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Interval observers design for discrete-time linear switched systems

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***Abstract*— The main goal of this paper is to present a methodology to design interval observers for discrete-time linear switched systems affected by disturbances and measurement noises which are considered bounded but unknown. The steps to design these observers are detailed. Furthermore, to relax some restrictive conditions, a change of coordinates is recommended.**

Keywords: Interval observer, discrete-time, linear systems, switched systems, LMI.

I. INTRODUCTION

Interval observers design is a technique introduced for the first time in [18]. It presents an issue to deal with real systems, as in biological applications, whose models are often affected by various types of disturbances and measurement noises. Due to uncertainties, the state estimators of these systems need additional information, which is not usually evident. In such a context, interval observers can be considered as an interesting alternative. They rely on the design of a dynamic extension endowed with two outputs giving an upper and a lower bounds for the state of the considered system [18]. Such observers require that the bounds of the initial state, of the noises and of the disturbances are available. Then, two trajectories, which define the interval observer, are computed to enclose the actual state. In this context, an interval observer can be considered as a classical one where the centre of the interval is a point estimation and its radius is the uncertainty on the estimate.

As presented for instance in [31], [11], [23], [25], the most important reason for the increased popularity of this class of observers is that, on the one hand, they cope with large uncertainties, which is crucial for instance for biological systems. On the other hand, interval observers have been successfully applied to many real-life problems as illustrated in the papers [3], [17], [1]. Several works dealing with this technique of estimation have been proposed for linear [7], [22], [23], [24] and nonlinear systems [2], [29], [31], [32]. A common feature of these results is that they apply mainly to continuous-time systems. The design of interval observers in the case of discrete-time is evoked for instance in [25], [11]. The importance of discrete-time systems comes from the fact that discretization methods used to transform continuous-time systems into discrete-time ones affect the transformed systems by disturbances, which motivates the development of robust state estimation techniques such as interval observers. However, the design of interval observers in the case of dynamical systems that exhibit both continuous and discrete dynamic behavior (i.e. hybrid systems) is not yet popular. We mention here a few recent works [26], [13], [14], [19], [20].

Switched systems are a class of hybrid systems. They consist of a finite number of subsystems governed by switching signals [4], [16]. A number of contributions presenting the analysis and the control of this class of systems have been proposed during the last decade [5], [6]. Nonetheless, the use of interval observers in the case of the switched systems is not fully evoked and well treated in the literature[30], [8]. In this paper, the main objective is to design an interval observer for the family of discrete-time linear switched systems affected by disturbances and measurement noises which are assumed unknown but bounded with known bounds.

This paper starts with some preliminaries in Section II. In section III, two methodologies to design interval observers for discrete-time linear switched systems are treated. With a numerical example, in section IV, the techniques used are illustrated. Section V concludes the paper.

II. PRELIMINARIES

The set of natural numbers, integers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of nonnegative real numbers and nonnegative integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, respectively. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{Z} \to \mathbb{R}$, the symbol $\|u\|_{[t_0,t_1]}$ denotes its L_∞ norm:

$$\|u\|_{[t_0,t_1]} = \sup\{|x|, t \in [t_0,t_1]\}.$$

If $t_1 = \infty$ then we will simply write $\|u\|$. We denote $\mathscr{L}_\infty$ as the set of all inputs u with the property $\|u\| < \infty$. We denote the sequence of integers $1,\ldots,N$ as $\overline{1,N}$. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1},...,x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1},...,x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \{1,...,n\}$, $x_{a,i} \leq x_{b,i}$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, define Q^+, $Q^- \in \mathbb{R}^{n\times n}$ such as $Q^+ = \max(Q,0)$ and $Q^- = Q^+ - Q$. The superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be nonnegative if each entry of this matrix is nonnegative. I_n is the identity matrix of $n \times n$ dimension. Any $n\times m$ (resp. $p\times 1$) matrix, whose entries are all 1 is denoted $E_{n\times m}$ (resp. E_p). $P \in \mathbb{R}^{n\times n}$ is positive (resp. negative) definite is denoted as $P \succ 0$ (resp. $P \prec 0$).

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Lemma 1: [10] Consider vectors $x, \overline{x}, \underline{x} \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$ and a constant matrix $A \in \mathbb{R}^{n\times n}$, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}, \qquad (1)$$

with $A^+ = \max(0, A)$, $A^- = A^+ - A$.

Lemma 2: [28], [15] Consider a positive scalar δ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, then

$$2x^T y \leq \frac{1}{\delta} x^T P x + \delta y^T P^{-1} y, \; x, y \in \mathbb{R}^n. \qquad (2)$$

Lemma 3 (Schur Complement): Given the matrices $R = R^T$, $Q = Q^T$ and S with appropriate dimensions. The following LMIs are equivalent:

1)
$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0.$$

2)
$$R \succ 0; \; Q - SR^{-1}S^T \succ 0.$$

3)
$$Q \succ 0; \; R - S^T Q^{-1} S \succ 0.$$

A discrete-time system described by $x(k+1) = f(x(k))$ is nonnegative if for any integer k_0 and any initial condition $x(k_0) \geq 0$, the solution x satisfies $x(k) \geq 0$ for all integers $k \geq k_0$.

A system described by $x(k+1) = Ax(k) + u(k)$, with $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is nonnegative if and only if the matrix A is elementwise nonnegative, $u(k) \geq 0$ and $x(k_0) \geq 0$. In this case, the system is also called cooperative. This property is essential in the design of interval observers since the estimation errors should follow nonnegative dynamics.

III. Interval observer for discrete-time linear switched systems

Consider the following discrete-time linear switched system:

$$\begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k), \\ y(k) = C_q x(k) + v(k), \; q \in \overline{1,N}, \; N \in \mathbb{N}, \end{cases} \qquad (3)$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $d \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are respectively the disturbances and the measurement noises. q is the index of the active subsystem and N is the number of subsystems. A_q, B_q and C_q are time-invariant matrices of the corresponding dimensions.

In this section, the goal is to design interval observers for discrete-time linear switched systems. Two approaches are considered: the first one is based on the nonnegativity (cooperativity) of the estimation errors in the original coordinates, while the second is more general and follows a transformation of coordinates.

A. Interval observer design in the original coordinates

In this part, we introduce the following assumptions in order to design an interval observer without a transformation of coordinates

Assumption 1: The state vector $x \in \mathbb{R}^n$ is bounded, i.e. $x \in \mathcal{L}_\infty^n$.

Assumption 2: Let a function $\overline{d} \in \mathcal{L}_\infty^n$ such that for all $k \in \mathbb{Z}_+$

$$-\overline{d}(k) \leq d(k) \leq \overline{d}(k).$$

Assumption 3: There exists a constant $V > 0$, such that

$$-E_p V \leq v(k) \leq E_p V, \; \forall k \in \mathbb{Z}_+.$$

Assumption 4: $\exists \, L_q \in \mathbb{R}^{n\times p}$, $\forall q \in \overline{1,N}$, such that $A_q - L_q C_q$ are nonnegative.

Assumption 1 states that the state x is bounded. This assumption is common in the theory of observers design since the control design (stabilization) is not considered at this stage. Assumption 2 and 3 are basic in the literature of interval observers where the noises and disturbances are assumed bounded with known bounds. Assumption 4 is an important condition to design an interval observer, it is rather restrictive and it will be relaxed in subsection III-B.

Using the available information and considering that $\underline{X}_0 \leq x(0) \leq \overline{X}_0$ for some known $\underline{X}_0, \overline{X}_0 \in \mathbb{R}^n$, the objective is to calculate two estimates $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$, such that

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k), \; k \in \mathbb{Z}_+. \qquad (4)$$

As a solution to this problem, the following interval observer candidate is considered:

$$\begin{cases} \overline{x}(k+1) = (A_q - L_q C_q)\overline{x}(k) + B_q u(k) + \overline{d}(k) + L_q y \\ \qquad\qquad + |L_q| E_p V, \\ \underline{x}(k+1) = (A_q - L_q C_q)\underline{x}(k) + B_q u(k) - \overline{d}(k) + L_q y \\ \qquad\qquad - |L_q| E_p V, \\ \underline{x}(0) = \underline{X}_0, \; \overline{x}(0) = \overline{X}_0, \end{cases} \qquad (5)$$

where $L_q \in \mathbb{R}^{n\times p}$ which has to be computed, is an appropriate observer gain associated to the q-subsystem with $q \in \overline{1,N}$.

Theorem 1 (Framer): Let Assumptions 1-4 hold, the lower and upper interval estimates for the state $x(k)$ given by (5) satisfy (4), provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.

Proof. Let $\overline{e}(k) = \overline{x}(k) - x(k)$ and $\underline{e}(k) = x(k) - \underline{x}(k)$ be the upper observation and the lower observation errors respectively. The aim is to prove that $\overline{e}(k)$ and $\underline{e}(k)$ are nonnegatives. The dynamics of the upper error follow:

$$\overline{e}(k+1) = (A_q - L_q Cq)\overline{e}(k) + \overline{\Psi}(k), \qquad (6)$$

where

$$\overline{\Psi}(k) = \overline{d}(k) - d(k) + L_q v(k) + |L_q| E_p V. \qquad (7)$$

Similarly, the dynamics of the lower error are described by:

$$\underline{e}(k+1) = (A_q - L_q C_q)\underline{e}(k) + \underline{\Psi}(k), \qquad (8)$$

where

$$\underline{\Psi}(k) = \overline{d}(k) + d(k) - L_q v(k) + |L_q| E_p V, \qquad (9)$$

According to Assumption 2, we have $\overline{d}(k) + d(k) \geq 0$ and $\overline{d}(k) - d(k) \geq 0$. Also as Assumption 3 holds, $|v(k)| \leq E_p V$ then $-L_q v(k) + |L_q| E_p V \geq 0$ and $L_q v(k) + |L_q| E_p V \geq 0$. Therefore, $\underline{\Psi}(k) \geq 0$ and $\overline{\Psi}(k) \geq 0$.

Bearing in mind Assumption 4 and from the fact that $\overline{e}(0) = \overline{x}(0) - x(0) \geq 0$ and $\underline{e}(0) = x(0) - \underline{x}(0) \geq 0$, it follows that, for all $k \in \mathbb{Z}_+$, $\overline{e}(k) \geq 0$ and $\underline{e}(k) \geq 0$. Thus, for all $k \in \mathbb{Z}_+$, $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$. This allows us to conclude. □

Let us introduce the following theorem in order to introduce Linear Matrix Inequalities (LMIs) as guidelines for selecting L_q for (5).

Theorem 2 (Stability of the systems (3)-(5)): Let Assumptions 1-4 hold, if there exist $P \in \mathbb{R}^{n\times n}$, $P = P^T \succ 0$ and scalars $\delta_q > 0$, $\forall\, q \in \overline{1,N}$ such that:

$$\begin{bmatrix} P & S_qC_q \\ (S_qC_q)^T & \frac{1}{\alpha_q}P - A_q^T P A_q + A_q^T S_q C_q + (A_q^T S_q C_q)^T \end{bmatrix} \succ 0, \tag{10}$$

with $\alpha_q = 1 + \frac{1}{\delta_q}$, $S_q = PL_q$. Then, the states $\underline{x}, \overline{x}$ computed by (5) are bounded.

Proof. To study the stability of the interval observer, we propose to apply a common Lyapunov function to the estimation errors $\overline{e}(k)$, $\underline{e}(k)$ defined by:

$$V(\overline{e}(k)) = \overline{e}^T(k) P \overline{e}(k),\ P = P^T \succ 0, \tag{11}$$

in the case of the upper observation error.

From (6), we have

$$\begin{aligned}\Delta V(\overline{e}(k)) &= V((\overline{e}(k+1)) - V((\overline{e}(k)) \\ &= \overline{e}^T(k)\left[(A_q - L_qC_q)^T P (A_q - L_qC_q) - P\right]\overline{e}(k) \\ &\quad + \overline{\Psi}^T(k) P \overline{\Psi}(k) + 2\overline{e}(k)^T (A_q - L_qC_q)^T P \overline{\Psi}(k).\end{aligned} \tag{12}$$

According to Lemma 2, we obtain:

$$\begin{aligned}&\Delta V(\overline{e}(k)) \leq \\ &\quad \overline{e}^T(k)\left[(A_q - L_qC_q)^T P (A_q - L_qC_q)(1 + \frac{1}{\delta_q}) - P\right]\overline{e}(k) \\ &\quad + (1+\delta_q)\overline{\Psi}^T(k) P \overline{\Psi}(k),\end{aligned} \tag{13}$$

Additionally, from (10) and by applying Lemma 3, we have

$$P - \alpha_q\left[A_q^T P A_q - A_q^T S_q C_q - (A_q^T S_q C_q)^T + (S_qC_q)^T P^{-1}(S_qC_q)\right] \succ 0, \tag{14}$$

with $\alpha_q = 1 + \frac{1}{\delta_q}$, $S_q = PL_q$.

Then,

$$[(A_q - L_qC_q)^T P (A_q - L_qC_q)(1 + \frac{1}{\delta_q}) - P] \prec 0. \tag{15}$$

On the other hand, the noises and disturbances are bounded, it follows that $\overline{\Psi}$ is bounded. Therefore, by combining (15) with (13) we conclude that $\overline{e}$ is bounded (ISS stability) [15]. The same arguments show that $\underline{e}$ is also bounded. Finally, since $x \in \mathcal{L}_\infty^n$, then $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$. □

B. Design of an interval observer in new coordinates

Even though we have proposed the different steps to design an interval observer by Theorem 1, in some cases, it is not possible to find gains L_q such that the matrices $A_q - L_qC_q$ are nonnegative.

Bearing in mind that the stability property is preserved under a transformation of coordinates, then to overcome the problem of selecting gains, one can first choose the gains L_q ensuring the stability in the original coordinates x and the next step is to find a nonsingular transformation $z = Px$ such that the matrices $P(A_q - L_qC_q)P^{-1}$ are nonnegative. However, the existence of a common transformation P is not obvious because of the difficulties in solving nonlinear matrices inequalities, as shown in [19].

As discussed above, determining a nonsingular transformation matrix P to transform the system (3) into a nonnegative form such that $P(A_q - L_qC_q)P^{-1}$ ($q \in \overline{1,N}$) are nonnegative is almost impossible. A new methodology is proposed. It is based on the design, in the original base, of two conventional observers. The structure is inspired by the one proposed in [9]. Consider the discrete-time linear switched system (3) and two point observers as follows:

$$\begin{cases} \hat{x}^+(k+1) &= (A_q - L_qC_q)\hat{x}^+(k) + B_q u(k) \\ &\quad + P_q^{-1}(P_q^+ \overline{d}(k) + P_q^- \overline{d}(k)) \\ &\quad + L_q y(k) + P_q^{-1}|P_qL_q|E_pV, \\ \hat{x}^-(k+1) &= (A_q - L_qC_q)\hat{x}^-(k) + B_q u(k) \\ &\quad - P_q^{-1}(P_q^+ \overline{d}(k) + P_q^- \overline{d}(k)) \\ &\quad + L_q y(k) - P_q^{-1}|P_qL_q|E_pV, \end{cases} \tag{16}$$

where the changes of coordinates matrices P_q, $q = \overline{1,N}$, are chosen as in Theorem 3 and for suitably selected initial conditions:

$$\begin{cases} \hat{x}^+(0) = Q_q(P_q^+ \overline{x}(0) - P_q^- \underline{x}(0)), \\ \hat{x}^-(0) = Q_q(P_q^+ \underline{x}(0) - P_q^- \overline{x}(0)), \end{cases} \tag{17}$$

with

$$Q_q = P_q^{-1},\quad \forall q \in \overline{1,N},\ N \in \mathbb{N}. \tag{18}$$

We are now ready to introduce the theorem below to deduce an interval estimation from the estimates computed by (16).

Theorem 3: Consider matrices P_q, $q \in \overline{1,N}$, $N \in \mathbb{N}$ such that $F_q = P_q(A_q - L_qC_q)P_q^{-1}$ are nonnegative. If $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then

$$\begin{cases} \underline{x}(k) = Q_q^+ P_q \hat{x}^-(k) - Q_q^- P_q \hat{x}^+(k), \\ \overline{x}(k) = Q_q^+ P_q \hat{x}^+(k) - Q_q^- P_q \hat{x}^-(k), \end{cases} \tag{19}$$

is a framer for (3) satisfying $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$. Furthermore, if there exist a matrix $M = M^T \succ 0$ and positive scalars α_q, δ_q such that

$$\begin{bmatrix} M & S_qC_q \\ C_q^T S_q^T & \frac{1}{\alpha_q}M - A_q^T M A_q + A_q^T S_q C_q + (A_q^T S_q C_q)^T \end{bmatrix} \succ 0,\quad q \in \overline{1,N},\ N \in \mathbb{N}, \tag{20}$$

with

$$\alpha_q = 1 + \frac{1}{\delta_q}, \quad S_q = ML_q,$$

then, $\underline{x}$ and $\overline{x}$ given by (19) are bounded.

Proof. Let us prove that $\overline{x}(k) - x(k) \geq 0$ and $x(k) - \underline{x}(k) \geq 0$.

First, let us define errors $E_q^+(k)$ and $E_q^-(k)$ as:

$$E_q^+(k) = P_q\hat{x}^+(k) - P_q x(k), \tag{21}$$
$$E_q^-(k) = P_q x(k) - P_q\hat{x}^-(k). \tag{22}$$

Thus,

$$E_q^+(k+1) = P_q\hat{x}^+(k+1) - P_q x(k+1), \tag{23}$$
$$E_q^-(k+1) = P_q x(k+1) - P_q\hat{x}^-(k+1). \tag{24}$$

From (3) and (16), the dynamics of E_q^+ are given by:

$$E_q^+(k+1) = F_q E_q^+(k) + \gamma_q^+, \tag{25}$$

where

$$F_q = P_q(A_q - L_q C_q)P_q^{-1}, \tag{26}$$

and

$$\gamma_q^+ = [(P_q^+\overline{d}(k) + P_q^-\overline{d}(k)) - P_q d(k)] + P_q L_q v(k) + |P_q L_q| E_p V. \tag{27}$$

Similarly, the dynamics of E_q^- are given by:

$$E_q^-(k+1) = P_q x(k+1) - P_q\hat{x}^-(k+1) = F_q E_q^-(k) + \gamma_q^-, \tag{28}$$

with

$$\gamma_q^- = [P_q d(k) - (-P_q^+\overline{d}(k) - P_q^-\overline{d}(k))] - P_q L_q v(k) + |P_q L_q| E_p V. \tag{29}$$

Bearing in mind Lemma 1, then

$$-P_q^+\overline{d}(k) - P_q^-\overline{d}(k) \leq P_q d(k) \leq P_q^+\overline{d}(k) + P_q^-\overline{d}(k). \tag{30}$$

By using (30) we have $[P_q d(k) - (-P_q^+\overline{d}(k) - P_q^-\overline{d}(k))] \geq 0$, $[(P_q^+\overline{d}(k) + P_q^-\overline{d}(k)) - P_q d(k)] \geq 0$. Also, with Assumption 3, we obtain $-P_q L_q v(k) + |P_q L_q| E_p V \geq 0$, $P_q L_q v(k) + |P_q L_q| E_p V \geq 0$, then we deduce that $\gamma_q^- \geq 0$ and $\gamma_q^+ \geq 0, \forall k \geq 0$.

Moreover we have $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then $E_q^-(0)$ and $E_q^+(0)$ are nonnegative. As $F_q = P_q(A_q - L_q C_q)P_q^{-1}$ is nonnegative, we deduce that $E_q^-(k) \geq 0$ and $E_q^+(k) \geq 0$, $\forall k \geq 0$.

Consequently, we obtain

$$P_q\hat{x}^-(k) \leq P_q x(k) \leq P_q\hat{x}^+(k). \tag{31}$$

Then, it can be verified that

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k),$$

where

$$\begin{cases} \underline{x}(k) = Q_q^+ P_q\hat{x}^-(k) - Q_q^- P_q\hat{x}^+(k), \\ \overline{x}(k) = Q_q^+ P_q\hat{x}^+(k) - Q_q^- P_q\hat{x}^-(k). \end{cases}$$

To prove the boundedness of $\underline{x}, \overline{x}$, we have to show that $E_q^-(k)$ and $E_q^+(k)$ are bounded, which comes to prove that $e^+(k) = \hat{x}^+(k) - x(k)$ and $e^-(k) = x(k) - \hat{x}^-(k)$ are bounded.

As a common Lyapunov function, we propose:

$$V(e^+(k)) = e^+(k)^T M e^+(k), \; M \in \mathbb{R}^{n\times n} = M^T \succ 0, \tag{32}$$

where

$$e^+(k) = \hat{x}^+(k) - x(k). \tag{33}$$

And

$$V(e^-(k)) = e^-(k)^T M e^-(k), \; M \in \mathbb{R}^{n\times n} = M^T \succ 0, \tag{34}$$

with

$$e^-(k) = x(k) - \hat{x}^-(k). \tag{35}$$

Using the same reasoning as in Section III-A, we ensure that the dynamics of the Lyapunov function are, with bounded inputs, decreasing with respect to the errors e^+ and e^-. Let us study the case of e^+ (the same steps are used to prove the boundedness of e^-).

We have

$$\Delta V(e^+(k)) = V(e^+(k+1)) - V(e^+(k)). \tag{36}$$

Thus,

$$\begin{array}{l} \Delta V(e^+(k)) = \\ e^+(k)^T(A_q - L_q C_q)^T M (A_q - L_q C_q) e^+(k) \\ +2e^+(k)^T(A_q - L_q C_q)^T M P_q^{-1}\gamma_q^+ \\ +(P_q^{-1}\gamma_q^+)^T M (P_q^{-1}\gamma_q^+) - e^+(k)^T M e^+(k). \end{array} \tag{37}$$

Bearing in mind Lemma 2:

$$\begin{array}{l} \Delta V(e^+(k)) \leq \\ e^+(k)^T \left[(A_q - L_q C_q)^T M (A_q - L_q C_q)(1 + \frac{1}{\delta_q}) - M \right] e^+(k) \\ +(1+\delta_q)(P_q^{-1}\gamma_q^+)^T M (P_q^{-1}\gamma_q^+). \end{array} \tag{38}$$

From (27), $(1+\delta_q)(P_q^{-1}\gamma_q^+)^T M (P_q^{-1}\gamma_q^+)$ is bounded. Then, it is obvious to note that the observation error e^+ is bounded if the following inequality is satisfied

$$\left[(A_q - L_q C_q)^T M (A_q - L_q C_q)(1 + \frac{1}{\delta_q}) - M \right] \prec 0.$$

Then by applying the Schur complement, due to (20), we can conclude the stability of e^+. The same arguments allow us to prove that e^- is also bounded. □

Remark 1: The second approach based on changes of coordinates is general since it is always possible to transform any real square matrix into a nonnegative form. The existence of such a transformation is not a strong assumption. For instance, it has been shown in [11] that there always exists an invertible matrix P such that in the coordinates $z(k) = Px(k)$, the matrix $E = P(A - LC)P^{-1}$ is nonnegative. In addition, it has been shown in [27] that based on the Jordan canonical form, it is always possible to transform any square constant matrix into a nonnegative form with a constant or a time-varying transformation.

Remark 2: Even by using changes of coordinates, the interval observer is designed in the original coordinates (equations (16), (17)). This approach allows one to avoid jumping of the observer state in the coordinates $z(k) = Px(k)$ and a hybrid behavior.

IV. A NUMERICAL EXAMPLE

Consider the discrete-time linear switched system:

$$\begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k), \\ y(k) = C_q x(k) + v(k), \ q \in \overline{1,2}, \end{cases} \tag{39}$$

where $d(k)$ and $v(k)$ are respectively the disturbances and the measurement noises, with $d(k) = 0.04[\sin(0.1k)\ \cos(0.2k)]^T$, $v(k) = 0.2\sin(0.1k)$,

$$A_1 = \begin{bmatrix} 0.6 & -0.6 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.7 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.3 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

It was not possible to compute a common matrix P such as $P(A_q - L_q C_q)P^{-1}$ to be nonnegative. Therefore, changes of coordinates matrices P_q such that the matrices $P_q(A_q - L_q C_q)P_q^{-1}$ are nonnegative, are computed. The next step is to design for the system (39) the observer defined by (16). Then we have to solve the LMI given by (20) with the *Yalmip* toolbox; as a result, we have:

$$L_1 = \begin{bmatrix} 0.7308 \\ -0.0557 \end{bmatrix}, \ L_2 = \begin{bmatrix} -0.0513 \\ -0.1117 \end{bmatrix},$$
$$M = \begin{bmatrix} 0.9908 & -0.0893 \\ -0.0893 & 1.2006 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 0.7290 \\ -0.1322 \end{bmatrix}, \ S_2 = \begin{bmatrix} -0.0409 \\ -0.1296 \end{bmatrix},$$
$$\delta_q = 90.9091, \forall q \in \overline{1,2}.$$

The matrices P_q, ensuring that $P_q(A_q - L_q C_q)P_q^{-1}$ are nonnegative, are given by:

$$P_1 = \begin{bmatrix} -0.0850 & 1.2136 \\ 0.0850 & -0.2136 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.2634 & -0.1359 \\ -0.2634 & 1.1359 \end{bmatrix}.$$

As mentioned above, all conditions of Theorem 3 are verified, then, the interval observer given by (17) is ISS stable. The results of simulations of the interval observer are depicted in Figure 1, where solid lines represent the actual state and dashed lines represent the estimated bounds.

The signal which governs the switching between the two subsystems is plotted in Figure 2.

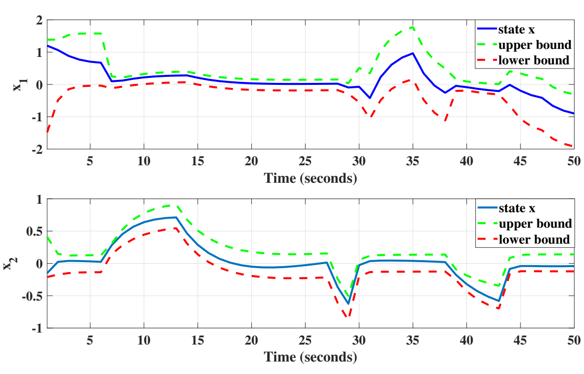

Fig. 1. State and estimate bounds

The simulations show that, despite the presence of the disturbances and of the measurement noises, the state belongs inside the interval formed by the upper and the lower trajectories. The interval observer stability is ensured. Finally, as shown in Figure 1, the interval observer remains stable despite the switching instants.

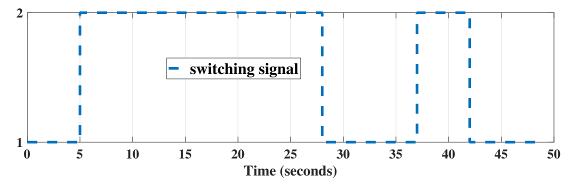

Fig. 2. Switching signal

V. CONCLUSION

In this paper, two techniques to design interval observers for a class of discrete-time linear switched systems in the presence of additive disturbances and measurement noises are proposed. The assumptions given in the first one are not always feasible. Therefore, a second approach based on changes of coordinates is proposed to relax the condition of nonnegativeness of $A_q - L_q C_q$, $q = \overline{1,N}$. In this context, two copies of classical observers associated with suitably selected initial conditions are reformulated in the coordinates "x". The observer gains can be computed in term of LMIs. Two tracks can be considered as perspectives to this work. The first one, is the use of the H_∞ formalism to compute optimal gains, by using this technique we can obtain a tighter framer. The second perspective, is the synthesis of interval observers for switched systems with unknown switching instants.

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