A note on discrete robust optimization without the full recourse property

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Let (P) be the following mixed-integer linear program (MILP):

$$
(P)\left|\begin{array}{ll}
\min\limits_{x,y} \alpha x + \beta y & \\
\quad Ax + By \geq d & (1)\\
\quad Cx \geq b & (2)\\
\quad x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p,\ y \in \mathbb{R}^q_+. & (3)
\end{array}\right.
$$

where $A \in \mathbb{Q}^{T\times p},\ B \in \mathbb{Q}^{T\times q},\ d \in \mathbb{Q}^T$, $C \in \mathbb{Q}^{n\times p},\ b \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}^p_+,\ \beta \in \mathbb{Q}^q_+$, and $\mathbb{Q}$ is the set of rational numbers.
The associated robust problem (PR) can be written as:

$$
(PR)\left|\begin{array}{l}
\min\limits_{x} \ \alpha x \ + \ \max\limits_{d\in\mathfrak{D}} \min\limits_{y} \beta y \\
\qquad\qquad\qquad\qquad By \geq d - Ax \\
\qquad\qquad\qquad\qquad y \in \mathbb{R}^q_+ \\
\quad Cx \geq b \\
\quad x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p.
\end{array}\right.
$$

We denote, for all feasible x, by $R(x)$ the associated robust problem when x is fixed, and we denote by $R(x, d)$ the associated robust problem when x and d are fixed.
In this note we assume that the deterministic problem doesn't satisfied the full recourse property. We prove how to obtain a new MILP denoted $(\tilde{P})$ such that the robust associated problem has the same optimal solution as the initial robust problem and $(\tilde{P})$ satisfies the full recourse property. We only assume that the robust problem (PR) has a finite optimal solution, i.e. there exists M such that $v(PR) \leq M$. In addition, the method detects if the problem has no solutions. To obtain $(\tilde{P})$, we add new recourse variables $w_t,\ t = 1, ..., T$. Let ε be a given strictly positive value, we define the following MILP:

$$
\tilde{P}_\varepsilon \left|\begin{array}{ll}
\min\limits_{x,y,w} \alpha x + \beta y + \dfrac{M}{\varepsilon}\sum\limits_{t=1}^{T} w_t & \\
\quad Ax + By + w \geq d & (1\varepsilon)\\
\quad Cx \geq b & (2)\\
\quad x_i \in \mathbb{N},\ i = 1, ..., p & (3\varepsilon)\\
\quad y \in \mathbb{R}^q_+,\ w \in \mathbb{R}^T_+ & (4\varepsilon)
\end{array}\right.
$$

We notice that since the variables $w_t,\ t = 1, ..., T$, are not bounded, $(\tilde{P}_\varepsilon)$ satisfies the full recourse property. We denote by $(\tilde{PR})_\varepsilon$, the robust problem associated to $(\tilde{P}_\varepsilon)$, and by $(\tilde{R}_\varepsilon(x))$, $(\tilde{R}_\varepsilon(x, d))$ and $(\tilde{DR}_\varepsilon(x))$ the associated subproblems as those defined above. Notice that since all the inputs, $A, B, C, b, d, \alpha, \beta, \Delta$, of (PR) have rational coefficients, we can reduce $(\tilde{PR})_\varepsilon$ and all the corresponding subproblems to programs where all the inputs are integer. Therefore we assume from now that all the inputs are integer.

Proposition 1. $v(\tilde{PR}_\varepsilon)$ *satisfies* $0 \leq v(\tilde{PR}_\varepsilon) \leq v(PR) \leq M$.

Proof. Let $(\hat{x}, \hat{y})$ be an optimal solution of (PR), By hypothesis, $v(PR) \leq M$ thus $v(R(\hat{x})) \leq M$. Let $\bar{d}$ be a scenario in $\mathfrak{D}$. Since $v(R(\hat{x})) = \max\limits_{d \in \mathfrak{D}} v(\hat{R}(\hat{x}, d))$, we have $v(R(\hat{x}, \bar{d})) \leq M$. Let $\bar{y}$ be an optimal solution of $R(\hat{x}, \bar{d})$, we notice that $(y, w) = (\bar{y}, 0)$ is a feasible solution of $(\tilde{R}_\varepsilon(\hat{x}, \bar{d}))$ with the same cost. Thus $v(\tilde{R}_\varepsilon(\hat{x}, \bar{d})) \leq v(\hat{R}(\hat{x}, \bar{d}))$, for any $\bar{d} \in \mathfrak{D}$, which implies $v(\tilde{R}_\varepsilon(\hat{x})) \leq v(R(\hat{x}))$, and $0 \leq v(\tilde{PR}_\varepsilon) \leq v(PR) \leq M$. □

We know that for all x, the worst scenario for x is at an extreme point of $\mathfrak{D}$. Let $\{d^1, ..., d^S\}$, be the set of extreme points of $\mathfrak{D}$ (Notice that in our case all the extreme points are integers). We can rewrite the problem $(\tilde{PR})_\varepsilon$ as the following MILP:

$$
(PR'_\varepsilon) \left|\begin{array}{ll}
\min\limits_{\substack{x,r\\ y^1,...,y^S\\ w^1,...,w^S}} & \alpha_1 x_1 + \alpha_2 x_2 + r\\
 & r \geq \beta y^s + \dfrac{M}{\varepsilon}\sum\limits_{t=1}^{T} w^s_t,\ s = 1, ..., S\\
 & A_1 x_1 + A_2 x_2 + B y^s + w^s \geq d^s,\ s = 1, ..., S\\
 & C_1 x_1 + C_2 x_2 \geq b\\
 & x_1 \in \mathbb{N}^{p_1},\ x_2 \in \mathbb{R}^{p-p_1}_+\\
 & y^s \in \mathbb{R}^q_+,\ w^s \in \mathbb{R}^T_+,\ s = 1, ..., S
\end{array}\right.
$$

Where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ A = (A_1, A_2)\,,\ C = (C_1, C_2)\,.$
Let $(x_1^*, , x_2^*, r^*, y^{1*}, ..., y^{S*}, w^{1*}, ..., w^{S*})$ be an optimal solution of $(PR')_\varepsilon$. We notice that from Proposition 1, we have for all $s = 1, ..., S$:

$$\alpha_1 x_1^* + \alpha_2 x_2^* + \beta y^{s*} + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^{s*} \leq \alpha_1 x_1^* + \alpha_2 x_2^* + r^* \leq M.$$

Therefore, since α et β are positive, $\frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^{s*} \leq M$ and $w_t^{s*} \leq \varepsilon$, for all t, s. We now prove that for ε small enough, we have $w_t^{s*} = 0$.

We notice that $(x_2^*, r^*, y^{1*}, ..., y^{S*}, w^{1*}, ..., w^{S*})$ is an optimal solution of:

$$PR'_\varepsilon(x_1^*) \left| \begin{array}{lr} \min\limits_{\substack{x_2, r \\ y^1, ..., y^S \\ w^1, ..., w^S}} \ \alpha_2 x_2 + r & \\ r \geq \beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s,\ s = 1, ..., S & (4) \\ A_2 x_2 + B y^s + w^s \geq d^s - A_1 x_1^*,\ s = 1, ..., S & (5) \\ C_2 x_2 \geq b - C_1 x_1^* & (6) \\ x_2 \in \mathbb{R}_+^{p-p_1} & (7) \\ y^s \in \mathbb{R}_+^q,\ w^s \in \mathbb{R}_+^T,\ s = 1, ..., S & (8) \end{array} \right.$$

Dualizing the constraints $r \geq \beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s,\ s = 1, ..., S$ of the program above, we prove that the optimal value of the program above is equal to the optimal value of the following program:

$$\left| \begin{array}{l} \max\limits_{\lambda \geq 0} \ \min\limits_{\substack{x_2, r \\ y^1, ..., y^S \\ w^1, ..., w^S}} \ \alpha_2 x_2 + r + \sum_{s=1}^{S} \lambda_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - r \right) \\ A_2 x_2 + B y^s + w^s \geq d^s - A_1 x_1^*,\ s = 1, ..., S \\ C_2 x_2 \geq b - C_1 x_1^* \\ x_2 \in \mathbb{R}_+^{p-p_1} \\ y^s \in \mathbb{R}_+^q,\ w^s \in \mathbb{R}_+^T,\ s = 1, ..., S \end{array} \right.$$

Let $(\lambda', x_2', r', y'^1, ..., y'S, w'^1, ..., w'^S)$ be an optimal solution of the program above, then $(x_2', r', y'^1, ..., y'^S, z'^1, ..., z'^S)$ is an optimal solution of the following program:

$$
Aux(\lambda') \left| \begin{array}{ll}
\min\limits_{\substack{x_2,r\\ y^1,\dots,y^S\\ w^1,\dots,w^S}} & \alpha_2 x_2 + r + \sum_{s=1}^{S} \lambda'_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - r \right) \\
& A_2 x_2 + B y^s + w^s \geq d^s - A_1 x_1^*,\ s = 1,\dots,S \qquad (9) \\
& C_2 x_2 \geq b - C_1 x_1^* \qquad (10) \\
& x_2 \in \mathbb{R}_+^{p-p_1} \qquad (11) \\
& y^s \in \mathbb{R}_+^q,\ w^s \in \mathbb{R}_+^T,\ s = 1,\dots,S. \qquad (12)
\end{array} \right.
$$

We have that $v(Aux(\lambda')) = v(PR'_\varepsilon(x_1^*))$, therefore $(x_2^*, r^*, y^{1*}, \dots, y^{S*}, w^{1*}, \dots, w^{S*})$ is also an optimal solution of $Aux(\lambda')$, since it is a feasible solution of $Aux(\lambda')$ and that $\sum_{s=1}^{S} \lambda'_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - r \right) \leq 0$, for all feasible solution of $PR'_\varepsilon(x_1^*)$ Considering the slack variable e^s, $s = 1, \dots, S$ and f, we can rewrite the problem above as

$$
Aux(\lambda') \left| \begin{array}{ll}
\min\limits_{\substack{x_2,r\\ y^1,\dots,y^S\\ w^1,\dots,w^S\\ e^1,\dots,e^S,f}} & \alpha_2 x_2 + r + \sum_{s=1}^{S} \lambda'_s \left(\beta y^s + \frac{M}{\varepsilon} \sum_{t=1}^{T} w_t^s - r \right) \\
& A_2 x_2 + B y^s + w^s - e^s = d^s - A_1 x_1^*,\ s = 1,\dots,S \\
& C_2 x_2 - f = b - C_1 x_1^* \\
& x_2 \in \mathbb{R}_+^{p-p_1} \\
& y^s \in \mathbb{R}_+^q,\ w^s, e^s \in \mathbb{R}_+^T,\ s = 1,\dots,S.
\end{array} \right.
$$

Proposition 2. *Every optimal solution, $(x_2^*, r^*, y^{*1}, \dots, y^{*S}, w^{*1}, \dots, w^{*S}, e^{*1}, \dots., e^{*S}, f)$, of $Aux(\lambda')$ satisfies $w^{*s} = 0,\ s = 1, \dots, S$.*

Proof. We can rewrite the constraint of $Aux(\lambda')$ as $Lu = v$, where

$$
L = \begin{pmatrix}
A_2 & B & I_T & -I_T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
A_2 & 0 & 0 & 0 & B & I_T & -I_T & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
A_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & B & I_T & -I_T & 0 \\
C_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -I_T
\end{pmatrix} \in \mathbb{N}^{(ST+n)\times(p-p_1+S(q+2T)+T)},
$$

has rank $ST+n$,

$$u = \begin{pmatrix} x_2 \\ y^1 \\ w^1 \\ e_1 \\ \vdots \\ y^S \\ w^S \\ e_S \\ f \end{pmatrix}$$

and

$$v = \begin{pmatrix} d^1 - A_1 x_1^* \\ \vdots \\ d^S - A_1 x_1^* \\ b - C_1 x_1^* \end{pmatrix}$$

Assume that the optimal solution is a basic optimal solution, there exists a basic matrix $E = (e_{ij}) \in \mathbb{N}^{(ST+n)\times(ST+n)}$ and a basic vector u_E such that

$$u_E = \frac{1}{\det(E)} adj(E) v.$$

Assume there exists s and t such that w_t^{*s} belongs to the basic vector u_E, then there exists t' such that $w_t^{*s} = \frac{1}{\det(E)}(com(E)v)_{t'}$. Since $w_t^{*s} \leq \varepsilon$, we have that $|(com(E)v)_{t'}| \leq \varepsilon |\det(E)|$.
According to Hadamard's inequality,

$$|\det(E)| \leq \prod_{j=1}^{ST+n} \sqrt{\sum_{k=1}^{ST+n} e_{kj}^2}.$$

Therefore if we note $l_M = \max l_{ij}$, we have

$$|(com(E)v)_{t'}| \leq \varepsilon l_M^{(ST+n)} (ST+n)^{(ST+n)/2}.$$

Notice that in our case, $S = \binom{T}{\bar{z}}$.
Set ε to $\frac{1}{2}\frac{1}{l_M^{(ST+n)}(ST+n)^{(ST+n)/2}}$, then $|(com(E)v)_{t'}| \leq 1/2$. Since all the coefficients of E and v are integers, $|(com(E)v)_{t'}| \in \mathbb{N}$, therefore $|(com(E)v)_{t'}| = 0$ and $w_t^{*s} = 0$. Therefore

$$w^{*s} = 0, \ \forall s = 1, ..., S.$$

Assume now that u is not a basic optimal solution of $Aux(\lambda')$. Therefore we can write u as a the sum of a convex combinaison of basic optimal solutions and a positive combinaison of extreme rays of the constraints polyhedron of $Aux(\lambda')$. Necessarily all the basic optimal solutions,w^s, in the convex combinaison must satisfy $w^s = 0$ for all s. Furthermore, since all optimal solutions of $Aux(\lambda')$ satisfy $w^s \leq \varepsilon$ for all $s = 1, ..., S$, no extreme ray in the u decomposition can has a w^s coordinate different from 0. Therefore the proposition is proved. $\square$

According to Proposition 1, for all ε, $v(\tilde{PR_\varepsilon}) \leq v(PR)$. However, according to Proposition 2, taking $\varepsilon < \frac{1}{l_M^{(ST+n)}(ST+n)^{(ST+n)/2}}$, implies that $v(\tilde{PR_\varepsilon}) \geq v(PR)$ and

$$v(\tilde{PR_\varepsilon}) = v(PR).$$