Solving a class of two-stage robust inventory management problems

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1 Introduction

In this paper we address the problem of optimizing multi-period inventory in the case of uncertain demands. At each time period, the company produces a certain quantity of goods which is used to serve a client demand. The unit selling price was fixed in advance by contract for the time horizon and an expected value of the quantity to deliver at each time period is known. In case of overproduction, goods are added to the stock. In case of underproduction the missing goods are either taken from the stock or bought on the international market. In addition at each time period the manager can decide to buy more goods and add them to the stock or to sell a part of the goods in stock, on the international market. The unit price of purchase (or sale) of these goods on the international market are estimated in advance for every period, according to the previous years.

But in fact, the demand and the purchasing costs on the international market are uncertain and may differ from their expected values. Following the Betsimas and Thiele approach, we assume that there is no known probabilistic distribution of these values, but each one may vary in a given interval. We also assume that the variation of the purchasing or selling costs is small, while the real demand can be far enough from its expected value. Then the prices on the international market can be approximate in the following way: the unit purchasing cost is set to its expected value plus the maximum possible gap and the unit selling cost is set to its expected value minus the maximum possible gap. Doing so, we guarantee a lower bound on the profits.

The manager takes decisions in two stages: first he before discovering the actual value taken by the demand, second once uncertainty has been revealed.

In this paper we address the problem of optimizing multi-period inventory in the case of uncertain demands. We consider a wholesaler who purchases goods on the international market and stocks them in a warehouse before selling them to local customers. To serve the demand he can either demand at each time period, the manager decides the quantity to buy His decisions are made in two stages: first before discovering the actual value taken by the demand, second once uncertainty has been revealed.

2 Problem definition

2.1 The deterministic case

We consider a inventory problem where We assume that we have the possibility to store the goods, that are not delivered for the demand, into a storehouse at a certain cost. At each time period, we have the possibility to buy some goods from the market, furthermore we can also sell some goods to the market (at a much lower price). More precisely, the data of the problem are:

- $T \in \mathbb{N}$, is the time horizon of the problem
- For each $t \in [1, ..., T]$, we denote respectively by $d_t \in \mathbb{R}_+$ and $p_t \in \mathbb{R}_+$ the demand and the production at time t.
- $\alpha \in \mathbb{R}_+^T$ and $\beta \in \mathbb{R}_+^T$ are the vector of costs, respectively to buy some goods from the market and to sell some goods to the market. Implicitly, $\alpha > \beta$.
- $c \in \mathbb{R}_+$ is the unit cost to store one unit of good in the storehouse during a time period
- $K \in \mathbb{R}_+$ is the capacity of the storehouse and $s_0 \in \mathbb{R}_+$ is the initial stock (we could also define bounds on maximum/minimum load/unload of goods during a time period but for the sake of simplicity we do not consider this case here).

In this setting we aim to look for the best policy to minimize the costs. The decision variables of the problem are:

- $x \in \mathbb{R}_+^T$, where for each t, x_t denotes the quantity of goods we buy from the market at time t
- $y \in \mathbb{R}_+^T$, where for each t, y_t denotes the quantity of goods we sell to the market at time t
- $e^i \in \mathbb{R}_+^T$, where for each t, e_t^i denotes the quantity of goods we store into the storehouse at time t
- $e^o \in \mathbb{R}_+^T$, where for each t, e_t^o denotes the quantity of goods we unload from the storehouse at time t
- $s \in \mathbb{R}_+^T$, where for each t, s_t denotes the quantity of goods stored in the storehouse at time t

The LP formulation of the problem is the following:

$$
(P)\begin{cases}
\min \sum_{t=1}^{T}(\alpha_t x_t - \beta_t y_t) + c\sum_{t=1}^{T} s_t \\
\quad x_t - y_t - e_t^i + e_t^o = d_t - p_t,\ \forall t = 1, ..., T \\
\quad s_t = s_{t-1} + e_t^i - e_t^o,\ \forall t = 1, ..., T \\
\quad 0 \le s \le K \\
\quad x, y, s, e^i, e^o \in \mathbb{R}_+^T
\end{cases}
$$

2.2 The robust case

In practice, the demand d is uncertain. We assume that for each t, d_t belongs to the interval $[\bar{d}_t - \Delta_t, \bar{d}_t + \Delta_t]$. Similarly, to the EJOR paper, the robust model can be formulated into the following mathematical program:

$$
(RP)\begin{cases}
\max\limits_{\substack{\sum_{t=1}^{T} \delta_t \le \bar{\delta} \\ \delta \in \{0,1\}^T}} \ \min \sum_{t=1}^{T}(\alpha_t x_t - \beta_t y_t) + c\sum_{t=1}^{T} s_t \\
\qquad x_t - y_t - e_t^i + e_t^o = \bar{d}_t + \delta_t \Delta_t - p_t,\ \forall t = 1, ..., T \\
\qquad s_t = s_{t-1} + e_t^i - e_t^o,\ \forall t = 1, ..., T \\
\qquad 0 \le s \le K \\
\qquad x, y, s, e^i, e^o \in \mathbb{R}_+^T
\end{cases}
$$

where $\bar{\delta}$ denotes the *uncertainty budget*, i.e. the number of time periods where the demand d_t is allowed to take its worst value $\bar{d}_t + \Delta_t$. We aim therefore to look for the $\bar{\delta}$ time periods that induce the maximum cost of the problem (P).

3 Solving the robust problem

We prove, in this section, that the robust problem (RP) can be solved in polynomial time using a nested dynamic programming approach. After defining the *Restricted*-RP, we study some properties of a class of piecewise linear function, then we present the dynamic programming algorithm and prove that it converges to an optimal solution of (RP) in polynomial time.

Let $\tau \in [1, ..., T]$, $\zeta \in [0, ..., \bar{\delta}]$ and $\sigma \in [0; K]$, we define a *truncated recourse problem* $R(\tau, \zeta, \sigma)$ defined on the $(T - \tau + 1)$ last time periods assuming that the uncertainty budget on that period is ζ and that initially (at $t = \tau - 1$), the initial stock in the storehouse is σ. The mathematical formulation of $R(\tau,\zeta,\sigma)$ is:

$$R(\tau,\zeta,\sigma)\left\{\begin{array}{ll}\max\limits_{\substack{\sum_{t=\tau}^{T}\delta_t\le\bar{\delta}\\ \delta\in\{0,1\}^{T-\tau+1}}} & \min\sum_{t=\tau}^{T}(\alpha_t x_t-\beta_t y_t)+c\sum_{t=\tau}^{T}s_t\\ & x_t-y_t-e^i_t+e^o_t=\bar{d}_t+\delta_t\Delta_t-p_t,\ \forall t=\tau,...,T\\ & s_t=s_{t-1}+e^i_t-e^o_t,\ \forall t=\tau,...,T,\ s_{\tau-1}=\sigma\\ & 0\le s\le K\\ & x,y,s,e^i,e^o\in\mathbb{R}_+^{T-\tau+1}\end{array}\right.$$

Let $v(\tau,\zeta,\sigma)$ be the optimal value of $R(\tau,\zeta,\sigma)$. We will prove that, for all (τ,ζ), the function $\sigma\mapsto v(\tau,\zeta,\sigma)$ belongs to the class of function that we introduce below.

3.1 The class $\mathcal{C}$ of non-increasing piecewise linear function

Let $f:[a,b]\mapsto\mathbb{R}$, we say that $f\in\mathcal{C}^{a,b}$ if there exist $f_0\in\mathbb{R}$, $a=a_0<a_1<...<a_n=b$ and $\gamma_0>\gamma_1>...>\gamma_{n-1}\ge 0$ such that $f(a)=f_0$ and for all $i=0,...,n-1$ f is an affine function on $[a_i,a_{i+1}]$ with coefficient $-\gamma_i$, i.e.

$$f(x)=f_0-\sum_{i=0}^{n-1}\gamma_i\max(\min(x-a_i,a_{i+1}-a_i),0)$$

where $n\in\mathbb{N}$ defines the size, size(f), of f (see Figure 3.1). We say that $(\gamma_0,...,\gamma_{n-1})$ defines the coefficients of f.

Proposition 1. *Let $f\in\mathcal{C}^{a,b}$ with coefficients $(\gamma_0,...,\gamma_{n-1})$:*
$f(x)=f_0-\sum_{i=0}^{n-1}\gamma_i\max(\min(x-a_i,a_{i+1}-a_i),0)$. *Let $L\le 0\le U$ and let us consider the following function $g:[a,b]\mapsto\mathbb{R}$:*

$$g(x)=\min_{L\le h\le U}g_0-\gamma h+f(x-h)$$

Then $g\in\mathcal{C}^{a,b}$ with coefficients $(\gamma_0,...,\gamma,...,\gamma_{n-1})$ (size(g) = size$(f)+1$).

Proof. Let $x\in[a,b]$. There exists $i=0,...,n-1$ such that $x\in[a_i,a_{i+1}]$. Let $g_x(h)=g_0-\gamma h+f(x-h)$ and let $L\le h^*\le U$ achieve the minimum of g_x. We consider two cases:

- $\gamma>\gamma_i$:
  Notice first that for all $h<0$, there exists $j\ge i$ such that $x-h\in[a_j,a_{j+1}]$, therefore, since $\gamma_j\le\gamma_i$, $f(x-h)\ge f(x)+\gamma_i h$ .Hence for all $h<0$, $g_x(h)\ge g_0-\gamma h+f(x)+\gamma_i h=g_0+f(x)+(\gamma_i-\gamma)h\ge g_0+f(x)=g_x(0)$, therefore the minimum of g_x is achieved for $h^*\ge 0$.

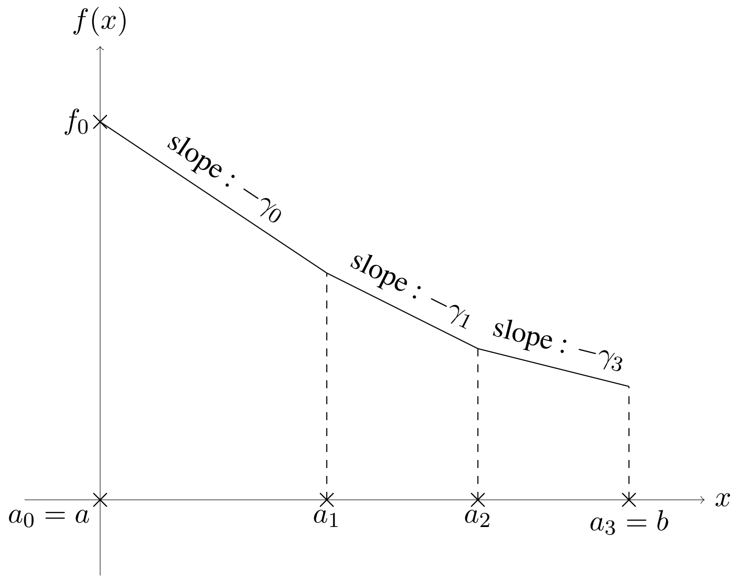

Figure 1: Example of function belonging to $\mathcal{C}$ of size 3

Let $\bar{h} \geq 0$ be the smallest $h \geq 0$ such that $x - \bar{h} = a_j$ with $j \leq i$ and $\gamma_{j-1} \geq \gamma$. If $h \geq \bar{h}$, then $x - h$ belongs to some interval $[a_k, a_{k+1}]$ where $\gamma_k \geq \gamma$. Similarly to the case "$\gamma > \gamma_i$ and $h < 0$" we can prove that the function g_x is non-decreasing on $[\bar{h}, h]$, therefore $h^* \leq \bar{h}$. We now prove that the function g_x is non-increasing on $[0, \bar{h}]$. Let $h \in [0, \bar{h}]$, there exists $k \leq i$ such that $x - h \in [a_k, a_{k+1}]$. Assume that $x - h \in]a_k, a_{k+1}[$. g_x admits a derivative at h and $g_x'(h) = -\gamma + \gamma_k \leq 0$ by definition of $\bar{h}$. By continuity of g_x we conclude that g_x is non decreasing on the whole interval $[0, \bar{h}]$, therefore $h^* = \begin{cases} U & \text{if } \bar{h} \geq U \\ \bar{h} & \text{otherwise} \end{cases}$ and

$$g(x) = \begin{cases} g_0 - \gamma U + f(x - U) & \text{if } \bar{h} \geq U \\ g_0 - \gamma \bar{h} + f(x - \bar{h}) & \text{otherwise} \end{cases} \tag{1}$$

- $\gamma_i \geq \gamma$:
  Similarly to the previous case, we prove that the minimum of g_x is attained for $h^* \leq 0$. Let $\bar{h} \leq 0$ be the biggest $h \leq 0$ such that $x - \bar{h} = a_j$ with $j > i$ and $\gamma_j \leq \gamma$. Similarly to the previous case, we have that $h^* = \begin{cases} L & \text{if } \bar{h} \leq L \\ \bar{h} & \text{otherwise} \end{cases}$ and

$$g(x) = \begin{cases} g_0 - \gamma L + f(x - L) & \text{if } \bar{h} \leq L \\ g_0 - \gamma \bar{h} + f(x - \bar{h}) & \text{otherwise} \end{cases} \tag{2}$$

We can now give the general expression of $g(x)$:
Let $i \in [0, ..., n-1]$ be the smallest index such that $\gamma > \gamma_i$. Then, by for all $x \in [a_i, a_i + U]$, $\bar{h} = x - a_i$, hence by Equation 1:

$$\begin{aligned} g(x) &= g_0 - \gamma(x - a_i) + f(x - (x - a_i)) \\ &= g_0 + \gamma a_i + f(a_i) - \gamma x \end{aligned}$$

If $x > a_i + U$, then $\bar{h} = U$, hence by Equation 1:

$$g(x) = g_0 - \gamma U + f(x - U)$$

Similarly, for all $x \in [a_i - L, a_i]$, $\bar{h} = x - a_i$, hence by Equation 2:

$$\begin{aligned} g(x) &= g_0 - \gamma(x - a_i) + f(x - (x - a_i)) \\ &= g_0 + \gamma a_i + f(a_i) - \gamma x \end{aligned}$$

and for all $x < a_i - L$

$$g(x) = g_0 - \gamma L + f(x - L)$$

We conclude therefore that g is a function belonging to $\mathcal{C}^{a,b}$ with coefficients $(\gamma_0, ..., \gamma_{i-1}, \gamma, \gamma_i ..., \gamma_{n-1})$ $\square$

Corollary 1. *Let $f \in \mathcal{C}^{a,b}$ with coefficients $(\gamma_0, ..., \gamma_{n-1})$:*
$f(x) = f_0 - \sum_{i=0}^{n-1} \gamma_i \max(\min(x - a_i, a_{i+1} - a_i), 0)$. *Let $L_1 \leq 0 \leq U_1$, $L_2 \leq 0 \leq U_2$ and let us consider the following function $g : [a, b] \mapsto \mathbb{R}$:*

$$g(x) = \min_{\substack{L_1 \leq h_1 \leq U_1 \\ L_2 \leq h_2 \leq U_2}} g_0 - \gamma h_1 - \gamma' h_2 + f(x - h_1 - h_2)$$

with $\gamma' < \gamma$. Then $g \in \mathcal{C}^{a,b}$ with coefficients $(\gamma_0, ..., \gamma, ..., \gamma', ..., \gamma_{n-1})$.

Proof. Let $g^1(x) = \min_{L_1 \leq h \leq U_1} g_0 - \gamma h + f(x - h)$. By Proposition 1, $g^1 \in \mathcal{C}^{a,b}$ with coefficients $(\gamma_0, ..., \gamma, ..., \gamma_{n-1})$. Hence, since
$g(x) = \min_{L_2 \leq h \leq U_2} -\gamma' h + g^1(x - h)$, we conclude that $g \in \mathcal{C}^{a,b}$ with coefficients $(\gamma_0, ..., \gamma, ..., \gamma', ..., \gamma_{n-1})$. $\square$

Let $f_1, f_2 \in \mathcal{C}^{a,b}$ respectively with coefficients $(\gamma_0, ..., \gamma_{n-1})$, $(\gamma'_0, ..., \gamma'_{m-1})$, and define $f = \max(f_1, f_2)$. It is obvious that $f \in \mathcal{C}^{a,b}$. Let $I \subseteq [0, ..., m-1]$ be the set of indexes i such that γ'_i is not one coefficient of f_1. Then we notice that the size of f is bounded by:

$$\text{size}(f) \leq \text{size}(f_1) + |I|$$

3.2 The dynamical program

Let (τ, ζ, σ) fixed, we define $f(d_t - p_t, \sigma_1, \sigma_2)$ has the cost induced at time $t = \tau$ if $\sigma - K \leq \sigma_1 \leq \sigma$ quantity of goods are (un)loaded from the storehouse and if σ_2 quantity of goods produced at $t = \tau$ are sold to the market. We have:

$$f(d_t - p_t, \sigma_1, \sigma_2) = \begin{cases} \alpha_\tau \max(d_t - p_t - \sigma_1, 0) - \beta_\tau \sigma_2 + c(\sigma - \sigma_1 - \sigma_2) & \text{if } d_t \geq p_t \\ \alpha_\tau \max(\sigma_1, 0) + c(\sigma + \sigma_2) & \text{otherwise} \end{cases} \tag{3}$$

The Bellman's equation verified by $v(\tau, \zeta, \sigma)$ is:

$$v(\tau, \zeta, \sigma) = \max_{\delta_\tau \in \{0,1\}} \left(\min_{\substack{\sigma - K \leq \sigma_1 \leq \min(\sigma, \max(d_t - p_t, 0)) \\ 0 \leq \sigma_2 \leq \max(p_t - d_t, 0)}} \Big(f(d_t - p_t, \sigma_1, \sigma_2) + v(\tau + 1, \zeta - \delta_\tau, \sigma - \sigma_1 - \sigma_2) \Big) \right) \tag{4}$$

where $d_t = \bar{d}_t + \delta_t \Delta_t$

Using the results of the previous section, we can prove (10 pages later ...), with the same kind of ideas than the ones developed in the EJOR article that the nested dynamical program solve the robust inventory problem in polynomial time.