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# Fixed-time state estimation for a class of switched nonlinear time-varying systems

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**Abstract**

This paper deals with the state estimation problem of a class of nonlinear time-varying systems with switched dynamics. Based on the concept of fixed-time stability, an observer is designed to reconstruct the continuous state of switched nonlinear time-varying systems with state jumps, satisfying the minimal dwell-time condition. Using the past input and output values of the studied system, some sufficient conditions are provided to estimate the state before the next switching. Some numerical results illustrate the effectiveness of the proposed scheme.

**KEYWORDS**

nonlinear time-varying systems, state estimation, switched systems

## 1 | INTRODUCTION

One solution to estimate the system state when some variables are not directly accessible by measurements is to use a real-time estimation algorithm, usually called an observer. Thanks to observers, one can estimate useful information on dynamical systems with the aim of monitoring, fault detection and feedback control design. Therefore, in control theory, the state estimation problem has become a fundamental one which has been addressed in many works. For instance, Luenberger observers [1] are traditional estimators which compute point estimates of the state from input-output data. Interval observers [2] which were proposed two decades ago, are another cutting-edge technique of guaranteed state estimation. They have been developed when upper and lower bounds of the initial state are known, see [3–5] and the references therein. Sliding mode observers have been proposed to estimate in finite time the state of the system using the concept of sliding surface and equivalent control [6–8]. Using the homogeneity properties of nonlinear systems, a finite-time observer has been designed in [9]. However, the mentioned works focus on asymptotic convergence, where the settling time is infinite or finite-time convergence, where the settling time depends on the initial states.

Switched systems are a class of hybrid systems exhibiting changes along the time among a finite number of possible dynamical behaviors. The problem of designing observers for switched systems has attracted an ever growing attention and has been acknowledged as an important topic of research (see e.g. [10–12] to name a few). The study has been analyzed depending on whether the switching signal is known or unknown. If it is known, we focus on estimating the state after a finite number of switchings [13,14]. If not, the observability of the state and of the switching signal has been shown to be mutually independent properties [15]. Besides, it should be noted that finite-time convergence is an interesting property, mainly for switched systems [16]. Indeed, the observation problem can be easily solved if the observer estimates the state before the next switching. However, using finite-time observers, the bound of the settling time depends on the initial states, which prevents us from an appropriate tuning of the observer gains. To solve this problem, the concept of fixed-time stability has been defined to investigate algorithms which ensure an upper bound of the settling time regardless of the initial conditions [17]. Finite-time stabilization for nonlinear discrete-time singular Markov jump systems was studied in [18]. Uniform robust exact differentiators were proposed in [19–22] based on a Lyapunov analysis or homogeneity properties. A fixed-time observer, with linear matrix inequalities for tuning the observer parameters, was introduced in [23] for linear systems. Based on uniform robust exact differentiators, a uniformly convergent sliding mode observer for switched linear systems was proposed in [24]. Recently, a fixed-time convergent observer was designed for a class of linearizable systems in [25]. Although the settling time estimate does not depend on the initial conditions of the system in many works, it cannot be easily tuned and it is very over-estimated. Nevertheless, to the best of our knowledge, the existing fixed-time observers cannot be applied to nonlinear time-varying systems which are affine in the unmeasured part of the state vector.

In this paper, based on fixed-time stability, we propose a new approach for the state estimation of a class of nonlinear time-varying systems which are affine in the unmeasured part of the state vector. Here, contrary to many existing works, the bound of the initial conditions is not assumed to be a priori known. Important results on the observer design for this class of systems in the time-invariant case have been proposed (see for instance [26,27]). Recently, interval observers have also been studied in [28–31]. However, to the best of our knowledge, no observer has been proposed for nonlinear time-varying systems which are affine in the unmeasured part of the state vector and with switched dynamics. Motivated by the work of [32–34] for linear and nonlinear time-invariant systems, the idea is to incorporate past input and output values of the studied system. Here, some sufficient conditions are provided to reconstruct the state of switched nonlinear time-varying systems with state jumps, satisfying the minimal dwell-time condition. Using the concept of fixed-time stability, the proposed estimator estimates the continuous state of the system before the next switching.

The paper is organized as follows. After recalling some basics on fixed-time stability, Section 2 introduces the state estimation problem of a class of nonlinear time-varying systems with switched dynamics. The proposed estimator is derived in Section 3. In Section 4, some numerical results illustrate the effectiveness of the proposed scheme.

## 2 | PROBLEM FORMULATION AND PRELIMINARIES

### 2.1 | Problem formulation

Let us consider the following class of nonlinear time-varying systems with switched dynamics:

$$\dot{x}(t) = \alpha_k(y,t)x(t) + \beta_k(y,t,u(y,t)),\ t \in [t_{k-1}, t_k), \tag{1a}$$

$$y(t) = C_k x(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^q$ is the known input and $y(t) \in \mathbb{R}^p$ is the output vector, $\alpha_k \in \mathbb{R}^{n\times n}$ and $\beta_k \in \mathbb{R}^n$ are continuous nonlinear functions. The index $k \in \{1; \dots; N\}$ determines the active subsystem over the interval $[t_{k-1}, t_k)$ and the system trajectories are right-continuous. The switching mode $k \in \{1; \dots; N\}$ and the switching times $\{t_k\}$ may be governed by a supervisory logic controller, or determined internally depending on the system state, or considered as an external input [13]. In any case, it is assumed in this paper that the active subsystem as well as the switching times $\{t_k\}$ are known. The objective of this paper is to design an observer which provides an estimate of the state $x(t)$ of system (1a). Based on formulas incorporating past values of the input and the output of the studied plant, an observer is introduced for a class of switched nonlinear systems. Here, the following assumptions are considered.

**Assumption 1.** System (1a) satisfies the minimal dwell time condition and the dwell time is a known constant. That means there exists a known $T_S > 0$ such that time instants $t_k$ satisfy $t_k - t_{k-1} \geq T_S$ for all $k \in \{1; \dots; N\}$.

**Assumption 2.** For all modes of operation, i.e. $\forall k = \{1; \dots; N\}$, there exist two $C^1$ functions $V_1 : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$, $V_2 : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ and a corresponding continuous function $\lambda_{*k} : \mathbb{R}^p \times [0,\infty) \to \mathbb{R}^{n\times p}$ such that:

$$\underline{V}_1(\xi) \leq V_1(\xi,t) \leq \overline{V}_1(\xi), \tag{2}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial \xi}(\xi)\alpha_k(y,t)\xi \leq -\omega_1(\xi), \tag{3}$$

$$\underline{V}_2(\xi) \leq V_2(\xi,t) \leq \overline{V}_2(\xi), \tag{4}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial \xi}(\xi)H_k(y,t)\xi \leq -\omega_2(\xi), \tag{5}$$

for all $\xi \in \mathbb{R}^n$, $t \geq 0$, $y \in \mathbb{R}^p$ where $\underline{V}_1$, $\overline{V}_1$, $\underline{V}_2$, $\overline{V}_2$ are continuous positive definite functions and radially unbounded, $\omega_1$, $\omega_2$ are continuous positive definite functions and

$$H_k(y,t) = \alpha_k(y,t) + \lambda_{*k}(y,t)C_k. \tag{6}$$

**Assumption 3.** There exists a positive constant $\tau_*$ such that $\tau_* < T_S$ with $T_S$ defined in Assumption 1 and for all $y \in \mathbb{R}^p$, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \dots ; N\}$, $\zeta \in [t - \tau_*, t]$, matrices

$$E_{\tau_{*k}}(t) = e^{\int_t^{t-\tau_*} H_k(y(\zeta),\zeta)d\zeta} - e^{\int_t^{t-\tau_*} \alpha_k(y(\zeta),\zeta)d\zeta} \in \mathbb{R}^{n\times n}$$

are invertible. Moreover two couples of matrices $\alpha_k(y,t)$ and $\int_t^{t-\tau_*} \alpha_k(y(\zeta),\zeta)d\zeta$ as well as $H_k(y,t)$ and $\int_t^{t-\tau_*} H_k(y(\zeta),\zeta)d\zeta$ satisfy the commutative properties, i.e.

$$\alpha_k(y,t)\times \int_t^{t-\tau_*} \alpha_k(y(\zeta),\zeta)d\zeta = \int_t^{t-\tau_*} \alpha_k(y(\zeta),\zeta)d\zeta \times \alpha_k(y,t), \quad (7)$$

$$H_k(y,t)\times \int_t^{t-\tau_*} H_k(y(\zeta),\zeta)d\zeta = \int_t^{t-\tau_*} H_k(y(\zeta),\zeta)d\zeta \times H_k(y,t), \quad (8)$$

where matrices $H_k(y,t)$ are defined in (6).

*Remark* 1. Let us discuss the considered assumptions:

- Assumption 1 guarantees that no Zeno phenomenon, which roughly consists of high frequency switchings at finite time instants, occurs. In fact, minimal dwell time $T_S$ assures that the system stays on each mode during a period greater than or equal to $T_S$. This condition is usually used when tackling with stability and stabilization problems as well as observer design for switched systems (see e.g., [35–37] and the references therein).
- Assumption 2 is usually used in designing estimators for switched systems (see for instance in [38,39]). Without taking into account the effect of external inputs, Assumption 2 establishes conditions of internal stability ($V_1$, $V_2$ are called common Lyapunov functions [40]). In fact, Assumption 2 implies that, for any constant vector $y \in \mathbb{R}^p$, the origin of $\dot{\chi} = \alpha_k(y,t)\chi$ and the origin of $\dot{\chi} = H_k(y,t)\chi$ are globally asymptotically stable.
- Assumption 3 is a technical assumption. Commutative properties (7) and (8) help give an analytic expression of the state transition matrices when dealing with time-varying systems. Actually, consider the equation $\dot{\chi} = \alpha(y,t)\chi$, $t \geq 0$, $\chi(0) = \chi_0$. Assuming that $\alpha(y,t)$ and $\int_0^t \alpha(y(\zeta),\zeta)d\zeta$ commute, we have $\chi(t) = e^{\int_0^t \alpha(y(\zeta),\zeta)d\zeta}\chi(0)$ because $\frac{d}{dt}\chi(t) = \frac{d}{dt}e^{\int_0^t \alpha(y(\zeta),\zeta)d\zeta}\chi(0) = \alpha(y,t)e^{\int_0^t \alpha(y(\zeta),\zeta)d\zeta}\chi(0) = \alpha(y,t)\chi(t)$. Remember that for time-invariant systems, $\alpha(y,t) = \alpha$ and $\int_0^t \alpha(y(\zeta),\zeta)d\zeta = \alpha t$ commutes with $\alpha$. Note also that two matrices that are simultaneously diagonalizable are always commutative. Consequently, if the couple of matrices $\alpha_k(y,t)$ and $\int_t^{t-\tau_*} \alpha_k(y(\zeta),\zeta)d\zeta$ (respectively $H_k(y,t)$ and $\int_t^{t-\tau_*} H_k(y(\zeta),\zeta)d\zeta$) is simultaneously diagonalizable, (7) (respectively (8)) is satisfied.
- For the couple $\alpha_k(y,t)$ and $\int_t^{t-\tau_*} \alpha_k(y(\xi),\xi)d\xi$ (same for the couple $H_k(y,t)$ and $\int_t^{t-\tau_*} H_k(y(\xi),\xi)d\xi$), for all $k \in \{1; \dots ; N\}$, each of the following situations will guarantee that the two matrices are pairwise commuting:
  - $\alpha_k(y,t)$ is constant;
  - $\alpha_k(y,t) = a_k(y,t)M_k$ where $a_k(y,t)$ is a scalar function, and $M_k \in \mathbb{R}^{n\times n}$ is a constant matrix;
  - $\alpha_k(y,t) = \sum_i a_{k_i}(y,t)M_{k_i}$ where $\{M_{k_i}\}$ are constant matrices that commute: $M_{k_i}M_{k_j} = M_{k_j}M_{k_i}$, and $a_{k_i}(y,t)$ are scalar functions;
  - $\alpha_k(y,t)$ has a time-invariant basis of eigenvectors spanning $\mathbb{R}^n$;
  - $\alpha_k(y,t)$ has special structures such as $\alpha_k(y,t) = \begin{bmatrix} a_k(t) & b_k(t) \\ -b_k(t) & a_k(t) \end{bmatrix}$; etc.

  Moreover it is worth noting that for the couple $H_k(y,t)$ and $\int_t^{t-\tau_*} H_k(y(\xi),\xi)d\xi$, we have one more degree of freedom due to the gains $\lambda_{*k}(y,t)$. In some particular cases of the output $y$, one can choose $\lambda_{*k}(y,t)$ such that $H_k(y,t) = \alpha_k(y,t) + \lambda_{*k}(y,t)C$ is a diagonal matrix hence, $\int_t^{t-\tau_*} H_k(y(\xi),\xi)d\xi$ is also a diagonal matrix.
- Assumptions 1 and 3 ensure the exact state estimation before the next switching.

## 3 | MAIN RESULTS

Let us state and prove the following result:

**Theorem 1.** *Let system (1a) satisfy Assumptions 1-3. Consider a stack of N dynamic extensions, each one of dimension n associated to a different mode of operation: for all* $k = \{1; \dots ; N\}$

$$\dot{z}(t) = \alpha_k(y,t)z(t) + \beta_k(y,t,u(y,t)) \quad (9)$$

*and*

$$\dot{z}_*(t) = H_k(y,t)z_*(t) + \beta_k(y,t,u(y,t)) - \lambda_{*k}(y)y(t), \quad (10)$$

*with initial conditions*

$$z(t_{k-1}) = z_{0_{k-1}} \in \mathbb{R}^n, \quad (11)$$

$$z_*(t_{k-1}) = z_{*0_{k-1}} \in \mathbb{R}^n, \tag{12}$$

*for all $k \in \{1; \dots; N\}$, $z_{0_{k-1}}$ and $z_{*0_{k-1}}$ are constants which can be arbitrarily selected, $t_0 = 0$ and $t_k$ are the switching time instants. Then, for a given piecewise continuous input $u(y,t)$, the state observer of dimension n,*

$$\hat{x}(t) = E^{-1}_{\tau_{*k}}(t)\left(e^{\int_t^{t-\tau_*} H_k(y,\ell)d\ell} z_*(t) - z_*(t-\tau_*) - e^{\int_t^{t-\tau_*} \alpha_k(y,\ell)d\ell} z(t) + z(t-\tau_*)\right) \tag{13}$$

*provides an estimation of x in each $[t_{k-1} + \tau_*, t_k)$, i.e.*

$$\hat{x}(t) = x(t),\ t \in [t_{k-1}+\tau_*, t_k),\ k \in \mathbb{N}. \tag{14}$$

*Remark* 2. The proposed fixed-time observer does not require an appropriate knowledge of the initial conditions while most of existing approaches in the literature need such a knowledge. Contrary to finite-time observers (where the settling time estimate is finite but depends on the initial conditions), the proposed fixed-time observer guarantees a finite settling time with uniform convergence with respect to the initial conditions. For many applications such as switched systems with unknown state jumps, i.e., $x(t_k) = F_k x(t_k^-)$, this approach is more convenient since the trajectories of the estimation error reach the origin within a fixed time, which can be defined in advance as a function of the system parameters. Matrix $F_k$ corresponds to the jump parameters of the continuous state $x$ at the switching times $t_k$, $k \in \{1; \dots; N\}$ and is naturally assumed to be unknown. The notation $t_k^-$ means the time just before the switching times (for more details one can refer to the reset condition given in [41]).

*Remark* 3. To simplify the exposition, we set by convention $t_0 = 0$ but the initial time can be arbitrary nonzero. Moreover, it is worth noticing that at each switching time instant, we reset the initial conditions of dynamics $z$ and $z_*$.

*Remark* 4. Assumption 2 guarantees asymptotic stability of the origin for the zero-input system. Without Assumption 2, then even if $x(t)$ is a bounded solution and $u(t)$ is a bounded input, system (9) admits unstable solutions. This can be a drawback in many cases. Notice also that $z$ given in (9) is an exact copy of (1a) whereas $z_*$ given in (10) corresponds to an observer for (9).

*Remark* 5. It is worth pointing out that the proposed estimators (9)- (10)- (13) are not directly derived from the observers constructed in [32] although some of the key ideas of [32] are used along our design. The fact that matrix $\alpha_k$ is time-varying and not constant makes the problem tougher than in [32]. Additionally, we employ in this paper only one standard observer (10) with only $\lambda_{*k}$ which is considered as a parameter to be selected, whereas [32] combined two classical observers with two separate gains which needed to be carefully chosen.

*Proof.* Let us consider the case $t \in [t_{k-1}, t_k)$. Consider a solution $(z(t), z_*(t))$ of (9)- (10) associated with a solution $x(t)$ of the corresponding subsystem of (1a) defined over $[t_{k-1}, t_k)$, $k \in \{1; \dots; N\}$. Note that $\hat{x}(t)$ is always defined for all $t \in [t_{k-1}, t_k)$, $k \in \{1; \dots; N\}$ because $z(t)$ and $z_*(t)$ are bounded at the origin thanks to (i) Assumption 2 and (ii) the fact that $\beta_k$ and $\lambda_{*k}$ are continuous functions.

From the output (1b), for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \dots; N\}$ the corresponding subsystem of (1a) can be rewritten in two different ways:

$$\dot{x}(t) = \alpha_k(y,t)x(t) + \beta_k(y,t,u(y,t)), \tag{15a}$$

$$\dot{x}(t) = H_k(y,t)x(t) + \beta_k(y,t,u(y,t)) - \lambda_{*k}(y,t)y(t). \tag{15b}$$

By integrating (15a) and (15b) between two values $v_1 \geq 0$ and $v_2 \geq 0$ and noting that (7) and (8) are satisfied, we obtain the equalities:

$$x(v_1) = e^{\int_{v_2}^{v_1} \alpha_k(y(\ell),\ell)d\ell} x(v_2) + \int_{v_2}^{v_1} e^{\int_\ell^{v_1} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell,$$

$$x(v_1) = e^{\int_{v_2}^{v_1} H_k(y(\ell),\ell)d\ell} x(v_2) + \int_{v_2}^{v_1} e^{\int_\ell^{v_1} H_k(y(\rho),\rho)d\rho} \left[\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)\right] d\ell.$$

Now, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \dots; N\}$, let us select $v_2 = t \geq 0$ and $v_1 = t - \tau_* \geq 0$. Hence one can get:

$$x(t-\tau_*) = e^{\int_t^{t-\tau_*} \alpha_k(y(\ell),\ell)d\ell} x(t) + \int_t^{t-\tau_*} e^{\int_\ell^{t-\tau_*} \alpha_k(y(\rho),\rho)d\rho} \times \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell, \tag{16a}$$

$$x(t-\tau_*) = e^{\int_t^{t-\tau_*} H_k(y(\ell),\ell)d\ell} x(t) + \int_t^{t-\tau_*} e^{\int_\ell^{t-\tau_*} H_k(y(\rho),\rho)d\rho} \left[\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)\right] d\ell. \tag{16b}$$

Consequently from (16a) and (16b), one obtains that

$$E_{\tau_{*k}}(t)x(t) = \int_{t}^{t-\tau_*} e^{\int_{\ell}^{t-\tau_*} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell - \int_{t}^{t-\tau_*} e^{\int_{\ell}^{t-\tau_*} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell. \tag{17}$$

Thus,

$$E_{\tau_{*k}}(t)x(t) = -\int_{t-\tau_*}^{t} e^{\int_{\ell}^{t-\tau_*} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell + \int_{t-\tau_*}^{t} e^{\int_{\ell}^{t-\tau_*} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell. \tag{18}$$

Since from Assumption 3, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \ldots ; N\}$, matrix $E_{\tau_{*k}}(t)$ is invertible, one has

$$x(t) = -E_{\tau_{*k}}^{-1}(t) \int_{t-\tau_*}^{t} e^{\int_{\ell}^{t-\tau_*} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell + E_{\tau_{*k}}^{-1}(t) \int_{t-\tau_*}^{t} e^{\int_{\ell}^{t-\tau_*} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell. \tag{19}$$

On the other hand, by integrating (9) and (10) and bearing in mind that (7) and (8) are fulfilled, we deduced that, for all constants $v_1 \geq 0$ and $v_2 \geq 0$, the equalities

$$z(v_1) = e^{\int_{v_2}^{v_1} \alpha_k(y(\ell),\ell)d\ell} z(v_2) + \int_{v_2}^{v_1} e^{\int_{\ell}^{v_1} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell,$$

$$z_*(v_1) = e^{\int_{v_2}^{v_1} H_k(y(\ell),\ell)d\ell} z_*(v_2) + \int_{v_2}^{v_1} e^{\int_{\ell}^{v_1} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell,$$

are satisfied. Hence, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \ldots ; N\}$, we select $v_2 = t - \tau_* \geq 0$, $v_1 = t \geq 0$ and we have:

$$z(t) = e^{\int_{t-\tau_*}^{t} \alpha_k(y(\ell),\ell)d\ell} z(t-\tau_*) + \int_{t-\tau_*}^{t} e^{\int_{\ell}^{t} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell, \tag{20a}$$

$$z_*(t) = e^{\int_{t-\tau_*}^{t} H_k(y(\ell),\ell)d\ell} z_*(t-\tau_*) + \int_{t-\tau_*}^{t} e^{\int_{\ell}^{t} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell. \tag{20b}$$

It follows that, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \ldots ; N\}$,

$$\int_{t-\tau_*}^{t} e^{\int_{t}^{t-\tau_*} \alpha_k(y(\ell),\ell)d\ell} e^{\int_{\ell}^{t} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell = e^{\int_{t}^{t-\tau_*} \alpha_k(y(\ell),\ell)d\ell} z(t) - z(t-\tau_*), \tag{21}$$

$$\int_{t-\tau_*}^{t} e^{\int_{t}^{t-\tau_*} H_k(y(\ell),\ell)d\ell} e^{\int_{\ell}^{t} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell = e^{\int_{t}^{t-\tau_*} H_k(y(\ell),\ell)d\ell} z_*(t) - z_*(t-\tau_*). \tag{22}$$

Then, for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \ldots ; N\}$,

$$\int_{t-\tau_*}^{t} e^{\int_{\ell}^{t-\tau_*} \alpha_k(y(\rho),\rho)d\rho} \beta_k(y(\ell),\ell,u(y(\ell),\ell))d\ell = e^{\int_{t}^{t-\tau_*} \alpha_k(y(\ell),\ell)d\ell} z(t) - z(t-\tau_*), \tag{23}$$

$$\int_{t-\tau_*}^{t} \mathrm{t}\; e^{\int_{\ell}^{t-\tau_*} H_k(y(\rho),\rho)d\rho} [\beta_k(y(\ell),\ell,u(y(\ell),\ell)) - \lambda_{*k}(y(\ell),\ell)y(\ell)]\, d\ell = e^{\int_{t}^{t-\tau_*} H_k(y(\ell),\ell)d\ell} z_*(t) - z_*(t-\tau_*). \tag{24}$$

Finally, from (19), (23) and (24), one can immediately deduce that for all $t \in [t_{k-1} + \tau_*, t_k)$, $k \in \{1; \ldots ; N\}$,

$$x(t) = E_{\tau_{*k}}^{-1}(t) \left( e^{\int_{t}^{t-\tau_*} H_k(y(\ell),\ell)d\ell} z_*(t) - z_*(t-\tau_*) - e^{\int_{t}^{t-\tau_*} \alpha_k(y(\ell),\ell)d\ell} z(t) + z(t-\tau_*) \right) = \hat{x}(t).$$

This concludes the proof. □

*Remark* 6. The estimate (13) is independent of the initial condition $x_0$ of system (1a). It is also important to note that the observation error exactly converges to zero after the settling time $\tau_*$ following any switches even in the case unknown state jumps which may occur in (1a) (see discussions about unknown state jumps in Remark 2). Furthermore, one can highlight that the settling time $\tau_*$ is not over-estimated contrary to [20]. It also does not depend on the initial observation error at each switching time and can be easily made arbitrarily small.

## 4 | ILLUSTRATIVE EXAMPLE

In this section, let us consider the nonlinear switched system (1a) with $N = 2$ to illustrate Theorem 1. The two distinct subsystems are defined as follows:

Subsystem 1:

$$\dot{x}_1 = -(t+2)x_1 - \frac{3}{2}x_2 + u_1 + y^2\sin(y), \tag{25a}$$

$$\dot{x}_2 = \frac{3}{2}x_1 - (t+2)x_2 + u_2 + y^2\sin(y), \tag{25b}$$

$$y = x_1 + x_2. \tag{25c}$$

Subsystem 2:

$$\dot{x}_1 = -(t+4)x_1 - \frac{3}{2}x_2 + u_1 + \sin(y), \tag{26a}$$

$$\dot{x}_2 = \frac{3}{2}x_1 - (t+4)x_2 + u_2 + \sin(y), \tag{26b}$$

$$y = x_1 + x_2. \tag{26c}$$

System (25) corresponds to $k = 1$, called sub-model 1 and system (26) corresponds to $k = 2$, called sub-model 2. Systems (25)- (26) are of the form (1) with $C_1 = C_2 = [11]$,

$$\alpha_1(t) = \begin{bmatrix} -(t+2) & -\frac{3}{2} \\ \frac{3}{2} & -(t+2) \end{bmatrix} \text{ and} \tag{27}$$

$$\beta_1(y,u) = \begin{bmatrix} u_1 + y^2\sin(y) \\ u_2 + y^2\sin(y) \end{bmatrix}, \tag{28}$$

$$\alpha_2(y) = \begin{bmatrix} -(t+4) & -\frac{3}{2} \\ \frac{3}{2} & -(t+4) \end{bmatrix} \text{ and} \tag{29}$$

$$\beta_2(y,u) = \begin{bmatrix} u_1 + \sin(y) \\ u_2 + \sin(y) \end{bmatrix}. \tag{30}$$

It is worth pointing out that even the above-mentioned example is quite simple, the technique proposed in [32] cannot be applied for it. The example confirms that our methodology to design fixed-time observers is significantly different from the one introduced in [32] as discussed in Remark 5. Now, let us choose $\lambda_{*1} = \lambda_{*2} = \left[\frac{3}{2} - \frac{3}{2}\right]^{\top}$. Hence, one can obtain

$$H_1(t) = \alpha_1(t) + \lambda_{*1}C_1 = \begin{bmatrix} -t - \frac{1}{2} & 0 \\ 0 & -t - \frac{7}{2} \end{bmatrix}, \tag{31}$$

$$H_2(t) = \alpha_2(t) + \lambda_{*2}C_2 = \begin{bmatrix} -t - \frac{5}{2} & 0 \\ 0 & -t - \frac{11}{2} \end{bmatrix}. \tag{32}$$

Therefore, Assumption 2 is satisfied. Through long but simple calculations, one can obtain

$$E_{\tau_{*1}}(t) = \begin{bmatrix} \varepsilon_{11}(t) & \varepsilon_{12}(t) \\ \varepsilon_{21}(t) & \varepsilon_{22}(t) \end{bmatrix}, \tag{33}$$

with

$$\varepsilon_{11}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*}\left[e^{\frac{\tau_*}{2}} - e^{2\tau_*}\cos\left(\frac{3}{2}\tau_*\right)\right],$$

$$\varepsilon_{12}(t) = -e^{-\frac{\tau_*^2}{2}+t\tau_*+2\tau_*}\sin\left(\frac{3}{2}\tau_*\right),$$

$$\varepsilon_{21}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*+2\tau_*}\sin\left(\frac{3}{2}\tau_*\right),$$

$$\varepsilon_{22}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*}\left[e^{\frac{7\tau_*}{2}} - e^{2\tau_*}\cos\left(\frac{3}{2}\tau_*\right)\right].$$

and

$$E_{\tau_{*2}}(t) = \begin{bmatrix} \varepsilon'_{11}(t) & \varepsilon'_{12}(t) \\ \varepsilon'_{21}(t) & \varepsilon'_{22}(t) \end{bmatrix}, \tag{34}$$

with

$$\varepsilon'_{11}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*}\left[e^{\frac{5\tau_*}{2}} - e^{4\tau_*}\cos\left(\frac{3}{2}\tau_*\right)\right],$$

$$\varepsilon'_{12}(t) = -e^{-\frac{\tau_*^2}{2}+t\tau_*+4\tau_*}\sin\left(\frac{3}{2}\tau_*\right),$$

$$\varepsilon'_{21}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*+4\tau_*}\sin\left(\frac{3}{2}\tau_*\right),$$

$$\varepsilon'_{22}(t) = e^{-\frac{\tau_*^2}{2}+t\tau_*}\left[e^{\frac{11\tau_*}{2}} - e^{4\tau_*}\cos\left(\frac{3}{2}\tau_*\right)\right].$$

Next, let us compute the corresponding determinants

$$\det E_{\tau_{*1}}(t) = e^{-\tau_*^2+2t\tau_*+2\tau_*} \times \left[2e^{2\tau_*} - \cos\left(\frac{3}{2}\tau_*\right)\left(e^{\frac{\tau_*}{2}} + e^{\frac{7\tau_*}{2}}\right)\right], \tag{35}$$

$$\det E_{\tau_{*2}}(t) = e^{-\tau_*^2+2t\tau_*+4\tau_*} \times \left[2e^{4\tau_*} - \cos\left(\frac{3}{2}\tau_*\right)\left(e^{\frac{5\tau_*}{2}} + e^{\frac{11\tau_*}{2}}\right)\right]. \tag{36}$$

When $\tau_* = \{0.5; 0.7; 1\}$, one can check that $\det E_{\tau_{*1}}(t)$ and $\det E_{\tau_{*2}}(t)$ are different from 0 for all $t \geq 0$. That means that matrices $E_{\tau_{*1}}(t)$ and $E_{\tau_{*2}}(t)$ are invertible for all $t \geq 0$. Then, Assumption 3 is satisfied for $\tau_* = \{0.5; 0.7; 1\}$.

Hence, the concrete form of $\hat{x}$ for subsystems (25)- (26) is given as follows for $k = \{1; 2\}$, $\tau_* = \{0.5; 0.7; 1\}$,

$$\hat{x}(t) = E^{-1}_{\tau_{*k}}(t)\left(\mathbb{H}_{\tau_{*k}}(t)z_*(t) - z_*(t-\tau_*) - \mathbb{A}_{\tau_{*k}}(t)z(t) + z(t-\tau_*)\right) \tag{37}$$

where $E_{\tau_{*1}}, E_{\tau_{*2}}$ defined in (33), (34), respectively and

$$\mathbb{H}_{\tau_{*1}}(t) = \begin{bmatrix} e^{-0.5\tau_*^2+t\tau_*+0.5\tau_*} & 0 \\ 0 & e^{-0.5\tau_*^2+t\tau_*+3.5\tau_*} \end{bmatrix}, \tag{38}$$

$$\mathbb{H}_{\tau_{*2}}(t) = \begin{bmatrix} e^{-0.5\tau_*^2+t\tau_*+2.5\tau_*} & 0 \\ 0 & e^{-0.5\tau_*^2+t\tau_*+5.5\tau_*} \end{bmatrix}, \tag{39}$$

$$\mathbb{A}_{\tau_{*1}}(t) = \begin{bmatrix} a_1(t)\cos(1.5\tau_*) & a_1(t)\sin(1.5\tau_*) \\ -a_1(t)\sin(1.5\tau_*) & a_1(t)\cos(1.5\tau_*) \end{bmatrix}, \tag{40}$$

$$\mathbb{A}_{\tau_{*2}}(t) = \begin{bmatrix} a_2(t)\cos(1.5\tau_*) & a_2(t)\sin(1.5\tau_*) \\ -a_2(t)\sin(1.5\tau_*) & a_2(t)\cos(1.5\tau_*) \end{bmatrix}, \tag{41}$$

with $a_1(t) = e^{-0.5\tau_*^2+t\tau_*+2\tau_*}$, $a_2(t) = e^{-0.5\tau_*^2+t\tau_*+4\tau_*}$.

The two following cases will now be simulated. The first case is the non-switched case. Here, only the nonlinear system (26) is considered (i.e. $k := \text{Subsystem}2, \forall t \geq 0$). We apply Theorem 1 with $u_1 = u_2 = 30$ and the initial conditions $x(0) = (2.3, 1)^\top$, $z(0) = (4.3, 2)^\top$, $z_*(0) = (3.3, 1.5)^\top$. Figure 1 and Figure 2 illustrate the error between the real state and the estimated state of system (26) with $\tau_* = 0.5$ and $\tau_* = 1$ respectively. One can see that the observation error exactly converges to zero after the settling time $\tau_*$. It can be easily arbitrarily tuned.

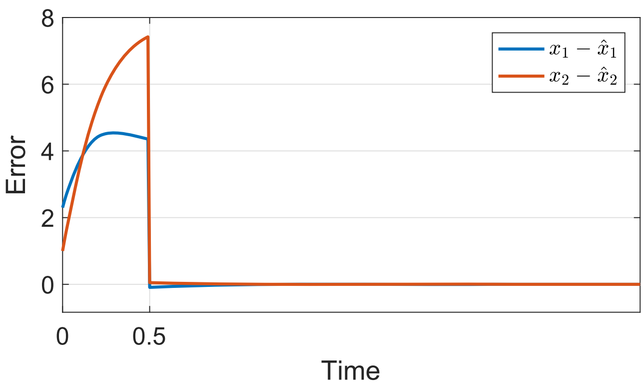


**FIGURE 1** The error between the real state and the exact estimation of (26) with $\tau_* = 0.5$ [Color figure can be viewed at wileyonlinelibrary.com]

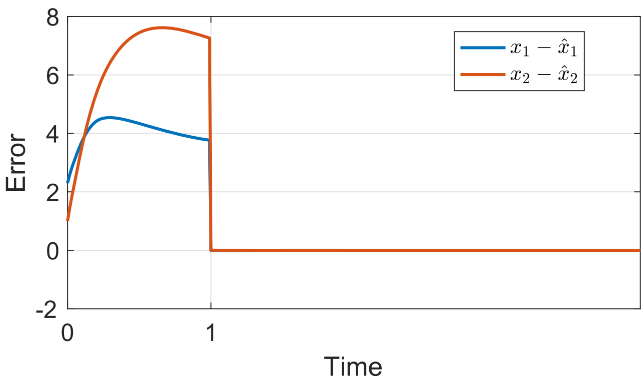


**FIGURE 2** The error between the real state and the exact estimation of (26) with $\tau_* = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

The second case is the switched case with $k$ set according to:

$$k := \begin{cases} \text{Subsystem} \quad 2 & \text{if} \quad 0 \leq t \leq 2, \\ \text{Subsystem} \quad 1 & \text{if} \quad 2 \leq t \leq 3, \\ \text{Subsystem} \quad 2 & \text{if} \quad 3 \leq t \leq 4, \\ \text{Subsystem} \quad 1 & \text{if} \quad t \geq 4. \end{cases} \tag{42}$$

The switching law is depicted in Fig. 3. We apply Theorem 1 with

$$u_1 = u_2 = \begin{cases} 50 & \text{if} \quad k := \text{Subsystem} \quad 1, \\ 30 & \text{if} \quad k := \text{Subsystem} \quad 2, \end{cases}$$

$x(t_{k-1}) = (2.3, 1)^\top$, $z(t_{k-1}) = (4.3, 2)^\top$, $z_*(t_{k-1}) = (3.3, 1.5)^\top$, where $k = \{1; 2; 3; 4\}$. The switching time instants are $t_1 = 2$, $t_2 = 3$ and $t_3 = 4$.

Figures 4 and 5 illustrate the error between the real state and the estimated state of the switched system defined by the two distinct subsystems (25), (26) and the switching signal (42) with $\tau_* = 0.5$, $\tau_* = 0.7$ respectively. Note that the minimal dwell-time condition is satisfied and the

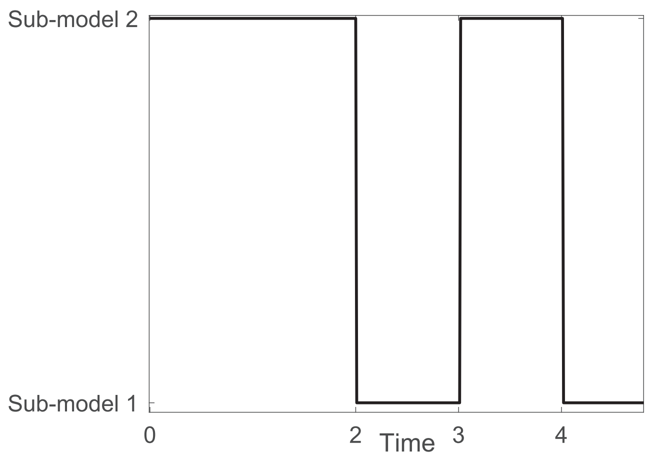


**FIGURE 3** Switching law

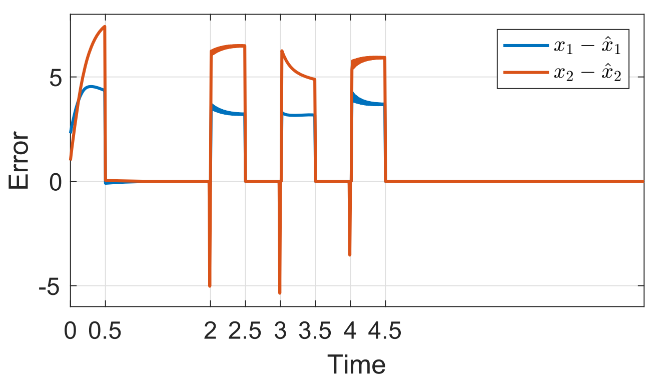


**FIGURE 4** The error between the real state and the exact estimation of switched system defined by two distinct subsystems (25), (26) and the switching signal (42) with $\tau_* = 0.5$ [Color figure can be viewed at wileyonlinelibrary.com]

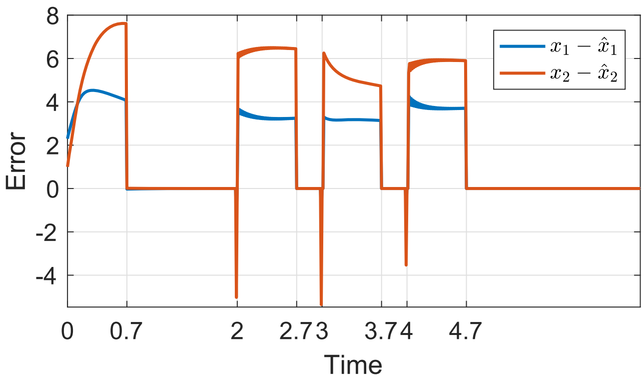


**FIGURE 5** The error between the real state and the exact estimation of switched system defined by two distinct subsystems (25), (26) and the switching signal (42) with $\tau_* = 0.7$ [Color figure can be viewed at wileyonlinelibrary.com]

settling time $\tau_*$ can be easily arbitrarily tuned. We observe that the estimation is exact for all $t$, $t_{k-1} + \tau_* \leq t \leq t_k$, $k = \{1; 2; 3\}$ and $t \geq t_3$.

## 5 | CONCLUSION

In this paper, the state estimation problem has been solved for a class of nonlinear time-varying systems with switched dynamics. Using the past input and output values of the studied system, some sufficient conditions are provided to estimate the state before the next switching. Extensions to systems with disturbances and with unknown switching signals are expected.

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