 \text{sign}(s) \), it has been proved in [15] that for \( s \) to be zero in finite time, \( k_1 > |B_2| |d| \).

In case of fractional reaching law,

\[ s(z, t) = c_1 z_1(t) + z_2(t) \]  

(13)

Using fractional derivative of \( s \) in (13), (10) becomes,

\[ c \frac{d}{dt} z_1(t) = (A_{11} - A_{12} c_1) z_1(t) + A_{12} \]
\[ c \frac{d}{dt} s = c_1 t_i \frac{d^{\alpha-1}}{dt^{\alpha-1}} s + c_2 \frac{d}{dt} z_2(t) \]

\[ = c_1 ((A_{11} - A_{12} c_1) z_1(t) + A_{12} s) \]
\[ + A_{21} z_1(t) + A_{22} z_2(t) + B_2 u(t) + B_2 d \]

Here, the control \( u(t) \) is chosen as, \( u(t) = B_2^{-1} (v - c_1 (A_{11} - A_{12} c_1) z_1(t) + A_{12} s + A_{21} z_1(t) - A_{22} z_2(t)) \) where, \( v = -k_1 \text{sign}(s) \). After applying the control, the closed-loop system becomes,

\[ c \frac{d}{dt} z_1(t) = (A_{11} - A_{12} c_1) z_1(t) + A_{12} s \]
\[ c \frac{d}{dt} s = -k_1 \text{sign}(s) + B_2 d \]  

(14)

Now, the following theorem is important:

**Theorem 3.** The sliding surface \( s \) in (13) becomes zero in finite time if \( k_1 > |B_2| |d| \).

**Proof.** The Lyapunov function is chosen as \( V = \frac{1}{2} s^2 \).

Then, \( c \frac{d}{dt} V = \frac{1}{2} c \frac{d}{dt} s^2 \). Using (8),

\[ c \frac{d}{dt} V \leq s^T \frac{d}{dt} s = s (-k_1 \text{sign}(s) + B_2 d) \]
\[ \leq -k_1 |s| + |s||B_2|d = |s|(k_1 - |B_2|d) \]
\[ = -(2\eta \frac{1}{(k_1 - |B_2|d)}) \]

where, \( k_1 = -k_1 \text{sign}(s) + B_2 d \). Taking fractional integral of order \( \alpha \) on both sides,

\[ \frac{d^\alpha}{dt^\alpha} s(t) = k_1 \frac{d^\alpha}{dt^\alpha} \text{sign}(s) + B_2 \frac{d^\alpha}{dt^\alpha} d \]  

(15)

Equation (15) becomes after finite time \( t = T \),

\[ s(T) - \frac{d^\alpha}{dt^\alpha} s(0) \frac{T^{\alpha-1}}{\Gamma(\alpha)} = -k_1 \text{sign}(s(0)) \frac{T^\alpha}{\Gamma(\alpha + 1)} \]
\[ + B_2 \frac{d^\alpha}{dt^\alpha} d \]

Multiplying with \( \text{sign}(s(0)) \) and using \( s(T) = 0 \),

\[ \frac{d^\alpha}{dt^\alpha} s(0) \frac{T^{\alpha-1}}{\Gamma(\alpha)} = -k_1 \frac{T^\alpha}{\Gamma(\alpha + 1)} \]
\[ + B_2 \frac{d^\alpha}{dt^\alpha} d \]

Using the inequality,

\[ \int t_i \frac{d^\alpha}{dt^\alpha} s(0) \frac{T^{\alpha-1}}{\Gamma(\alpha)} \leq t_i \frac{d^\alpha}{dt^\alpha} d \leq \frac{d_0}{\Gamma(\alpha + 1)} \]

Equation (16) becomes,

\[ -\frac{d^\alpha}{dt^\alpha} s(0) \frac{T^{\alpha-1}}{\Gamma(\alpha)} \leq -(k_1 - B_2 d_0) \frac{T^\alpha}{\Gamma(\alpha + 1)} \]
which further results into,

$$T \leq \frac{\Gamma(\alpha + 1)c\int_0^t s(0)\text{sign}(s(0))}{\Gamma(\alpha)(k_1 - B_2d_0)}$$

(16)

which is always finite.

Remark 2. It is clear that sliding mode has taken place after a finite time $t \geq T$ where, $T$ is such that $s(z,T) = 0$. Further, a modified sliding surface is proposed in which sliding mode starts from $t \geq t_0$ such that the reduced-order design methodology is preserved.

## 4 | MAIN RESULTS

Consider the same system as in Equation (10). The sliding surface for integer reaching law is,

$$s = c_i \int_t^1 \{ (c_1z_1(t) + z_2(t)) - (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \}$$

where $\lambda > 0$ and $c_i \in \mathbb{R}^{1 \times n-1}$ are the design parameters.

Note that $s = 0$ at initial time $t = t_0$. Then, the system (10) is transformed as,

$$\dot{c_i} D^\alpha_t z_1(t) = (A_{11} - A_{12}c_i)z_1(t) + A_{12} \left\{ c_i D^\alpha_t s + (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \right\}$$

$$\dot{s} = c_i D^\alpha_t \{ (c_1z_1(t) + z_2(t)) - (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \}$$

$$= c_i [(A_{11} - A_{12}c_i)z_1(t) + A_{12} \left\{ c_i D^\alpha_t s + (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \right\}]$$

The control input is selected as,

$$u(t) = B_2^{-1} [v - c_i \left\{ (A_{11} - A_{12}c_i)z_1(t) + A_{12} \times \left\{ c_i D^\alpha_t s + (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \right\} \right\} - B_2^{-1} (A_{21}z_1(t_0) + A_{22}z_2(t_0))$$

where, $v = -k_1 \text{sign}(s)$. Hence, $\dot{s} = -k_1 \text{sign}(s) + B_2d + \Xi$, where $\Xi = B_2^{-1} (-\lambda)^\alpha ((c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)})$. It is important to note that $|\Xi|$ is always bounded for any initial condition $z(t_0)$.

Further, it is proved that the trajectories remain on the sliding surface $s$ once they start from it at $t = t_0$ and then, asymptotically converge to $z_1(t) = z_2(t) = 0$.

**Lemma 1.** If $k_1 > |B_2d| + |\Xi|$, then the trajectories are maintained on the sliding surface $s = 0$, $\forall t \geq t_0$.

**Proof.** Consider the Lyapunov function, $V = \frac{1}{2}s^2$. By taking the time derivative of Lyapunov function along closed-loop subsystem $\dot{s} = -k_1 \text{sign}(s) + B_2d + \Xi$,

$$\dot{V} = \dot{s}s = s(-k_1 \text{sign}(s) + B_2d + \Xi)$$

$$\leq -k_1 |s| + sB_2d + s\Xi$$

$$\leq -(\lambda^\alpha)|s| + |B_2d| + |\Xi|$$

$$\leq -\eta(\lambda^\alpha)^{1/2}$$

where, $\eta = k_1 - |B_2d| - |\Xi| > 0$. Lyapunov stability theory ($V = 0$ and $\dot{V} \leq 0 \Rightarrow V = 0$, $\forall t \geq t_0$ implies $s = 0$, $\forall t \geq t_0$. This completes the proof.

The expression for the finite time, $T$ can be derived as follows:

$$\int_0^T \frac{dv}{dt} \leq -\eta\sqrt{2V(T)}$$

$$\int_0^T dt \leq -\int_0^V \frac{dV}{\eta\sqrt{2V}} = \frac{\sqrt{2V(0)}}{\eta}$$

**Lemma 2.** If the matrix $(A_{11} - A_{12}c_i)$ is negative definite, then the closed-loop system is asymptotically stable.

**Proof.** Take the Lyapunov function, $V = \frac{1}{2}z_1^2(t)$. Then,

$$\dot{c_i} D^\alpha_t V = c_i D^\alpha_t z_1^2(t)z_1(t)$$

Using (8),

$$\dot{c_i} D^\alpha_t V \leq z_1^2(t)c_i D^\alpha_t z_1(t) \leq z_1^2(t)(A_{11} - A_{12}c_i)z_1(t)$$

$$+ z_1^2(t)A_{12} \left\{ c_i D^\alpha_t s + (c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \right\}$$

$$\nabla$$

As $s = 0$ from time $t = t_0$, the term $(c_1z_1(t_0) + z_2(t_0)) e^{-j(t-t_0)} \rightarrow 0$ as $t \rightarrow \infty$, $z_1(t)$ and hence, the system is asymptotically stable if the matrix $(A_{11} - A_{12}c_i)$ is negative definite. This completes the proof.

**Remark 3.** It is important to note that if we select $v = -\lambda |s|^{1/2} \text{sign}(s) - \alpha \int_0^t \text{sign}(s)dt$, where $\alpha = 1.1\Delta$ and $\lambda = 1.5\sqrt{A}$ such that $B_2|d(t)| \leq \Delta$, where $\Delta$ is some a priori known constant, then the proposed control (17) generates continuous signal and it is also better for the chattering minimization problem, which is commonly encountered during the practical implementation of discontinuous control. The above suggested controller is known as Super-Twisting in the literature. Again,

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1For negative definiteness, the leading principle minors of $-(A_{11} - A_{12}c_i)$ should be positive i.e. the leading principle minors of $(A_{11} - A_{12}c_i)$ should have alternating signs, with the odd-numbered minors being negative and the even-numbered minors being positive [31].
the trajectories once start from the sliding surface, will remain there for the subsequent time (for more detailed explanation, see [30] and the references cited therein).

Now, using fractional reaching law approach, the sliding surface is defined as,

\[ s = c_1 \ddot{z}_1(t) + z_2(t) - (c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \] (18)

Note that \( s = 0 \) when \( t = t_0 \). Using (18), (10) becomes,

\[ \dot{\xi}_s D^\alpha s = \dot{\xi}_s D^\alpha z_1(t) + \xi_s D^\alpha z_2(t) \]

\[ - (c_1 \dot{z}_1(t_0) + z_2(t_0)) \dot{\xi}_s D^\alpha (z_1(t_0)) \]

\[ = c_1 [(A_{11} - A_{12} c_1) \xi_s + A_{12} (s + (c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)})] \]

\[ + A_{21} \xi_s + A_{22} z_2(t) + B_2(u(t) + d(t)) \]

\[ - (-\lambda)^\alpha ((c_1 \dot{z}_1(t_0) + z_2(t_0)))e^{-\lambda(t-t_0)} \] (19)

The control input is designed as,

\[ u(t) = B_2^{-1} \left[ v - c_1 [(A_{11} - A_{12} c_1) \xi_s + A_{12} (s + (c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)})] \right] \]

\[ - B_2^{-1} (A_{21} \xi_s + A_{22} z_2(t)) \]

where, \( v = -k_1 \text{sign}(s) \). From (19) and (20), \( \dot{\xi}_s D^\alpha s = -k_1 \text{sign}(s) + B_2 d + \Xi \) where, \( \Xi = B_2^{-1} (-\lambda)^\alpha ((c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)}) \). Again, the trajectories remain on the sliding surface \( s = 0 \) from the very initial time \( t = t_0 \), provided \( k_1 > |B_2||d| + |\Xi| \). This can be shown as follows:

Consider the Lyapunov function,

\[ V = \frac{1}{2} \xi_s^2 (t) \]

Taking the fractional derivative,

\[ \dot{\xi}_s D^\alpha V \leq \xi_s (t) \xi_s D^\alpha \xi_s(t) \leq \xi_s (t) (A_{11} - A_{12} c_1) \xi_s(t) \]

\[ + \xi_s (t) A_{12} \left[ s + (c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \right] \]

As \( s = 0 \) from time \( t = t_0 \), the term \( (c_1 \dot{z}_1(t_0) + z_2(t_0))e^{-\lambda(t-t_0)} \rightarrow 0 \) as \( t \rightarrow \infty \). Further, \( z_2(t) \) and hence the system is asymptotically stable if \( (A_{11} - A_{12} c_1) \) is negative definite.

5 | ILLUSTRATIVE EXAMPLE

A commensurate fractional-order uncertain system is considered to illustrate the theoretical results obtained in the paper. The example of a fractional inverted pendulum system is taken. In this system, an inverted pendulum is mounted on the top of a cart such that the pendulum is attached to an extension immersed in a viscoelastic solution [32]. The cart is able to move back and forth. The whole system can be represented by,

\[ \ddot{x} = \frac{1}{(m_c + m_p)} \left( \frac{1}{2} m_p l \ddot{\theta} \cos \theta - (\dot{\theta})^2 \sin \theta - f \dot{x} + F \right) \]

\[ \ddot{\theta} = \frac{1}{(J + \frac{1}{2} m_p l^2)} \left( \frac{1}{2} m_p l \ddot{x} \cos \theta + g \sin \theta + r \right) \]

where, \( x \) is the position of the cart, \( \theta \) is the angle of deflection of the pendulum, \( m_c \) is the mass of the cart, \( m_p \) is the mass of the pendulum, \( f \) is the friction coefficient of the cart, \( r \) is the applied torque, \( k \) is the damping coefficient of the viscoelastic solution, \( a \) is the derivation order of the damper, \( \omega_l \) and \( \omega_h \) are the lower and higher frequencies of the bandwidth of the fractional derivative. The state vector is chosen as,

\[ X = \begin{bmatrix} x \ d^\alpha x \ d^\alpha^2 x \ d^\alpha^3 x \ d^\alpha^4 x \ d^\alpha^5 x \ d^\alpha^6 x \ d^\alpha^7 x \end{bmatrix}^T \]

The above equations can be linearized about the equilibrium point of the system resulting in pseudo state-space form having commensurate order, \( a = 0.5 \),

\[ \frac{d^\alpha X(t)}{dt^a} = AX(t) + B(u(t) + d(t)) \] (21)

where,

\[ A = a \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ B = a \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.338 \end{bmatrix} \]

where, \( \alpha = \frac{1}{4(l_0 + m_c + m_p) l_0} \), \( a_{43} = -4 f - f m_p l_0 \), \( a_{45} = \frac{m_p^2 g}{4(l_0 + m_p) l_0} \), \( a_{46} = 2 m_p l_0 \), \( a_{47} = -k \omega_l \), \( a_{48} = -k \omega_h \), \( a_{49} = -\alpha \omega_l \), \( a_{49} = m_p^2 l_0 \), \( a_{48} = 2 m_p l_0 \), \( a_{49} = 2 m_p l_0 \), \( k = 0.1 \) N.m.sec\(^{-1} \), \( J = 0.065 \) kg.m\(^2 \), \( k = 0.1 \) N.m.sec\(^{-1} \), \( \omega_l = 0.1 \) rad.sec\(^{-1} \), \( \omega_h = 10 \) rad.sec\(^{-1} \), \( g = 9.81 \) m.sec\(^{-2} \), \( d(t) = 0.1 \) sin(t). The sliding surface is chosen as,

\[ s(t) = (|c_1 c_2 c_3 c_4 c_5 c_6 c_7 c_8| X(t)) \]

where, \( c_1 \) to \( c_8 \) are the gain values selected such that the reduced-order dynamics is stable. The controller parameter \( k_1 \) has to be selected such that \( k_1 > |B_2 d| + |\Xi| \). We know that \( |B_2 d| = 0.1 \) and \( |\Xi| \) is also small. Hence, we choose \( k_1 = 10 \) and \( \lambda = 0.4 \). The evolution of states, sliding surface and control input with time are shown in Figure 1, Figure 2 and Figure 3 respectively. In Figure 1, the state trajectories are shown to converge to the equilibrium point in finite time in the presence of matched disturbance/uncertainties. From Figure 2, it is clear that starting from \( t = 0 \), the trajectories are always maintained on the sliding surface.
FIGURE 1  Evolution of States $(x_1 \text{ to } x_9)$ w.r.t. time [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 2  Evolution of Sliding Surface $(s)$ w.r.t. time [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 3  Evolution of Control Input $(u)$ w.r.t. time [Color figure can be viewed at wileyonlinelibrary.com]
6 | CONCLUSION

The work presented in the paper proposes a new sliding surface based controller for uncertain fractional-order systems. Two different control schemes, one based on integer reaching law and the other on fractional reaching law have been used in order to maintain the trajectories on the sliding surface from the very initial time preserving robustness. The simplicity of the technique lies in the control design being based on the reduced order subsystem as in the case of classical sliding mode control. The effectiveness of the proposed approach is verified through numerical simulation for the case of a fractional inverted pendulum system.

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