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Interval observers for global feedback control of nonlinear systems with robustness with respect to disturbances☆

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ABSTRACT

This paper develops criteria for designing interval observers to guarantee robustness with respect to disturbances for feedback control of nonlinear systems. Intervals in which components of the state vector are guaranteed to stay are estimated based on the information of the range of the initial state and the disturbances. For formulating desirable properties of boundedness and convergence of estimated intervals in the presence of disturbances, the notion of integral input-to-state stability is introduced to interval observer design. Guaranteed properties of the observer-based feedback designed in the formulation are not only global, but also address nonlinearities which are broader than those covered by previously existing approaches.

1. Introduction

The Luenberger observer and similar traditional observers estimate the state variables of a system from its input-output data. The estimation is undoubtedly useful for feedback control purposes. The Luenberger-type observers are stable mechanisms which give estimates of the state vectors as time tends to infinity. Due to the stable mechanisms, this asymptotic estimation remains valid in the presence of sufficiently small errors in system parameters. However, there is no guarantee during transient periods that the Luenberger-type observers give useful information of the unmeasured state. Indeed, they cannot provide us with any readily usable estimate when system parameters or disturbances are changing or large. In applications, another important demand in estimation is to monitor and detect faults of systems. Usefulness of the Luenberger-type observers is limited when some guarantees for monitoring and detection are needed in transient periods.

About two decades ago, the notion of interval observers was introduced as a new paradigm for monitoring unmeasured variables all times in the presence of large and fluctuating disturbances [11]. Interval observers produce component-wise upper and lower bounds of state vectors of considered systems at every instant. In the absence of disturbances, interval observers guarantee the difference between the upper and lower bounds to converge to zero. Such state estimators without the convergence are called framers. Framers and interval observers belong to a specific class of estimation methods called guaranteed state estimation methods. The capability of coping with large uncertainties has been useful for biological models, and framers and interval observers have been successfully applied to many real-life problems (see e.g., [1,5,10,19] and references therein). Designing framers and interval observers has been studied for both continuous-time and discrete-time systems. Some works are devoted to various classes of finite or infinite-dimensional linear systems [8,17,18,20,22], and others concern some classes of nonlinear systems [21,23,25,26].

Recently, in the context of systems with control input and output measurement, an interval observer has been proposed for a class of nonlinear systems which are affine in the unmeasured part of the state variables [7]. Compared with other approaches, it employs a simpler structure consisting of two modified Luenberger observers. A sufficient condition under which the constructed simple structure is guaranteed to function as an interval observer globally is presented there. Notably, the modified Luenberger-type construction enables us to use the interval observer for global feedback control as in the ordinary observer case. Thus, a single interval observer can play both roles of control and monitoring in a simple way. While continuous-time measurement is assumed

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in [7], an interval observer can be constructed with discrete-time measurement [6]. The technique employed for the discrete-time measurement is, however, basically effective only for strongly limited bilinearlities, and generalizing the result to nonlinearities covered by the continuous-time measurement case is not trivial. Although continuous-time measurement allows one to cover a larger class of nonlinear systems in feedback control, nonlinearities satisfying the sufficient condition proposed in [7] are not satisfactorily broad. The sufficient condition requires the feedback control input to be mild in accordance with the observer, which restricts the use of nonlinear damping. The necessity of global Lipschitzness assumed in [7] is not clear. Moreover, the convergence of the difference between the upper and lower bounds is guaranteed to converge to zero only when disturbances are identically zero.

The present paper continues to study the class of nonlinear control systems tackled in [7] with continuous-time measurement, and aims at guaranteeing the convergence of the difference between the bounds to zero in the presence of converging disturbances. This paper demonstrates that exploiting iISS in interval observer design not only opens a new door helping the observer proposed in [7] be more powerful, but also allows one to relax assumptions on inputs for feedback control. A preliminary result leading to the main theorems in this paper was reported in [15], where a preliminary idea of achieving iISS/ISS properties is presented as additional discussions without proofs.

Notation

The set of real numbers is denoted by $\mathbb{R}$. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$, i.e., $\mathbb{R}_{\geq 0} := [0, \infty)$. The symbol I denotes the identity matrix in $\mathbb{R}^{n\times n}$ of any dimension n. A square matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive definite and written as $M \succ 0$ if $v^\top M v > 0$ holds for all $v \in \mathbb{R}^n - \{0\}$. The symbol $|\cdot|$ denotes Euclidean norm of vectors of any dimension. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, \dots, x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, \dots, x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \{1, \dots, n\}$, $x_{a,\,i} \leq x_{b,\,i}$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, let the matrix $Q^+ \in \mathbb{R}^{n\times n}$ denote $Q^+ = \left(\max\{q_{i,j}, 0\}\right)_{i,j=1,1}^{n,n}$, where the notation $Q = \left(q_{i,j}\right)_{i,j=1,1}^{n,n}$ is used. Let $Q^- \in \mathbb{R}^{n\times n}$ be defined by $Q^- = Q^+ - Q$. This notation is limited to square matrices, and the superscripts + and - for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be Metzler if each off-diagonal entry of this matrix is nonnegative. For functions $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, by $\alpha(s) \equiv \beta(s)$ we mean $\alpha(s) = \beta(s)$ for all $s \in \mathbb{R}_{\geq 0}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be positive definite and written as $\alpha \in \mathcal{P}$ if α is continuous and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s \in (0, \infty)$. A function $\alpha \in \mathcal{P}$ is said to be of class $\mathcal{K}$ if α is strictly increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. In this paper, ess sup is written as sup for brevity.

2. Problem setup and definitions

Consider the system

$$\dot{x}(t) = A(y(t))x(t) + \beta(y(t), u(t)) + \delta(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

with time $t \in \mathbb{R}_{\geq 0}$, the state $x(t) \in \mathbb{R}^n$, the output $y(t) \in \mathbb{R}^p$, the input $u(t) \in \mathbb{R}^q$ and the initial condition $x(0) = x_0$, where $C \in \mathbb{R}^{p\times n}$ is a constant matrix. The functions $A : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ and $\beta : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ are supposed to be locally Lipschitz. Let $\alpha(x) := A(Cx)x$. The disturbance vector $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is supposed to be any Lebesgue measurable locally essentially bounded function, which is denoted as $\delta \in \mathcal{D}$. The input u is supposed to belong to $\mathcal{U}$ which denotes the set of piecewise continuous functions. The maximal open subinterval (of $\mathbb{R}_{\geq 0}$) in which the unique $x(t)$ exists is denoted by $[0, T_{x_0,u,\delta})$. That means $T_{x_0,u,\delta} = \sup\{t \in \mathbb{R}_{\geq 0} : |x(t)| < \infty\}$ is the escape time for given x_0, u and δ. The solution $x(t)$ does not escape if $T_{x_0,u,\delta} = \infty$. In fact, system (1b) is said to be forward complete [3] if $T_{x_0,u,\delta} = \infty$ holds for any $x_0 \in \mathbb{R}^n$, any $\delta \in \mathcal{D}$ and any $u \in \mathcal{U}$.

Consider

$$\dot{z}(t) = f(z(t), y(t), u(t), \delta^+(t), \delta^-(t)) \tag{2}$$

$$x^+(t) = h^+(z(t)), \quad x^-(t) = h^-(z(t)) \tag{3}$$

defined with the dimensions $z(t) \in \mathbb{R}^{2n}$, $x^+(t) \in \mathbb{R}^n$ and $x^-(t) \in \mathbb{R}^n$ for the initial condition

$$z(0) := z_0 = g(x_0^+, x_0^-), \tag{4}$$

where the function $f : \mathbb{R}^{2n} \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n}$ is locally Lipschitz, and the functions h^+, $h^- : \mathbb{R}^{2n} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n}$ are continuous. The variables $\delta^+(t)$, $\delta^-(t) \in \mathbb{R}^n$ in (2) and the constants x_0^+ and $x_0^- \in \mathbb{R}^n$ in (4) are used to define the following notions:

Definition 1. Let the vectors $x_0^-, x_0^+ \in \mathbb{R}^n$ and the piecewise continuous functions $\delta^+, \delta^- : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be such that

$$x_0^- \leq x_0^+ \tag{5}$$

$$\delta^-(t) \leq \delta^+(t), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{6}$$

Then the system consisting of (2)–(4) is called

(i) a framer for (1) if a unique solution $z(t)$ to (2) exists in the interval $[0, T_{x_0,u,\delta})$ and satisfies

$$x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \in [0, T_{x_0,u,\delta}) \tag{7}$$

for any $u \in \mathcal{U}$, $x_0 \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$ satisfying

$$x_0^- \leq x_0 \leq x_0^+ \tag{8}$$

$$\delta^-(t) \leq \delta(t) \leq \delta^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \quad \text{(a.e.)}. \tag{9}$$

(ii) an interval observer for (1) if it is a framer, and satisfies the implication

$$T_{x_0,u,\delta} = \infty \;\Rightarrow\; \lim_{t\to\infty} |x^+(t) - x^-(t)| = 0 \tag{10}$$

for any $u \in \mathcal{U}$ and $x_0 \in \mathbb{R}^n$ satisfying (8) in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

The observer candidate (2)–(4) is not allowed to use $x(t)$ and $\delta(t)$ which are unknown. Instead, x_0^-, x_0^+, $\delta^-(t)$ and $\delta^+(t)$ are supposed to be known and allowed to be used. The above definition is basically the same as the one employed in [7]. This paper, however, does not exclude $T_{x_0,u,\delta} < \infty$.

Following the popular definition in the literature (see e.g., [7] and references therein), for being an interval observer, Definition 1 requires the convergence (10) for only $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$, although property (7) needs to be guaranteed for non-zero disturbances. As known in observer theory, requiring the convergence even only for zero disturbances is still practically meaningful since the convergent mechanism allows one to get rid of an open loop simulator which can exhibit no robustness. Indeed, even for a linear system, an arbitrarily small displacement in an open loop simulator can result in unbounded errors in estimation.

In the nonlinear case, robustness of estimation with respect to disturbances is not obvious even if the estimation is not open-loop. To evaluate robustness of an observer with respect to disturbances, we introduce the following vectors:

$$z(t)=\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad z_1(t), z_2(t) \in \mathbb{R}^n$$

$$Z_i(t)=\begin{bmatrix} z_i(t)-x(t) \\ x^+(t)-x^-(t) \end{bmatrix}, \ \hat{\rho}(t)=\begin{bmatrix} \delta^+(t)-\delta(t) \\ \delta(t)-\delta^-(t) \end{bmatrix}$$

The following notions are employed:

Definition 2. The system consisting of (2)–(4) is called

(i) an ISS interval observer for (1) if it is a framer and there exist $\hat{\theta} \in \mathcal{KL}$ and $\hat{\phi} \in \mathcal{K}$ such that

$$|Z_i(t)| \le \hat{\theta}(|Z_i(0)|, t) + \hat{\phi}\left(\sup_{\tau \in [0,t]} |\hat{\rho}(\tau)| \right), \ \forall t \in \mathbb{R}_{\ge 0}, \ i = 1, 2 \tag{11}$$

is guaranteed for any $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\delta \in \mathcal{D}$ satisfying (9) and $T_{x_0,u,\delta} = \infty$.

(ii) an iISS interval observer for (1) if it is a framer and there exist $\hat{\theta} \in \mathcal{KL}$, $\hat{\psi} \in \mathcal{K}$ and $\hat{\chi} \in \mathcal{K}_\infty$ such that

$$\hat{\chi}(|Z_i(t)|) \le \hat{\theta}(|Z_i(0)|, t) + \int_0^t \hat{\psi}(|\hat{\rho}(\tau)|) d\tau, \ \forall t \in \mathbb{R}_{\ge 0}, \ i = 1, 2 \tag{12}$$

is guaranteed for any $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\delta \in \mathcal{D}$ satisfying (9) and $T_{x_0,u,\delta} = \infty$.

Next, we consider the observer-based feedback control system consisting of (1), (2)–(4) and the control law

$$u(t) = u_s(y(t), z(t)), \tag{13}$$

where $u_s : \mathbb{R}^p \times \mathbb{R}^{2n} \to \mathbb{R}^q$ is a locally Lipschitz function. Stability properties of the feedback control system in the presence of disturbances are defined with

$$X(t)=\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \Delta(t)=\begin{bmatrix} \delta(t) \\ \delta^+(t) \\ \delta^-(t) \end{bmatrix}$$

as follows:

Definition 3. The entire control system consisting of (1), (2)–(4) and (13) is said to be

(i) 0-GAS if $[x^\top, z^\top]^\top = 0$ is globally asymptotically stable in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$.

(ii) ISS with restriction (9) if there exist $\theta \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that

$$|X(t)| \le \theta(|X(0)|, t) + \phi\left(\sup_{\tau \in [0,t]} |\Delta(\tau)| \right), \quad \forall t \in \mathbb{R}_{\ge 0} \tag{14}$$

is guaranteed for any $x_0 \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$ satisfying (9).

(iii) iISS with restriction (9) if there exist $\theta \in \mathcal{KL}$, $\psi \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$ such that

$$\chi(|X(t)|) \le \theta(|X(0)|, t) + \int_0^t \psi(|\Delta(\tau)|) d\tau, \ \forall t \in \mathbb{R}_{\ge 0} \tag{15}$$

is guaranteed for any $x_0 \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$ satisfying (9).

Roughly speaking, the formulation in Definitions 2 and 3 is identical to the notions of ISS and iISS introduced in [27,28] except that the disturbance δ is restricted to (9). However, if one checks the definitions carefully, there is a fundamental difference regarding the possible escape time $T_{x_0,u,\delta}$. In the case of the observer, $x(t)$ is an exogenous signal so that $T_{x_0,u,\delta} = \infty$ is assumed. In the case of the feedback system, $x(t)$ is endogenous, and $T_{x_0,u,\delta} = \infty$ is not assumed. The local Lipschitzness assumed for A, β, f and u_s guarantees the existence of $T_{x_0,u,\delta} > 0$ for all $x_0 \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$. Definition 3 requires the choice of (13) to secure $T_{x_0,u,\delta} = \infty$ for all $x_0 \in \mathbb{R}^n$ and $\delta \in \mathcal{D}$ satisfying (9). In fact, either (14) or (15) implies $T_{x_0,u,\delta} = \infty$. In the case of 0-GAS, it is replaced by $T_{x_0,u,0} = \infty$.

Due to the second element of Z_i, the function $\hat{\theta} \in \mathcal{KL}$ in (11) and (12) implies the convergence (10) of the estimated interval for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. Hence, an ISS interval observer is an interval observer. An iISS interval observer is an interval observer. The first element of Z_i with $\hat{\theta} \in \mathcal{KL}$ in (11) and (12) implies that either $z_1(t)$ or $z_2(t)$ serves as an asymptotic estimate of the state $x(t)$ for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$ in view of the standard definition of observers[1]. Therefore, the feedback input (13) is in the spirit of certainty equivalence if (2)–(4) is an ISS or iISS interval observer.

For linear systems, 0-GAS implies convergence of state variables to their unique equilibria even in the presence of disturbance signals as long as the disturbance signals are convergent. For nonlinear systems, one cannot expect this property. Indeed, state variables of a 0-GAS system can be unbounded or escape in finite time even for converging disturbances. Definition 2 is motivated by this point and introduced for securing reasonable usefulness of interval observers for nonlinear systems. In fact, as stated in the next proposition, it can be verified that the argument used in [14, Proof of Proposition 6] is applicable to (11) and (12), although the input magnitude is restricted.

Proposition 1.

(i) *If the system consisting of (2)–(4) is an ISS interval observer for (1), then*

$$\lim_{t\to\infty} |\hat{\rho}(t)| = 0 \ \Rightarrow \ \lim_{t\to\infty} |Z_i(t)| = 0, \ i = 1, 2 \tag{16}$$

holds for any $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\delta \in \mathcal{D}$ satisfying (9) and $T_{x_0,u,\delta} = \infty$.

(ii) *If the system consisting of (2)–(4) is an iISS interval observer for (1), then*

$$\int_0^\infty \hat{\psi}(|\hat{\rho}(\tau)|) d\tau < \infty \ \Rightarrow \ \lim_{t\to\infty} |Z_i(t)| = 0, \ i = 1, 2 \tag{17}$$

holds for any $x_0 \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\delta \in \mathcal{D}$ satisfying (9) and $T_{x_0,u,\delta} = \infty$, where $\hat{\psi}$ is a class $\mathcal{K}$ function satisfying (12).

Property (16) implies the convergence (10) and the convergence $\lim_{t\to\infty} |z_i(t) - x(t)| = 0$ even in the presence of disturbances as long as the disturbances are vanishing. Property (17) implies these convergence properties when the disturbances are vanishing sufficiently fast.

Remark 1. This paper defines the framer property (7) only up to the escape time $T_{x_0,u,\delta}$. The asymptotic property is stated as (10) accordingly. Indeed, requiring a property for an infinite time span is demanding since we not only consider the nonlinear plant (1), but also take into account disturbance δ. The interval $[0, T_{x_0,u,\delta})$ in (7) can always be replaced by the infinite time span $\mathbb{R}_{\ge 0}$ if (1) and (2) are linear. Assuming "global" Lipschitzness (i.e., α, β and u_s are globally Lipschitz) is a typical alternative to the linearity. Otherwise, one needs to assume that x, y and u are somehow restricted to compact sets for securing $T_{x_0,u,\delta} = \infty$ [9,31] unless the assumption $T_{x_0,u,\delta} = \infty$ is stated directly. The notion of input-to-output stability employed in [27,30] is useful for stating

[1] In Definition 2 stated for the general (2)–(4) both Z_1 and Z_2 are required to be checked. This requirement will lead to a particular choice of (2)–(4) for which one of Z_1 and Z_2 becomes redundant. In fact, for that particular choice in the next section, (11) (resp., (12)) holds for $i = 1$ if and only if (11) (resp., (12)) is satisfied for $i = 2$.

the assumption $T_{x_0,u\delta} = \infty$ on (1) in the presence of essentially bounded δ with respect to each given u, i.e., for formulating the observer design. Instead of mere observation, this paper focuses on feedback control addressing bounds over the infinite time interval and asymptotic properties of state variables. Hence, this paper uses (i)ISS which is stronger than input-to-output-type notions.

3. Main results

This section proposes an iISS-based approach to designing interval observers for feedback control. Its capability is demonstrated in the framework of Definitions 1 and 2.

Let $R \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and $S = R^{-1}$. Consider the differential equations

$$\dot{\hat{x}}^+ = A(y)\hat{x}^+ + \beta(y,u) + \Lambda(y)[C\hat{x}^+ - y] + S[R^+\delta^+ - R^-\delta^-] \quad (18a)$$

$$\dot{\hat{x}}^- = A(y)\hat{x}^- + \beta(y,u) + \Lambda(y)[C\hat{x}^- - y] + S[R^+\delta^- - R^-\delta^+] \quad (18b)$$

defined with the initial conditions

$$\hat{x}^+(0) = \hat{x}_0^+ := S[R^+x_0^+ - R^-x_0^-] \quad (19a)$$

$$\hat{x}^-(0) = \hat{x}_0^- := S[R^+x_0^- - R^-x_0^+] \quad (19b)$$

and the output equations

$$x^+ = S^+R\hat{x}^+ - S^-R\hat{x}^- \quad (20a)$$

$$x^- = S^+R\hat{x}^- - S^-R\hat{x}^+. \quad (20b)$$

Suppose that the feedback law (13) is in the form that there exists $\hat{u}_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ for which

$$u_s(y(t), z(t)) = \hat{u}_s(y(t),\ \ell\hat{x}^+(t) + (1-\ell)\hat{x}^-(t)), \quad \forall t \in \mathbb{R}_{\geq 0} \quad (21)$$

holds for a constant $\ell \in [0,\ 1]$. For example, the choice $\ell = 1$ represents $u_s(y(t), z(t)) = \hat{u}_s(y(t),\ \hat{x}^+(t))$, which was used in [7]. Eqs. (18)–(20) are in the form of (2)–(4) with $z_1 = \hat{x}^+$ and $z_2 = \hat{x}^-$. Eqs. (18)–(20) are proposed in [7][2] as an interval observer candidate. By employing the same structure of observers, this paper proposes a design method with new guarantees which are less restrictive and more capable. To this end, we first borrow two assumptions from [7].

Assumption 1. Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exists an invertible matrix $R \in \mathbb{R}^{n\times n}$ such that, for all $y \in \mathbb{R}^p$, the matrix

$$\Gamma(y) = R[A(y) + \Lambda(y)C]S \quad (22)$$

is Metzler.

Assumption 2. Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}, \overline{\nu} \in \mathcal{K}_\infty$ and $\omega \in \mathcal{P}$ such that

$$\underline{\nu}(|\xi|) \leq V(\xi) \leq \overline{\nu}(|\xi|) \quad (23)$$

$$\frac{\partial V}{\partial \xi}(\xi)[A(y) + \Lambda(y)C]\xi \leq -\omega(|\xi|) \quad (24)$$

hold for all $\xi \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$.

This paper introduces several iISS-type assumptions.

Assumption 3. There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathcal{P}$ and $\gamma \in \mathcal{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + \beta(Cx, \hat{u}_s(Cx, x+d))] \leq -\mu(|x|) + \gamma(|d|) \quad (25)$$

holds for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$.

The result in [7] relies on $\mu \in \mathcal{K}$, which is not necessarily assumed in this paper. This paper also employs the following two assumptions to broaden capabilities of the observer and the observer-based feedback:

Assumption 4. Given a locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, an invertible matrix $R \in \mathbb{R}^{n\times n}$, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}, \overline{\nu} \in \mathcal{K}_\infty$, $\omega \in \mathcal{P}$ and $\eta^+, \eta^- \in \mathcal{K}$ such that (23) and

$$\frac{\partial V}{\partial \xi}(\xi)\left\{[A(y) + \Lambda(y)C]\xi + S[R^+\rho^+ + R^-\rho^-]\right\}$$
$$\leq -\omega(|\xi|) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|) \quad (26)$$

hold for all $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $\rho^+ \in \mathbb{R}^n$ and $\rho^- \in \mathbb{R}^n$ satisfying

$$\rho_i^+, \rho_i^- \in [-D^\pm, +D^\pm] \cap \mathbb{R}, \quad i = 1, 2, \ldots, n, \quad (27)$$

where $D^\pm = \sup_{t\in\mathbb{R}_{\geq 0}} |\delta_i^+(t) - \delta_i^-(t)|$.

Assumption 5. There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathcal{P}$ and $\gamma, \zeta \in \mathcal{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + \beta(Cx, \hat{u}_s(Cx, x+d)) + \delta]$$
$$\leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \quad (28)$$

holds for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^n$ satisfying

$$\delta_i \in \left[\inf_{t\in\mathbb{R}_{\geq 0}} \delta_i^-(t),\ \sup_{t\in\mathbb{R}_{\geq 0}} \delta_i^+(t)\right] \cap \mathbb{R}, \quad i = 1, 2, \ldots, n. \quad (29)$$

Note that by definition, the fulfillment of Assumption 4 (resp. Assumption 5) implies that Assumption 2 (resp. Assumption 3) is satisfied. Readers who are familiar with [2,4] can notice that (25), (26) and (28) are in the form of dissipation inequalities characterizing iISS of some systems to be utilized later. Assumption 2 requires the pair (A, C) to be detectable when (1) is linear. In this sense, Assumption 2 plays the role of an observability-type condition. Assumption 4 estimates robustness of the observability-type condition with respect to disturbances. Assumption 3 requires u_s to be a fictitious static state-feedback law stabilizing the plant (1), i.e., implementing the idea of certainty equivalence. Some robustness of the feedback law with respect to disturbances is estimated by Assumption 5. The intersection with respect to $\mathbb{R}$ is taken in (27) and (29) to exclude $\pm\infty$ which is unnecessary.

Define

$$\eta := \eta^+ + \eta^-, \quad \delta^\pm := \delta^+ - \delta^-$$

$$\hat{\eta} = \begin{cases} \eta, & \ell \in \{0, 1\} \\ 2\eta, & \text{otherwise} \end{cases}$$

$$\hat{\omega}(s) = \begin{cases} \omega \circ \overline{\nu}^{-1}(s), & \ell \in \{0, 1\} \\ \min\limits_{r\in[0,s]} \omega \circ \overline{\nu}^{-1}(r) + \omega \circ \overline{\nu}^{-1}(s-r), & \text{otherwise.} \end{cases}$$

We are now in a position to state main results.

Theorem 1. *If Assumption 1 is satisfied, the following statements hold true:*

(i) *If Assumption 2 is satisfied, the system (18)–(20) is an interval observer for (1).*

[2] The development was inspired by [25].

(ii) *If Assumption 4 is satisfied, the system (18)–(20) is an iISS interval observer for (1).*

(iii) *If Assumption 4 is satisfied with*

$$\omega \in \mathcal{K}_\infty \vee \liminf_{s\to\infty} \omega(s) \geq \sup_{t\in\mathbb{R}_{\geq 0}} \eta(\sqrt{2}|\delta^{\pm}(t)|), \tag{30}$$

then the system (18)–(20) is an ISS interval observer for (1),

Theorem 2. *If Assumption 1 is satisfied, the following statements hold true:*

(i) *If Assumptions 2, 3 and $\mu \in \mathcal{K}$ are satisfied, then the entire control system consisting of (1), (18)–(20) and (21) is 0-GAS.*

(ii) *If Assumptions 2, 3 and*

$$\int_0^1 \frac{\gamma \circ \underline{\nu}^{-1}(s)}{\hat{\omega}(s)} ds < \infty \tag{31}$$

are satisfied, then the entire control system is 0-GAS.

(iii) *Assumptions 4 and 5, $\mu \in \mathcal{K}$ and*

$$\gamma \notin \mathcal{K}_\infty \vee \omega \in \mathcal{K}_\infty \vee \liminf_{s\to\infty} \omega(s) \geq \sup_{t\in\mathbb{R}_{\geq 0}} \hat{\eta}(\sqrt{2}|\delta^{\pm}(t)|) \tag{32}$$

are satisfied, then the entire control system is iISS with restriction (9).

(iv) *If Assumptions 4 and 5, $\omega \in \mathcal{K}$ and*

$$\exists c > 0, k \geq 1 \text{ s.t. } \ c\gamma \circ \underline{\nu}^{-1}(s) \leq [\hat{\omega}(s)]^k, \forall s \in \mathbb{R}_{\geq 0} \tag{33}$$

are satisfied, then the entire control system is iISS with restriction (9).

(v) *Assumptions 4 and 5 and*

$$\mu \in \mathcal{K}_\infty \wedge \left\{ \omega \in \mathcal{K}_\infty \vee \liminf_{s\to\infty} \omega(s) \geq \sup_{t\in\mathbb{R}_{\geq 0}} \hat{\eta}(\sqrt{2}|\delta^{\pm}(t)|) \right\} \tag{34}$$

are satisfied, then the entire control system is ISS with restriction (9).

Items (ii) and (iv) in Theorem 2 get rid of the requirement $\mu \in \mathcal{K}$ and allow the convergence rate μ to be radially vanishing at the cost of imposing a growth order constraint on the coupling between the observer (18) and the plant (1). Theorem 2 not only demonstrates this relaxation, but also clarifies its price in view of stability guarantees.

4. Proofs

4.1. Proof of Theorem 1

Consider the change of coordinates

$$\overline{\epsilon} = \hat{x}^+ - x, \quad \underline{\epsilon} = \hat{x}^- - x. \tag{35}$$

Due to (20), we have

$$x^+ - x^- = (S^+ + S^-)R(\hat{x}^+ - \hat{x}^-) = (S^+ + S^-)R(\overline{\epsilon} - \underline{\epsilon}). \tag{36}$$

From (1) and (18) we obtain

$$\dot{\overline{\epsilon}} = [A(y) + \Lambda(y)C]\overline{\epsilon} + S[R^+\delta^+ - R^-\delta^-] - \delta$$
$$\dot{\underline{\epsilon}} = [A(y) + \Lambda(y)C]\underline{\epsilon} + S[R^+\delta^- - R^-\delta^+] - \delta.$$

By virtue of $SR = I$ and $R = R^+ - R^-$ it holds that

$$\dot{\overline{\epsilon}} = [A(y) + \Lambda(y)C]\overline{\epsilon} + S[R^+\hat{\rho}^+ + R^-\hat{\rho}^-] \tag{37a}$$

$$\dot{\underline{\epsilon}} = [A(y) + \Lambda(y)C]\underline{\epsilon} - S[R^+\hat{\rho}^- + R^-\hat{\rho}^+], \tag{37b}$$

where $\hat{\rho}^+ = \delta^+ - \delta$ and $\hat{\rho}^- = \delta - \delta^-$. Hence, by virtue of Assumption 1, it can be verified as in [7] that equations (18)–(20) with the restrictions (8) and (9) achieve (7) for the system (1) for any $u \in \mathcal{U}$.

(i) Consider the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. Suppose $T_{x_0,u,\delta} = \infty$. Then applying Assumption 2 to (37a) and (37b) separately implies $\lim_{t\to\infty} |\overline{\epsilon}(t)| = 0$ and $\lim_{t\to\infty} |\underline{\epsilon}(t)| = 0$. Property (10) follows from (36).

(ii) Let $\delta \in \mathcal{D}$ be arbitrary under the constraint (9). Suppose that Assumption 4 and $T_{x_0,u,\delta} = \infty$ hold. Then $y(t)$ is guaranteed to be finite for all $t \in \mathbb{R}_{\geq 0}$, and system (37) is a time-varying system with the inputs $\hat{\rho}^+$ and $\hat{\rho}^-$. Regard ρ^+ and ρ^- in (26) as $\rho^+ = \hat{\rho}^+$ and $\rho^- = \hat{\rho}^-$. Since (26) is satisfied uniformly in y, the argument presented in [4] proves that system (37a) is iISS with respect to the state $\overline{\epsilon}$ and the input $\hat{\rho}$, provided that (9) holds. In the same way, regarding ρ^+ and ρ^- in (26) as $\rho^+ = -\hat{\rho}^-$ and $\rho^- = -\hat{\rho}^+$, property (26) yields iISS of system (37b) with respect to the state $\underline{\epsilon}$ and the input $\hat{\rho}$. Hence, there exist $\tilde{\theta} \in \mathcal{KL}$, $\tilde{\psi} \in \mathcal{K}$ and $\tilde{\chi} \in \mathcal{K}_\infty$ such that

$$\tilde{\chi}(|\mathcal{E}(t)|) \leq \tilde{\theta}(|\mathcal{E}(0)|, t) + \int_0^t \tilde{\psi}(|\hat{\rho}(\tau)|)d\tau, \quad \forall t \in \mathbb{R}_{\geq 0} \tag{38}$$

holds for any $x_0 \in \mathbb{R}^n$, where $\mathcal{E} = [\overline{\epsilon}^\top, \underline{\epsilon}^\top]^\top$. Now, recalling (36) we have

$$Z_1(t) = \begin{bmatrix} I & 0 \\ 0 & (S^+ + S^-)R \end{bmatrix} \begin{bmatrix} \hat{x}^+(t) - x(t) \\ \hat{x}^+(t) - \hat{x}^-(t) \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ (S^+ + S^-)R & -(S^+ + S^-)R \end{bmatrix} \begin{bmatrix} \overline{\epsilon}(t) \\ \underline{\epsilon}(t) \end{bmatrix}. \tag{39}$$

From (19) we also obtain

$$\mathcal{E}(0) = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} \hat{x}^+(0) - x_0 \\ \hat{x}^+(0) - \hat{x}^-(0) \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ I & -S(R^+ + R^-) \end{bmatrix} Z_1(0). \tag{40}$$

Due to (39) and (40), there exist $a > 0$ and $b > 0$ such that

$$a|Z_1(t)| \leq |\mathcal{E}(t)|, \quad \forall t \in \mathbb{R}_{\geq 0} \tag{41}$$

$$|\mathcal{E}(0)| \leq b|Z_1(0)|. \tag{42}$$

Defining

$$\hat{\chi}(s) = \tilde{\chi}(as), \quad \hat{\psi}(s) = \tilde{\psi}(s), \quad \forall s \in \mathbb{R}_{\geq 0} \tag{43}$$

$$\hat{\theta}(s, t) = \tilde{\theta}(bs, t), \quad \forall s, t \in \mathbb{R}_{\geq 0} \tag{44}$$

property (38) yields (12) for $i = 1$. A similar argument also yields (12) for $i = 2$. Therefore, the system consisting of (18)–(20) is an iISS interval observer.

(iii) Let $\delta \in \mathcal{D}$ be an arbitrary function satisfying (9). Suppose that Assumption 4 and $T_{x_0,u,\delta} = \infty$ hold. Substituting $\eta^+(|\rho^+|) + \eta^-(|\rho^-|) \leq \eta(|\rho|)$, $\rho = [\rho^{+\top}, \rho^{-\top}]^\top \in \mathbb{R}^{2n}$ into (26) gives

$$\frac{\partial V}{\partial \xi}(\xi)\left\{[A(y) + \Lambda(y)C]\xi + S[R^+\rho^+ + R^-\rho^-]\right\} \leq -\omega(|\xi|) + \eta(|\rho|). \tag{45}$$

Let $\rho^+ = \hat{\rho}^+$ and $\rho^- = \hat{\rho}^-$ for (37a), and $\rho^+ = -\hat{\rho}^-$ and $\rho^- = -\hat{\rho}^+$ for (37b). From (9), $\delta^{\pm} = \delta^+ - \delta^-$, $\hat{\rho}^+ = \delta^+ - \delta$ and $\hat{\rho}^- = \delta - \delta^-$, it follows that $|\rho(t)| \leq \sqrt{2}|\delta^{\pm}(t)|$ holds for all $t \in \mathbb{R}_{\geq 0}$. Thus, applying this to (45) yields

$$\dot{V}_{(37a)} \leq -\omega(|\overline{\epsilon}|) + \eta(\sqrt{2}|\delta^{\pm}|) \tag{46a}$$

$$\dot{V}_{(37b)} \leq -\omega(|\underline{\epsilon}|) + \eta(\sqrt{2}|\delta^{\pm}|). \tag{46b}$$

Here, where $\dot{V}_{(37a)}$ (resp., $\dot{V}_{(37a)}$) denotes time-derivative of V along the solution of (37a) (resp., (37b)). According to [29], with the help of (30), property (46a) implies that system (37a) is ISS with the state $\overline{\epsilon}$ and the input $\hat{\rho}$, provided that (9) holds. In the same way, property (46b) implies ISS of system (37b) with the state $\underline{\epsilon}$ and the input $\hat{\rho}$ under (9). This fact leads to the existence of $\tilde{\theta} \in \mathcal{KL}$ and $\tilde{\phi} \in \mathcal{K}$ satisfying

$$|\mathcal{E}(t)| \le \tilde{\theta}(|\mathcal{E}(0)|, t) + \tilde{\phi}\left(\sup_{\tau\in[0,t]} |\hat{\rho}(\tau)|\right), \quad \forall t \in \mathbb{R}_{\ge 0} \tag{47}$$

for any $x_0 \in \mathbb{R}^n$ and any δ satisfying (9). Defining $\hat{\phi}(s) \equiv \tilde{\phi}(s)$ and (44), property (47) yields (11) for $i = 1$. Since (11) is also verified for $i = 2$ in a similar manner, the system consisting of (18)–(20) is an ISS interval observer.

4.2. Proof of Theorem 2

From (1) and (13) with (21) we obtain

$$\dot{x} = A(y)x + \beta(y, \hat{u}_s(y, x + \ell\overline{\epsilon} + (1-\ell)\underline{\epsilon})) + \delta(t) \tag{48}$$

due to $\ell\hat{x}^+ + (1-\ell)\hat{x}^- = x + \ell\overline{\epsilon} + (1-\ell)\underline{\epsilon}$. The entire system consists of (48), (37a) and (37b). The local Lipschitzness imposed on the functions in (1) and the functions Λ and u_s ensures the existence of a unique maximal solution, local in time, to the entire system. The two subsystems (48) and (37) form a cascade in which (48) is driven by (37) through $\overline{\epsilon}$ and $\underline{\epsilon}$. In the case of $\ell = 1$ in (21), the subsystem (37a) drives (48), while for $\ell = 0$, the subsystem (37b) drives (48),

(i) Consider the zero disturbance case, i.e., $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$. First, let $\ell = 1$ in (21). Since Assumptions 2 and 3 are satisfied, the requirement $\mu \in \mathcal{K}$ allows [14, Corollary 1 (ii)] to be applied to the cascade of (48) and (37a) characterized by (25) and (24) with $d = \overline{\epsilon}$, respectively. The application yields the existence of continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\alpha_c \in \mathcal{K}$ such that for $i = 1, 2$,

$$\lambda_i(s) > 0, \forall s \in (0, \infty) \tag{49}$$

$$\int_1^\infty \lambda_i(s)ds = \infty \tag{50}$$

hold and

$$\dot{V}_c \le -\alpha_c(V_c) \tag{51}$$

is satisfied along the trajectories of (48) and (37a) for $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$ with

$$V_c(x, \overline{\epsilon}) = \int_0^{U(x)} \lambda_1(s)ds + \int_0^{V(\overline{\epsilon})} \lambda_2(s)ds. \tag{52}$$

Therefore, combining this inequality with (24) achieved by (37b) yields

$$\dot{W} \le -\alpha_c(V_c) - \omega(|\underline{\epsilon}|) \tag{53}$$

for

$$W(x, \overline{\epsilon}, \underline{\epsilon}) = V_c(x, \overline{\epsilon}) + V(\underline{\epsilon}). \tag{54}$$

Since W is a positive definite and radially unbounded function of $(x, \overline{\epsilon}, \underline{\epsilon})$ due to (49) and (50), inequality (53) with $\alpha_c \in \mathcal{K}$, $\omega \in \mathcal{P}$ implies that $(x, \overline{\epsilon}, \underline{\epsilon}) = (0, 0, 0)$ of the entire system consisting of (48) and (37) is 0-GAS. The treatment of the case $\ell = 0$ in (21) is the same as above since one can just switch the role of (37a) with that of (37b) with $d = \underline{\epsilon}$. Next, consider the case of $0 < \ell < 1$. Let

$$V_e(\overline{\epsilon}, \underline{\epsilon}) = V(\overline{\epsilon}) + V(\underline{\epsilon}). \tag{55}$$

Then property (24) applied to each of (37a) and (37b) gives

$$\dot{V}_{e,(37)} \le -\omega(\overline{\upsilon}^{-1}(V(\overline{\epsilon}))) - \omega(\overline{\upsilon}^{-1}(V(\underline{\epsilon}))) \le -\hat{\omega}(V_e), \tag{56}$$

where $\dot{V}_{e,(37)}$ denotes time-derivative of V_e along the solution of (37). On the other hand, due to $\gamma \in \mathcal{K}$, we have

$$\begin{aligned}\gamma(|\ell\overline{\epsilon} + (1-\ell)\underline{\epsilon}|) &\le \gamma(\ell|\overline{\epsilon}| + (1-\ell)|\underline{\epsilon}|)\\ &\le \gamma(\ell\underline{\upsilon}^{-1}(V(\overline{\epsilon})) + (1-\ell)\underline{\upsilon}^{-1}(V(\underline{\epsilon}))\\ &\le \gamma(\underline{\upsilon}^{-1}(V_e(\overline{\epsilon}, \underline{\epsilon})).\end{aligned} \tag{57}$$

Substitute (57) into (25). Property $\mu \in \mathcal{K}$ in (25) allows [14, Corollary 1 (ii)] to be applied to the cascade of (25) and (56), and it proves the existence of continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\alpha_w \in \mathcal{K}$ such that (49), (50) and

$$\dot{W} \le -\alpha_w(W) \tag{58}$$

hold, where (58) is along the solution of (48) and (37) for

$$W(x, \overline{\epsilon}, \underline{\epsilon}) = \int_0^{U(x)} \lambda_1(s)ds + \int_0^{V_e(\overline{\epsilon}, \underline{\epsilon})} \lambda_2(s)ds. \tag{59}$$

Since the above function W is a positive definite and radially unbounded, the entire system is 0-GAS.

(ii) The claim is proved by replacing [14, Corollary 1 (ii)] with [14, Corollary 1 (i)] in the proof of (i).

(iii) Suppose that Assumptions 4, 5 and $\mu \in \mathcal{K}$ and are satisfied. Recall that (26) implies (46a) and (46b) for $\delta^{\pm} = \delta^+ - \delta^-$. First, let $\ell = 1$. Consider the cascade described by (25) and (46a). The development in [14, Theorem 2] proves that the function V_c defined in (52) satisfies

$$\dot{V}_c \le -\alpha_c(V_c) + \sigma_c(|\delta|) + \sigma(|\delta^{\pm}|) \tag{60}$$

for some $\alpha_c, \sigma_c, \sigma \in \mathcal{K}$. Defining W as in (54), we obtain

$$\dot{W} \le -\alpha_c(V_c) - \omega(|\underline{\epsilon}|) + \sigma_c(|\delta|) + \sigma(|\delta^{\pm}|) + \eta(\sqrt{2}|\delta^{\pm}|) \tag{61}$$

along the trajectories of (48) and (37). Since (35) with $z_1 = \hat{x}^+$ and $z_2 = \hat{x}^-$ ensures that the map $[x^\top, \overline{\epsilon}^\top, \underline{\epsilon}^\top]^\top \mapsto X$ is bijection, the iISS of the entire system follows from the Lyapunov characterization developed in [4]. That is, there exist $\theta \in \mathcal{KL}$, $\psi \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$ such that (15) holds for any $x_0 \in \mathbb{R}^n$ and any $\delta \in \mathcal{D}$ satisfying (9). The above argument is also valid for the case of $\ell = 0$ by switching (46a) with (46b). Next, consider $0 < \ell < 1$. For V_e defined in (55), we have

$$\dot{V}_{e,(37)} \le -\hat{\omega}(V_e), +2\eta(\sqrt{2}|\delta^{\pm}|) \tag{62}$$

Consider the cascade described by (25) and (62). From the definition, it can be verified that $\liminf_{s\to\infty}\hat{\omega}(s) = \liminf_{s\to\infty}\omega(s)$. Thus, the application of [14, Theorem 2] to this cascade yields

$$\dot{W} \le -\alpha_w(W) + \sigma_c(|\delta|) + \sigma(|\delta^{\pm}|) \tag{63}$$

for some $\alpha_w, \sigma_c, \sigma \in \mathcal{K}$. Hence, following the argument used above, we arrive at the iISS of the entire system.

(iv) The claim is proved by applying the argument presented in [14, Remark 1] to the proof of (iii). It is straightforward except that $\hat{\omega}$ in (33) is not guaranteed to be of class $\mathcal{K}$. If $\hat{\omega}$ is not a class $\mathcal{K}$ function, the property $\gamma \circ \underline{\upsilon}^{-1} \in \mathcal{K}$ implies that (33) ensures $\liminf_{s\to\infty}\omega(s) > 0$ and

$$c\gamma \circ \underline{\upsilon}^{-1}(s) \le [\tilde{\omega}(s)]^k, \quad \forall s \in \mathbb{R}_{\ge 0}$$

with $\tilde{\omega}(s) = \inf_{r\in[s,\infty)} \hat{\omega}(r)$. The definition of $\hat{\omega}$ yields

$$\tilde{\omega}(s) = \begin{cases} \underline{\omega} \circ \overline{\upsilon}^{-1}(s), & \ell \in \{0, 1\} \\ \min\limits_{r\in[0,s]} \underline{\omega} \circ \overline{\upsilon}^{-1}(r) + \underline{\omega} \circ \overline{\upsilon}^{-1}(s-r), & \text{otherwise} \end{cases}$$

for $\underline{\omega}(s) = \inf_{r\in[s,\infty)} \omega(r)$. Therefore, replacing ω with $\underline{\omega}$ allows one to complete the proof.

(v) In the case of $\mu, \omega \in \mathcal{K}_\infty$, [14, Theorem 1 (ii)] applies to the cascade of (25) and (46a) (or (46b)) as well as the cascade of (25) and (62). It proves that (61) and (63) hold with $\alpha_c \in \mathcal{K}_\infty$ and $\alpha_w \in \mathcal{K}_\infty$, respectively. According to [29], (61) and (63) imply that the entire system admits the existence of $\theta \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that (14) holds. Hence, the entire system is ISS. In the case of $\liminf_{s\to\infty}\omega(s) \geq \sup_{t\in\mathbb{R}_{\geq 0}}\hat{\eta}(\sqrt{2}|\delta^{\pm}(t)|)$, property (62) guarantees the existence of a radially bounded function $\tilde{V}_e$ and $\tilde{\omega} \in \mathcal{K}_\infty$ such that

$$\dot{\tilde{V}}_{e,(37)} \leq -\tilde{\omega}(V_e), +\tilde{\eta}(|\delta^{\pm}|) \tag{64}$$

holds for some $\tilde{\eta} \in \mathcal{K}$ (see, e.g., Remark 4 in [13]), Therefore, with the help of $\mu \in \mathcal{K}_\infty$ in the case of $0<\ell<1$, ISS of the entire system is proved by applying the stability criterion in [14, Theorem 1 (ii)] to the cascade of (25) and (64). The above argument is also valid for proving the case of $\ell \in \{0, 1\}$ by replacing $\omega \in \mathcal{K}$ with some $\tilde{\omega} \in \mathcal{K}_\infty$ in (46a) and (46b).

5. Discussions

5.1. Broader nonlinearities

Under Assumptions 1, 2, 3 and global Lipschitzness assumption on functions of the plant and the observer, the result in [7] proposes

$$\exists \kappa \in \mathcal{K},\ l \in \mathbb{R}_{\geq 0} \text{ s.t.}$$

$$\gamma(\underline{\nu}^{-1}(s)) \leq (l+\kappa(s))\omega(\overline{\nu}^{-1}(s)),\ \ \forall s\in\mathbb{R}_{\geq 0} \tag{65}$$

as a sufficient condition[3] to establish that the system consisting of (18)–(20) is an interval observer with the guarantee of $T_{x_0,u,\delta}=\infty$ for the system (1) with $u=\hat{u}_s(y,\hat{x}^+)$. It can be verified that (65) is a special case of (31). In fact, the assumption (65) yields

$$\int_0^1 \frac{\gamma\circ\underline{\nu}^{-1}(s)}{\omega\circ\overline{\nu}^{-1}(s)}ds \leq \int_0^1 (l+\kappa(s))ds.$$

By virtue of $\kappa \in \mathcal{K}$, the above guarantees (31). Thus, property (65) assumed by [7] is a sufficient condition for (31) which guarantees 0-GAS of the entire system, in particular, $T_{x_0,u,0}=\infty$ required in Theorem 1 (i). It is worth stressing that (31) is only a local property around the origin of the functions γ and ω. Moreover, this paper does not require (31) when $\mu \in \mathcal{K}$ is satisfied. In other words, the restrictions on the coupling can be removed completely if the convergence rate of the plant is not vanishing as the magnitude of the state tends to infinity.

5.2. Verifying iISS and ISS Assumptions

This paper has replaced (24) and (25) , which are assumed in [7], with (26) and (28), respectively, to guarantee additional robustness with respect to disturbances. It can be verified that (26) (resp., (28)) is achieved whenever (24) (resp., (25)) is satisfied with a quadratic function $V(\xi)$ (resp., $U(x)$). In this sense, requiring (26) and (28) is not restrictive in popular situations of many previous studies. In other words, robustness with respect to disturbances and convergence in the presence of disturbances can be confirmed in popular cases without requiring new restrictive assumptions. The following two propositions demonstrate this point precisely:

Proposition 2. *Suppose that for a given locally Lipschitz function $\Lambda : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, there exist $\mathbb{R}^{n\times n}$ symmetric matrices $P\succ 0$ and $M\succ 0$ such that*

$$P[A(y)+\Lambda(y)C]+[A(y)+\Lambda(y)C]^\top P \preceq -M \tag{66}$$

holds for all $y\in\mathbb{R}^p$. Then there exist a positive definite radially unbounded C^1 function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}$, $\overline{\nu} \in \mathcal{K}_\infty$, $\omega \in \mathcal{K}_\infty$ and $\eta^+, \eta^- \in \mathcal{K}_\infty$ such that (23) and (26) are satisfied for all $\xi\in\mathbb{R}^n$, $y\in\mathbb{R}^p$, $\rho^+\in\mathbb{R}^n$ and $\rho^-\in\mathbb{R}^n$.

Proof. Let $V(\xi)=\xi^\top P\xi$. Then

$$\begin{aligned}
&\frac{\partial V}{\partial \xi}\left\{[A(y)+\Lambda(y)C]\xi + S[R^+\rho^+ + R^-\rho^-]\right\}\\
&\leq -\xi^\top M\xi + 2\epsilon\,\xi^\top P\xi + \frac{1}{\epsilon}(SR^+\rho^+)^\top P(SR^+\rho^+)\\
&\qquad + \frac{1}{\epsilon}(SR^-\rho^-)^\top P(SR^-\rho^-).
\end{aligned}$$

holds for any $\epsilon>0$. Due to $P\succ 0$ and $M\succ 0$, the inequalities (23) and (26) are satisfied with

$$\underline{\nu}(s)=p_{\min}s^2,\ \ \overline{\nu}(s)=p_{\max}s^2$$

$$\omega(s)=(m_{\min}-2\epsilon p_{\max})s^2$$

$$\eta^+(s)=\frac{b^+_{\max}}{\epsilon}s^2,\ \ \eta^-(s)=\frac{b^-_{\max}}{\epsilon}s^2$$

using $\epsilon < m_{\min}/(2p_{\max})$, where $p_{\min}>0$ (resp., $m_{\min}>0$) is the smallest eigenvalue of P (resp., M), and $p_{\max}>0$ is the largest eigenvalue of P. In the same way, $b^+_{\max}>0$ (resp., $b^-_{\max}>0$) is the largest eigenvalue of $(SR^+)^\top P(SR^+)$ (resp., $(SR^-)^\top P(SR^-)$). □

The condition (66) is not novel. It has been used extensively within the framework of linear parameter-varying systems. Proposition 2 is a restatement of its usefulness in view of (26).

Proposition 3. *Suppose that for a given locally Lipschitz function $u_s : \mathbb{R}^p\times\mathbb{R}^n\to\mathbb{R}^q$, there exist a $\mathbb{R}^{n\times n}$ symmetric matrix $Q\succ 0$, continuous functions $\tilde{\mu}\in\mathcal{P}$ and $\tilde{\gamma}\in\mathcal{K}$ such that*

$$\begin{aligned}
&x^\top\Big\{Q[A(Cx)x+\beta(Cx,u_s(Cx,x+d))]+\\
&[A(Cx)x+\beta(Cx,u_s(Cx,x+d))]^\top Q\Big\}x\\
&\quad\leq -\tilde{\mu}(|x|)+\tilde{\gamma}(|d|)
\end{aligned} \tag{67}$$

holds for all $x\in\mathbb{R}^n$ and $d\in\mathbb{R}^n$. Then there exist a positive definite radially unbounded C^1 function $U:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}$, $\overline{\nu}\in\mathcal{K}_\infty$, $\mu\in\mathcal{P}$ and $\gamma,\zeta\in\mathcal{K}$ such that (23) and (28) hold for all $x\in\mathbb{R}^n$, $d\in\mathbb{R}^n$ and $\delta\in\mathbb{R}^n$. Furthermore, if $\liminf_{s\to\infty}\tilde{\mu}(s)/s=\infty$ holds in addition, (28) can be achieved as above with $\mu\in\mathcal{K}_\infty$.

Proof. Let $U(x)=\log(1+x^\top Qx)$. Then $Q\succ 0$ ensures that (23) is satisfied with $\underline{\nu}$, $\overline{\nu}\in\mathcal{K}_\infty$ given by

$$\underline{\nu}(s)=\log(1+q_{\min}s^2),\quad \overline{\nu}(s)=\log(1+q_{\max}s^2),$$

where $q_{\min}>0$ (resp., $q_{\max}>0$) denotes the smallest (resp., largest) singular value of Q. From (67) we obtain

$$\begin{aligned}
&\frac{\partial U}{\partial x}[A(Cx)x+\beta(Cx,u_s(Cx,x+d))+\delta]\\
&\leq -\frac{\tilde{\mu}(|x|)}{(1+x^\top Qx)}+\frac{\tilde{\gamma}(|d|)}{(1+x^\top Qx)}+\frac{2x^\top Q\delta}{(1+x^\top Qx)}\\
&\leq -\frac{\tilde{\mu}(|x|)}{(1+q_{\max}|x|^2)}+\tilde{\gamma}(|d|)+\frac{2x^\top Q\delta}{(1+x^\top Qx)}.
\end{aligned} \tag{68}$$

Now, it can be verified that

$$(b^\top b)^{\frac{1}{2}} \geq \frac{2a^\top b}{1+a^\top a},\quad \forall a,b\in\mathbb{R}^n. \tag{69}$$

Indeed, using the complete square, we have

$$\{(1+a^\top a)(b^\top b)^{\frac{1}{2}}\}^2 \geq (4a^\top a)(b^\top b) \geq 4(a^\top b)^\top(a^\top b).$$

Applying (69) to (68) with $a=Q^{1/2}x$ and $b=Q^{1/2}\delta$ yields

$$\frac{\partial U}{\partial x}[A(Cx)x+\beta(Cx,u_s(Cx,x+d))+\delta]$$

[3] The expression (65) is not exactly the same as the one in [7], but they are qualitatively equivalent to each other.

$$\leq -\frac{\tilde{\mu}(|x|)}{(1+q_{\max}|x|^2)} + \tilde{\gamma}(|d|) + (\delta^\top Q\delta)^{\frac{1}{2}} \tag{70}$$

Thus, we arrive at (28) with

$$\mu(s) = \frac{\tilde{\mu}(s)}{(1+q_{\max}s^2)}, \quad \gamma(s) = \tilde{\gamma}(s), \quad \zeta(s) = q_{\max}^{\frac{1}{2}}s.$$

Next, assume $\liminf_{s\to\infty} \tilde{\mu}(s)/s = \infty$. By virtue of $\tilde{\mu} \in \mathcal{P}$ and $\liminf_{s\to\infty} \tilde{\mu}(s)/s = \infty$, there exists $\hat{\mu} \in \mathcal{K}_\infty$ such that $\hat{\mu}(s) \leq \tilde{\mu}(s)/2s$ holds for all $s \in \mathbb{R}_+$. Let $U(x) = x^\top Qx$. Then $Q \succ 0$ ensures the existence of $\underline{\nu}$, $\overline{\nu} \in \mathcal{K}_\infty$ satisfying (23). We also have

$$\frac{\partial U}{\partial x}[A(Cx)x + \beta(Cx, u_s(Cx, x+d)) + \delta]$$
$$\leq -2|x|\hat{\mu}(|x|) + \tilde{\gamma}(|d|) + 2|x||Q\delta|. \tag{71}$$

Due to $\hat{\mu} \in \mathcal{K}_\infty$, we have $ab \leq a\hat{\mu}(a) + \hat{\mu}^{-1}(b)b$ for all $a, b \in \mathbb{R}_+$. Applying this to (71) yields

$$\frac{\partial U}{\partial x}[A(Cx)x + \beta(Cx, u_s(Cx, x+d)) + \delta]$$
$$\leq -|x|\hat{\mu}(|x|) + \tilde{\gamma}(|d|) + \hat{\mu}^{-1}(2|Q\delta|)2|Q\delta|.$$

Thus, we arrive at (28) with

$$\mu(s) = s\hat{\mu}(s), \quad \gamma(s) = \tilde{\gamma}(s), \quad \zeta(s) = 2q_{\max}s\hat{\mu}^{-1}(2q_{\max}s),$$

where $\mu \in \mathcal{K}_\infty$ is established. □

Note that the choice $V(x) = x^\top Px$ with P assumed by Proposition 2 achieves (24) of Assumption 2. In the same way, the choice $U(x) = x^\top Qx$ with Q assumed by Proposition 3 achieves (25) of Assumption 3. According to Proposition 3 and Theorem 2, the quadratic function fulfilling (non-robust) Assumption 3 cannot always lead to ISS of the entire controlled system, although it can often lead to iISS. Proposition 2 allows a quadratic function V to provide the observer decay rate ω with the class $\mathcal{K}_\infty$ which is stronger than $\mathcal{P}$ that Proposition 3 yields for the decay rate μ of the feedback control. This is practically reasonable since the observer is implemented as codes on a microprocessor, while feedback control is implemented in a physical actuator which is often subject to limitations.

5.3. Comparison with other ways of formulations

The structure of (1) is often referred to as the observer canonical form up to input-output injection [12,16,24]. It is precisely the observer canonical form when matrix A is constant. The plant in the form (1) is the most popular class of systems for which the Luenberger-type observer

$$\dot{\hat{x}}(t) = A(y(t))\hat{x}(t) + \beta(y(t), u(t)) - \Lambda(y(t))(y(t) - C\hat{x}(t)) \tag{72}$$

has been studied in the literature of nonlinear control. Choosing $R = I$, $\delta^+(t) \equiv 0$ and $\delta^-(t) \equiv 0$, the system consisting of (18)–(20) becomes identical with (72) since $R^+ = I$, $R^- = 0$, $S = S^+ = I$ and $S^- = 0$. This paper demonstrates that such a simple modification of the Luenberger-type observer (72) provides us with not only an interval observer, but also an iISS/ISS interval observer which guarantees the convergence of the estimated state interval and the plant state to zero in the presence of convergent disturbances. It is worth mentioning that a system which is almost the same as (18) is employed in [9] for interval estimation in the framework of Linear Parameter Varying (LPV) systems[4] under the assumption that x, y and u are restricted to compact sets given a priori. The LPV modeling regards $y(t)$ in A as an exogenous signal instead of the plant output which is endogenous. This difference in the modeling typically appears in the part of feedback design (25) and (28) where x in $A(y) = A(Cx)$ is the same as the argument of $U(x)$ in this paper. If the LPV formulation is taken in (25) and (28), one needs to consider the vector y in $A(y)$ to be independent of the vector x of $U(x)$, and the right hand side of (25) and (28) is required to be uniform in the parameter y. In [31], under the compact set assumption of variables, interval observer design for nonlinear systems is studied by making use of Lipschitz constants which are valid in the compact sets. Based on a quadratic Lyapunov function, design guidelines are obtained as linear matrix inequalities without referring to the LPV formulation. The effectiveness of approaches using the compact set assumption is limited to small regions unless the effect of nonlinearities appearing in the plant is bounded.

This paper does not assume that x, y and u are somehow restricted to compact sets given a priori. To achieve global properties, the use of non-quadratic functions is natural for controlling nonlinear systems by feedback stabilization. The interval observer-based feedback design in this paper allows μ to be non-quadratic in (25) and (28), and ensures that the guarantees of control and estimation are global in time and state.

6. An example

Consider the system

$$\dot{x}_1 = x_2 + |x_1|(x_2 + u_1) + \delta_1 \tag{73a}$$

$$\dot{x}_2 = x_3 \tag{73b}$$

$$\dot{x}_3 = -6x_3 + u_2 + \delta_3 \tag{73c}$$

$$y = x_1. \tag{73d}$$

This system fits in (1) with

$$\beta(y,u) = \begin{bmatrix} |y| & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u, \quad A(y) = \begin{bmatrix} 0 & 1+|y| & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix}$$

and $\delta_2(t) \equiv 0$. Pick $\ell = 1$ in (21), and let the feedback control $u = \hat{u}_s$ and the observer gain Λ be

$$\hat{u}_s(y, \hat{x}^+) = \begin{bmatrix} -\hat{x}_2^+ \\ -8y - 12\hat{x}_2^+ \end{bmatrix}, \quad \Lambda(y) = \begin{bmatrix} -3-3|y| \\ -1-|y| \\ 0 \end{bmatrix}. \tag{74}$$

With

$$R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assumption 1 is satisfied since

$$\Gamma(y) = \begin{bmatrix} -2(1+|y|) & 1+|y| & 0 \\ 1+|y| & -(1+|y|) & 1 \\ 0 & 0 & -6 \end{bmatrix}$$

is Metzler for all $y \in \mathbb{R}$. An observer candidate is obtained as (18)–(20) with

$$R_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ R_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ S_+ = S, \ S_- = 0.$$

Let $V(\xi) = \xi^\top P\xi$ with

$$P = \begin{bmatrix} 1/3 & -1/2 & 0 \\ -1/2 & 11/6 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

[4] It is a special class of Linear Timer Varying (LTV) systems.

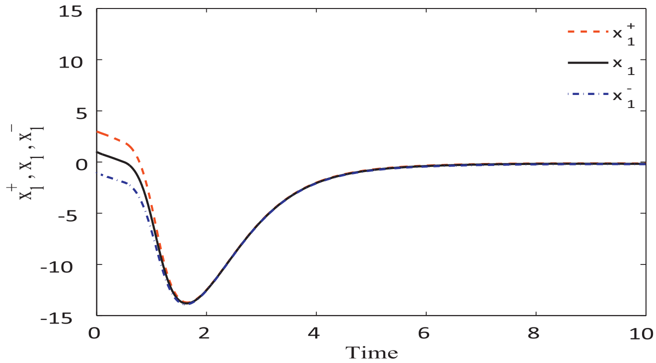

Fig. 1. Evolution of 1st component of x, x^+, x^- in the presence of disturbances (77). Note that the feedback control uses the measured variable x_1 instead of its estimate.

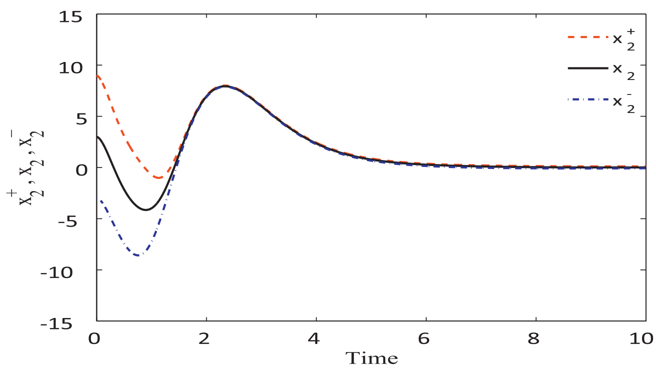

Fig. 2. Evolution of 2nd component of x, x^+, x^- in the presence of disturbances (77).

which satisfies (23) for $\underline{\nu}(s)=s^2/6$ and $\overline{\nu}(s)=2s^2$. We have

$$\frac{\partial V}{\partial \xi}(\xi)\left\{[A(y)+\Lambda(y)C]\xi+S[R^+\rho^+ + R^-\rho^-]\right\}$$
$$\le -\frac{5}{8}|\xi|^2+2\xi^\top PS[R^+\rho^+ + R^-\rho^-].$$

Since Young's inequality gives $2\xi^\top PSR^+\rho^+ \le 1/4|\xi|^2+300|\rho^+|^2$ and $2\xi^\top PSR^-\rho^- \le 1/4|\xi|^2+30|\rho^-|^2$, we have (26) with

$$\omega(s)=\frac{1}{8}s^2,\quad \eta^+(s)=300s^2,\quad \eta^-(s)=30s^2. \tag{75}$$

On the other hand, the choice $U(x)=\log(1+x^\top Qx)$ yields

$$\frac{\partial U}{\partial x}(x)[A(Cx)x+\beta(Cx,u_s(Cx,x+d))+\delta]$$
$$\le \frac{2}{1+x^\top Qx}x^\top Q\left(\begin{bmatrix}0&1&0\\0&0&1\\-8&-12&-6\end{bmatrix}x+\begin{bmatrix}-|x_1|d_2+\delta_1\\0\\-12d_2+\delta_3\end{bmatrix}\right).$$

Let p_{min} (resp., p_{max}) denote the smallest (resp., largest) singular value of a symmetric positive definite matrix Q. Due to the Hurwitz polynomial $s+6s^2+12s+8=(s+2)^3$, Young's inequality and $|x|^2/(1+x^\top Qx)\le 1/p_{min}$, there exist k_1, k_2, k_3, $k_4>0$ such that (28) is satisfied with

$$\mu(s)=\frac{k_1s^2}{1+p_{max}s^2},\quad \gamma(s)=k_2s+k_3s^2,\quad \zeta(s)=k_4s^2. \tag{76}$$

Therefore, Assumptions 4 and 5 are satisfied. According to Theorem 1 (iii), the system (18)–(20) is an ISS interval observer. Due to $\mu\in\mathcal{K}$ and $\omega\in\mathcal{K}_\infty$, Theorem 2 (iii) establishes iISS of the entire control system. Note that it is not necessary to check (65) proposed in [7], which is not satisfied with ω and γ in (75) and (76). The simulation result is plotted in Figs. 1–3 for the convergent disturbances

$$\delta(t)=\begin{bmatrix}2\sin(t)/(2+t)^2\\0\\2\cos(t)/(2+t)^2\end{bmatrix}. \tag{77}$$

The plots are computed with the initial conditions $x_0=[1,3,-1]^\top$, $x_0^+=[3,5,1]^\top$ and $x_0^-=[-1,1,-3]^\top$. The disturbance bounds are $\delta_1^+(t)=\delta_3^+(t)=2/(2+s)^2$ and $\delta_1^-(t)=\delta_3^-(t)=-2/(2+s)^2$. The component $x_2(t)$ which is not measured remains in the estimated interval $[x_2^-(t),x_2^+(t)]$ all the time in Fig. 2. The same applies to $x_3(t)$ shown in Fig. 3. Moreover, the length of the intervals converges to zero. It can also be confirmed by the plots that all the state variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ converge to the origin. These observations are consistent with Theorems 1, 2 and Proposition 1.

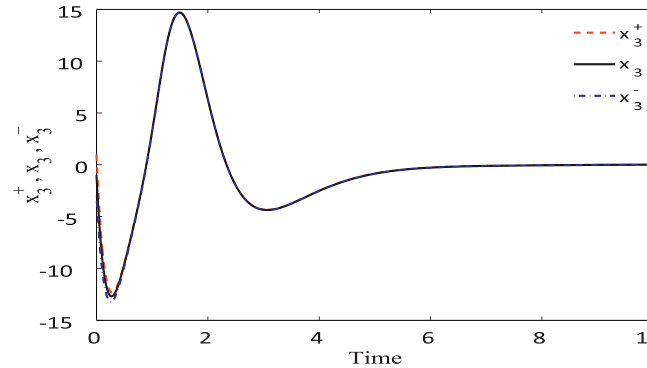

Fig. 3. Evolution of 3rd component of x, x^+, x^- in the presence of disturbances (77).

7. Concluding remarks

This paper has studied the problem of designing interval observers, and focused on the robustness with respect to disturbances and its usefulness for increasing the capability of dealing with nonlinearities in feedback control. The interval estimation and control this paper has achieved are valid in the entire state space, and initial and evolving states are not restricted to any regions assumed a priori. This paper has extended the earlier result in [7] in three points. Firstly, it is demonstrated that the iISS theory allows the interval observer design to ensure the convergence of the estimated intervals to zero even in the presence of disturbances if the disturbances converge to zero. Secondly, this paper has proposed iISS Lyapunov characterizations to replace the global Lipschitz conditions on which the approach in [7] relies in the presence of disturbances. The iISS framework covers a broad class of nonlinearities including saturations and bilinearities which are not covered by ISS. Thirdly, for feedback control design, restrictions on the coupling between the interval observer and the plant have been relaxed in several ways. For instance, a restriction imposed globally has been replaced by a local growth order condition around the origin. This paper has also demonstrated that the growth order condition can be completely removed when the convergence rate of the system is not radially vanishing.

It is worth mentioning that open-loop interval estimation achieving ISS was investigated in [19], and the issue of avoiding global Lipschitzness is addressed. For the monitoring which is not necessarily related to feedback control, an interval observer or a framer of that type generates a state interval based on the information of the initial condition and the disturbance without measurement. For assembling a framer recursively from components, triangular systems are targeted in [19] by hypothesizing that a framer is given for each component, which eliminates the need to assume global Lipschitzness. Since imposing appropriate monotonicity and ISS on each component allows the double copies of each component to be an interval observer component, recursive aggregation of the monotone ISS components leads to an interval observer of a given triangular system. Interestingly, the ISS assumption on each component yields ISS of the assembled observer from the disturbance interval to the estimated state interval.

To the best of the authors' knowledge, that study in [19] first introduced the notion of ISS to interval observer design, although it is open-loop estimation without exploiting measurement. In contrast, employing iISS, this present paper has pursued guidelines for designing closed-loop interval observers which not only give interval estimates, but also help to stabilize possibly unstable plants by feedback.

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