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Abstract: Due to the emergence of bandwidth-requiring services, telecommunication operators are being compelled to renew their fix access network, most of them favoring the Fiber To The Home (FTTH) technology. This paper focuses on the optimization of FTTH deployment, which is of prime importance due to the economic stakes. The key design issue here is locating splitters and routing fi bers in an existing network infrastructure to which is associated a graph with given capacities on the edges. No assumption is made on the structure of the graph. First we propose a mixed integer formulation for this decision problem. Then, valid inequalities and problem size reduction schemes are presented. Finally efficiency of solving approaches is assessed through extensive numerical tests performed on Orange real-life data.

Research Highlights

We propose an original model for a new real optical network design. The model is a linear program which is a generalization of a model of flow with multipliers (generalized flows). We give valid inequalities. We propose specific reductions of the graph and prove their validity. We test our model on real instances with up to thousand nodes: we use Cplex and obtain lower bounds within less than 2% from the optimum. We prove the usefulness of the approach (cuts and reductions).

Optimizing the deployment of a multilevel optical FTTH network

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Abstract

Due to the emergence of bandwidth-requiring services, telecommunication operators are being compelled to renew their fix access network, most of them favoring the Fiber To The Home (FTTH) technology. This paper focuses on the optimization of FTTH deployment, which is of prime importance due to the economic stakes. The key design issue here is locating splitters and routing fibers in an existing network infrastructure to which is associated a graph with given capacities on the edges. No assumption is made on the structure of the graph. First we propose a mixed integer formulation for this decision problem. Then, valid inequalities and problem size reduction schemes are presented. Finally efficiency of solving approaches is assessed through extensive numerical tests performed on Orange real-life data.

Keywords: Optical telecommunication network, location, integer programming, generalized flow.

1 Introduction

The increasing use of IP telecommunication networks for ever more bandwidth consuming services (file transfers, video or audio streaming, cloud computing, ...) leads telecommunication operators to seriously consider the high volume roll-out of optical-fiber-based access networks. They have to renew their access networks that are clearly become the bottleneck in terms of bandwidth. Therefore most of telecommunication operators are currently withdrawing their legacy copper network, giving way to optical fiber networks.

To allow faster connections, the optical fiber gets closer and closer to the subscriber. In Figure 1, several architectures are described : the optical fiber stops in the neighborhood (curb) of the subscribers (FTTC), or at their building (FTTB), or even at their home (FTTH = Fiber To The Home).

FTTH seems the most suitable choice for a long term objective: if the clients are wholly served by optical fibers, it will be easier to increase the bandwidth in

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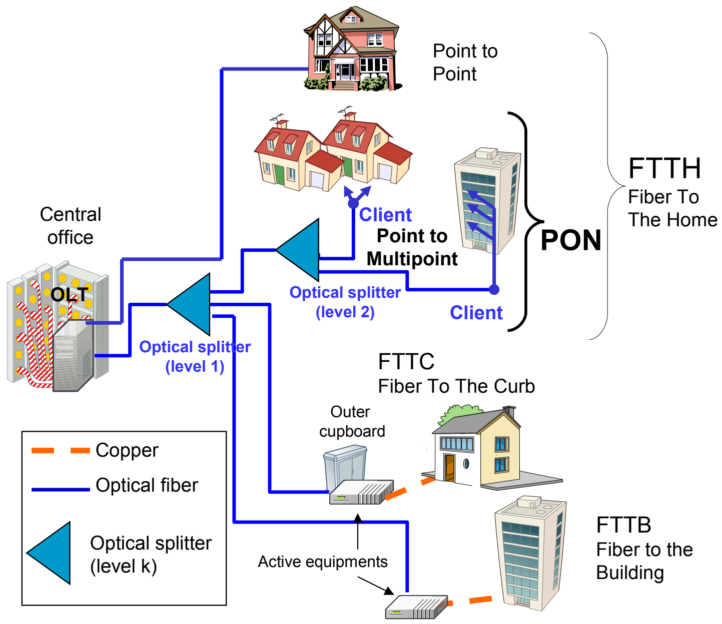

Figure 1: The different FTTx architectures

the future. That is why Orange (the main French telecommunication operator) has decided to gradually convert its copper wires network to an FTTH Passive Optical Network (PON). But this choice is obviously the one needing the most investments since the existing copper wire and splitters could not be reused. To avoid exorbitant civil engineering cost, the reuse of existing infrastructures, as sewers or ducts used at the moment for copper wires, is a major issue. A feasibility study made by Orange has shown that, in urban area, this existing civil engineering is quite abundant and allows to install the FTTH network without digging new trenches. As is generally the case, Orange's objective is to minimize the global cost of the network installation.

Access network design has been the subject of much researches dealing with different versions of the problem depending on the topologies and on which particular issue it is dealt with. [4], [6],[7] and [13], contain surveys on different existing models. These models often consider only one level of intermediate equipments and a star-star topology: a terminal is connected to an equipment and an equipment is directly connected to the central office.

One approach is the "Capacitated or Uncapacitated Local Access Network Design Problem" where one has to build a graph representing the topology of the network: an edge of the graph induces a fixed cost plus a variable cost depending on the capacity of the edge. The uncapacitated version is an NP-hard problem since it generalizes the classical Steiner Tree Problem and it is solved either by a heuristic or by integer linear programming. In [3], the authors study the polyhedron of the solutions and propose several valid inequalities. In [15], the authors compare several approaches. A variant considering a network using two technologies is found in [16]. These last models do not consider the cost of installing intermediate equipments and the topology of the obtained network is a tree. An efficient algorithm based on cutting-plane and column generation is proposed for survivable network design in [2]. A model for an access network hub location problem considering multiplexer installation cost is proposed in [18]; it is solved with a Lagrangian dual-based heuristic combined with an exact solution procedure. Another local access network design problem with stepwise link capacity is solved in [17]; the model is based on a flow formulation and the algorithm uses relaxations obtained by approximating the noncontinuous function by its convex envelope.

Problems involving more complex topologies, as tree-tree or topologies with three (or more) levels, are often split in several independent subproblems. See for instance [12] for a tree-tree network design problem: the authors first locate the equipments with a simple plant location model, second they solve an uncapacitated local access network design problem to connect the terminals to the equipments and third they connect the equipments to the central office.

In [5], the authors consider fixed and variable costs of the connections and they present a model dealing at the same time with the network topology and the location of the equipments. Their integer program is valid for more than one level of intermediate equipments and they propose a Branch&Bound algorithm that use lagrangean relaxation to compute lower bounds. Results are presented for one level of intermediate equipments and for graphs with at most 100 vertices and 400 edges.

There are few references devoted to FTTH networks. In [14], the authors present a heuristic to optimize the location of three types of equipments: cable splitting points, optical splitters, and distribution points. No existing infrastructure is considered and the heuristic locates each type of equipments separately. In [11], the authors propose a model optimizing location and capacities of one level of splitters; they solve a continuous location problem close to the multi-source Weber problem, with additional constraints on the fiber lengths and on the equipment capacities. The model closest to ours is proposed by Kim, Lee and Han [10]: the authors have to locate splitters and to assign fiber capacities in an FFTH-PON tree access network. They use the tree structure to propose a specific heuristic and integer programming models for one or two levels of splitters. They add some valid inequalities to the model to get lower bounds which allow them to guarantee their solutions. They obtain very small duality gaps for networks with twenty nodes or so. Some papers (see for instance [8]) deal with another FFTH optimization problem which is downstream from ours: the routing and wavelength assignment (RWA).

In this paper, we study an FTTH PON access network design problem where the network must be installed entirely in existing ducts. A fiber entering a splitter is split in several fibers of a higher level and there are two levels of splitters, i.e. three levels of fibers. The problem is to locate the splitters and to determine the route of the fibers in order to serve given demands, while respecting the capacity constraints of the ducts. All the demands are served by fibers at level three, the higher level of fibers. The cost to minimize includes the splitter costs and the optical fiber costs. We propose an integer model which can be viewed as an extension of the generalized flow problem (see [1]) where the multipliers are on the nodes (and not on the arcs as usual), where only a part of the flows entering a node is multiplied, and where we must distinguish several levels of flows that cannot be merged but share the same edge capacities. The proposed model and method consider any graph without specific structure.

First we describe the network architecture and the problem. Second, we propose an integer linear programming model for the problem. Then we propose some cuts and reductions reinforcing the model. Finally we give numerical results on several instances obtained from real life data for networks with up to three thousand nodes and a low density before concluding on the benefits of this model and future works to come.

2 The model

Orange has decided to use exclusively the existing network infrastructure that is supposed to be abundant enough to allow the development of the new optical connections required by the FTTH network. Let us modelize by an undirected graph $G = (V, E)$ this existing infrastructure. The vertices, or nodes, of V are denoted by v_i or simply i, $i = 0, ..., n$. v_0 is the Optical Line Termination (OLT). C is the set of nodes with a demand: $C = \{v_i \in V$ such that there is a Client at node $v_i\}$; a Client can represent several subscribers in the same building or neighborhood: in that case, a subscriber is linked to the Client by a copper fiber and the Client's demand is the sum of the subscribers demands (see Figure 1). The edges of G correspond to the existing ducts and have capacities equal to the maximum number of fibers that can be added in the duct. An edge is undirected but a fiber on an edge $[i, j]$ can be routed from i to j (if the fiber, on a path originated at OLT, meets i before j) or from j to i (otherwise).

We present in this section a location model for a three level architecture: from OLT to the S1 (splitters of level 1 connected to OLT by fibers of level 1), from the S1 to the S2 (splitters of level 2 connected to the S1 by fibers of level 2) and finally from the S2 to the Clients (all the Clients are served by fibers of level 3).

To each fiber of level k arriving in an optical splitter of level k ($k = 1, 2$) corresponds a fixed number m^k of fibers of level $k+1$ leaving the splitter; notice that, since m^k is a fixed parameter, a part of the m^k fibers produced by a splitter can be unused. There can be several splitters of different levels at a same node. The cost of the optical network contains the purchasing costs and the installation costs: it is the sum of the costs of the splitters, and the costs of the fibers that are linear functions of their lengths depending on the level. Costs of splitters of level 1 or 2 can be significantly different whereas linear costs of fibers of level 1, 2 or 3 are in practice quite homogeneous. The problem is to determine how locating the splitters and the fibers in order to minimize the global cost, while satisfying the demands and respecting the capacity constraints.

Let us now consider the whole problem.

The data are the following:

- l_{ij} : length of the edge $[i, j]$, $[i, j] \in E$;
- γ^k : linear cost of a level k fiber, $k = 1, 2, 3$.
- d_{ij}^k : cost of a fiber of level k routed on the edge $[i, j]$; $d_{ij}^k = l_{ij}\gamma^k$ for all $[i, j] \in E$, $k = 1, 2, 3$.
- C^k : cost of a splitter of level k, $k = 1, 2$.

- m^k : multiplying coefficient at level k, i.e. number of fibers of level $k+1$ leaving a splitter of level k (for only one fiber entering the splitter), $k = 1, 2$; $m^k \geq 2$.

- a_i : demand at node i , i.e. number of fibers of level 3 required by the Client at node i, $i = 0, ..., n$. $a_i = 0$ if $i \notin \mathcal{C}$;

- b_{ij} : capacity of the edge $[i, j]$, i.e. maximum number of fibers that can be added in the duct $[i, j]$, $[i, j] \in E$.

The integer variables are the following:

- $\mathbf{z}_i^k$: number of splitters of level k installed at node i, $i = 0, ..., n$, $k = 1, 2$.

- $\mathbf{f}_{ij}^k$: number of fibers of level k routed on the edge $[i, j]$ from i to j, $[i, j] \in E$, $k = 1, 2, 3$;

- $\mathbf{u}_i^k$: number of unused fibers of level k leaving a splitter of level $k-1$ at node i, $i = 0, ..., n$, $k = 2, 3$.

The FTTH location problem can be written as the following integer linear program:

$$
(\mathcal{PON}) \left\{
\begin{array}{lll}
\displaystyle \min_{\mathbf{f},\mathbf{z}} \quad \sum_{i=0}^{n} \sum_{k=1}^{2} C^k \, \mathbf{z}_i^k + \sum_{[i,j]\in E} \sum_{k=1}^{3} d_{ij}^k (\mathbf{f}_{ij}^k + \mathbf{f}_{ji}^k) & & \\
\text{such that :} & & \\
\displaystyle \sum_{j|[i,j]\in E} \mathbf{f}_{ji}^1 = \mathbf{z}_i^1 + \sum_{j|[i,j]\in E} \mathbf{f}_{ij}^1 & \forall i = 1, ..., n & (1) \\
\displaystyle \sum_{j|[i,j]\in E} \mathbf{f}_{ji}^2 + m^1 \, \mathbf{z}_i^1 = \mathbf{z}_i^2 + \sum_{j|[i,j]\in E} \mathbf{f}_{ij}^2 + \mathbf{u}_i^2 & \forall i = 0, ..., n & (2) \\
\displaystyle \sum_{j|[i,j]\in E} \mathbf{f}_{ji}^3 + m^2 \, \mathbf{z}_i^2 = a_i + \sum_{j|[i,j]\in E} \mathbf{f}_{ij}^3 + \mathbf{u}_i^3 & \forall i = 0, ..., n & (3) \\
\displaystyle \sum_{k=1}^{3} (\mathbf{f}_{ij}^k + \mathbf{f}_{ji}^k) \leq b_{ij} & \forall [i,j] \in E & (4) \\
\mathbf{z}_i^k \in \mathbb{N} \ \forall i = 0, ..., n, \ k = 1, 2; \qquad \mathbf{u}_i^k \in \mathbb{N} & \forall i = 0, ..., n, \ k = 2, 3 & \\
\mathbf{f}_{ij}^k \in \mathbb{N} \ \forall [i,j] \in E, k = 1, 2, 3 & &
\end{array}
\right.
$$

Constraints (1), (2) and (3) are similar to generalized flow equations (see [1]) at each node : here the multipliers are on the nodes and only a part of the flow entering the node is multiplied.

Constraints (1) ensure that the number of fibers of level 1 arriving at node i is equal to the number of split fibers, i.e. number of splitters of level 1 at node i, plus the number of unsplit fibers crossing i.

Constraints (2) ensure that the number of fibers of level 2 "arriving" at node i is equal to the number of split fibers, i.e. number of splitters of level 2 at node i, plus the number of unsplit fibers crossing i, plus the number of unused fibers. The number of fibers of level 2 (resp. 3) "arriving" at node i is the number of fibers of level 2 (resp. 3) really entering the node i, i.e. $\sum_{j|[i,j]\in E} \mathbf{f}_{ji}^2$ (resp.

$\sum_{j|[i,j]\in E} \mathbf{f}_{ji}^3$) plus the number of fibers of level 2 (resp. 3) leaving the splitters of level 1 (resp. level 2) installed at node i, i.e. $m^1\,\mathbf{z}_i^1$ (resp. $m^2\,\mathbf{z}_i^2$).

In the same way, constraints (3) ensure that the number of fibers of level 3 arriving at node i is equal to the demand at node i (0 if $i \notin C$), plus the number of unsplit fibers crossing i, plus the number of unused fibers.

Constraints (4) are the capacity constraints.

We present now some properties of any optimum solution.

Proposition 1. *In any optimum solution, we have:*

$$\mathbf{f}_{ij}^k \mathbf{f}_{ji}^k = 0 \ \forall [i,j] \in E, \ \ k = 1,2,3$$

Proof. Consider a solution $\mathcal{S}$ such that, for some $[\hat{\imath},\hat{\jmath}] \in E$ and some $\hat{k} \in \{1,2,3\}$, we have $0 < \mathbf{f}_{\hat{\imath}\hat{\jmath}}^{\hat{k}} \leq \mathbf{f}_{\hat{\jmath}\hat{\imath}}^{\hat{k}}$. We easily obtain a new solution $\mathcal{S}$' with a lower cost by setting: $\mathbf{f'}_{\hat{\imath}\hat{\jmath}}^{\hat{k}} = 0$, $\mathbf{f'}_{\hat{\jmath}\hat{\imath}}^{\hat{k}} = \mathbf{f}_{\hat{\jmath}\hat{\imath}}^{\hat{k}} - \mathbf{f}_{\hat{\imath}\hat{\jmath}}^{\hat{k}}$ and $\mathbf{f'}_{ij}^k = \mathbf{f}_{ij}^k$ for all $(i,j,k) \neq (\hat{\imath},\hat{\jmath},\hat{k})$. $\mathcal{S}$' verifies the constraints and the cost is decreased by $2d_{\hat{\imath}\hat{\jmath}}^{\hat{k}} f_{\hat{\imath}\hat{\jmath}}^{\hat{k}}$. □

More generally, in any optimum solution, we cannot have a circuit (directed cycle) such that $\mathbf{f}_{ij}^k > 0$ on all arcs on the circuit: let f_{inf} be the lowest value of $\mathbf{f}_{ij}^k$ on the circuit; we could get a better solution by decreasing by f_{inf} all the $\mathbf{f}_{ij}^k$ along the circuit. At least one of them becomes null and the circuit does not exist anymore.

In addition, we give the following property relative to the unused fibers:

Proposition 2. *In any optimum solution, we have:*

$$\mathbf{u}_i^{k+1} < m^k \ \forall i = 1,..,n \ \ k = 1,2$$

Proof. Consider a solution with $u_i^{k+1} \geq m^k$ for some k; all level k splitters being equivalent, we can consider that one of them produces m^k unused fiber, which is trivially useless. The solution is not optimal since there is another solution with (at least) one less splitter. □

To improve the efficiency in solving the mathematical program we add some inequalities to the model and we show how to reduce the initial problem.

3 Valid inequalities based on flow constraints

We present some valid inequalities following the mixed-integer rounding principle (see for instance [19]).

Preliminary result

Let Q be the set of points $(\mathbf{x},\mathbf{y})$ verifying :

$$\begin{cases} \beta\mathbf{x} & +\mathbf{y} & \geq \alpha \\ \mathbf{x} \in \mathbb{N} & \mathbf{y} \in \mathbb{N} & \end{cases}$$

with $\alpha \in \mathbb{N}, \beta \in \mathbb{N}$.

The polyhedra P, defined by the following inequalities : $\left\{ \begin{array}{lll} \beta\mathbf{x} & +\mathbf{y} & \geq \alpha \\ \mathbf{x} \geq 0 & \mathbf{y} \geq 0 & \end{array} \right.$, contains Q.

Now, let us denote by r the remainder and by q the quotient of the integer division of α by β: $q = \left\lfloor \frac{\alpha}{\beta} \right\rfloor$, and $r = \alpha - \left\lfloor \frac{\alpha}{\beta} \right\rfloor \beta$. We have the following proposition:

Proposition 3.

$$r\mathbf{x} + \mathbf{y} \geq r(q+1) \tag{5}$$

is a valid inequality for Q .

Proof. We divide $\beta\mathbf{x} + \mathbf{y} \geq \alpha$ by β and then we apply Proposition 8.6 in [19] (p. 127) to the inequality $x + \frac{y}{\beta} \geq \frac{\alpha}{\beta}$ with $x \in \mathbb{N}$ and $\frac{y}{\beta} \in \mathbb{R}^+$. □

The proposed inequality truncates the polyhedra P in the region defined by $q < \mathbf{x} < q+1$, as shown in Figure 2. In the following, we only consider cases where $r > 0$, else the inequality (5) is useless.

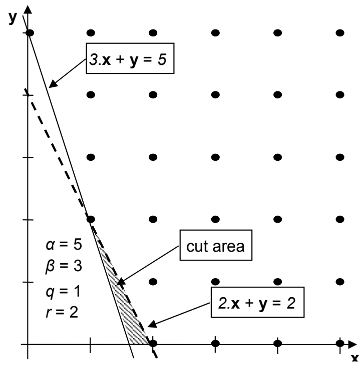

Figure 2: A valid inequality obtained with (5)

Applications to $(\mathcal{PON})$

Let A be a subset of nodes : $A \subset V \backslash \{v_0\}$. We have the identity:

$$\sum_{i \in A} \sum_{j \in A | [i,j] \in E} \mathbf{f}_{ji}^k = \sum_{i \in A} \sum_{j \in A | [i,j] \in E} \mathbf{f}_{ij}^k \quad k = 1, 2, 3$$

Then, when summing Equations (3) for all $i \in A$ we get:

$$\begin{aligned} \sum_{i \in A} u_i^3 &= \sum_{i \in A} m^2 \mathbf{z}_i^2 + \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} (\mathbf{f}_{ji}^3 - \mathbf{f}_{ij}^3) - \sum_{i \in A} a_i \geq 0 \\ \Rightarrow \sum_{i \in A} a_i &\leq \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} (\mathbf{f}_{ji}^3 - \mathbf{f}_{ij}^3) + m^2 \sum_{i \in A} \mathbf{z}_i^2 \\ \Rightarrow \sum_{i \in A} a_i &\leq \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} \mathbf{f}_{ji}^3 + m^2 \sum_{i \in A} \mathbf{z}_i^2 \qquad (6) \end{aligned}$$

By denoting: $\alpha = \sum_{i \in A} a_i$ (= total demand in A), $\beta = m^2$, $\mathbf{x} = \sum_{i \in A} \mathbf{z}_i^2$ and $\mathbf{y} = \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} \mathbf{f}_{ji}^3$, we get an equation of the type $\beta \mathbf{x} + \mathbf{y} \geq \alpha$, with $\mathbf{x}, \mathbf{y} \in \mathbb{N}$.
From the preliminary results, with $r = \alpha - \lfloor \frac{\alpha}{m^2} \rfloor m^2 > 0$ and then $q+1 = \lceil \frac{\alpha}{m^2} \rceil$, we get the following valid inequality:

Proposition 4.

$$\sum_{i \in A} \sum_{j \notin A | [i,j] \in E} \mathbf{f}_{ji}^3 + \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2 \right) \sum_{i \in A} \mathbf{z}_i^2 \geq \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2 \right) \left\lceil \frac{\alpha}{m^2} \right\rceil \qquad (7)$$

is a valid inequality for $(\mathcal{PON})$.

To get an interesting inequality, we must choose A such that $A \cap C \neq \emptyset$ $(\alpha > 0)$ and such that $\alpha = \sum_{i \in A} a_i$ is not multiple of m^2.

Now, we show how obtaining new valid inequalities.
From Equations (2) we have:

$$\mathbf{z}_i^2 + \mathbf{u}_i^2 = \sum_{j | [i,j] \in E} (\mathbf{f}_{ji}^2 - \mathbf{f}_{ij}^2) + m^1 \mathbf{z}_i^1$$

Since $\mathbf{u}_i^2 \geq 0$, by summing Equations (2) for all $i \in A$, we get:

$$\sum_{i \in A} \mathbf{z}_i^2 \leq \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} (\mathbf{f}_{ji}^2 - \mathbf{f}_{ij}^2) + m^1 \sum_{i \in A} \mathbf{z}_i^1$$

$$\Rightarrow \sum_{i \in A} \mathbf{z}_i^2 \leq \sum_{i \in A} \sum_{j \notin A | [i,j] \in E} \mathbf{f}_{ji}^2 + m^1 \sum_{i \in A} \mathbf{z}_i^1 \qquad (8)$$

Then, from Equations (6) and (8), we have:

$$\alpha = \sum_{i\in A} a_i \leq \sum_{i\in A} \sum_{j\notin A|[i,j]\in E} \left(\mathbf{f}^3_{ji} + m^2\mathbf{f}^2_{ji}\right) + m^2m^1 \sum_{i\in A} \mathbf{z}^1_i$$

Let $\mathbf{x} = \sum_{i\in A} \mathbf{z}^1_i$, $\mathbf{y} = \sum_{i\in A}\sum_{j\notin A|[i,j]\in E}(\mathbf{f}^3_{ji} + m^2\mathbf{f}^2_{ji})$, and $\beta = m^1m^2$. We get other valid inequalities of type $\beta\mathbf{x} + \mathbf{y} \geq \alpha$, with $\mathbf{x}, \mathbf{y} \in \mathbb{N}$. With $r = \alpha - \left\lfloor \frac{\alpha}{m^1m^2} \right\rfloor m^1m^2 > 0$ and $q + 1 = \left\lceil \frac{\alpha}{m^1m^2} \right\rceil$, we obtain:

Proposition 5.

$$\sum_{i\in A} \sum_{j\notin A|[i,j]\in E} (\mathbf{f}^3_{ji} + m^2\mathbf{f}^2_{ji}) + \left(\alpha - \left\lfloor \frac{\alpha}{m^1m^2} \right\rfloor m^1m^2\right) \sum_{i\in A} \mathbf{z}^1_i \geq \left(\alpha - \left\lfloor \frac{\alpha}{m^1m^2} \right\rfloor m^1m^2\right) \left\lceil \frac{\alpha}{m^1m^2} \right\rceil$$

is a valid inequality for $(\mathcal{PON})$.

Here, we shall choose A such that $A \cap C \neq \emptyset$ ($\alpha > 0$) and $\alpha = \sum_{i\in A} a_i$ not multiple of m^1m^2.

In the same way, with Equations (7) and (8), we get:

$$\sum_{i\in A} \sum_{j\notin A|[i,j]\in E} \mathbf{f}^3_{ji} + \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) \sum_{i\in A} \sum_{j\notin A|[i,j]\in E} \mathbf{f}^2_{ji} + \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) m^1 \sum_{i\in A} \mathbf{z}^1_i \geq \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) \left\lceil \frac{\alpha}{m^2} \right\rceil$$

Let $\alpha' = \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) \left\lceil \frac{\alpha}{m^2} \right\rceil$ and $\beta' = \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) m^1$. We get the following proposition:

Proposition 6. *Let*

- $q' = \left\lfloor \frac{\alpha'}{\beta'} \right\rfloor = \left\lfloor \frac{1}{m^1} \left\lceil \frac{\alpha}{m^2} \right\rceil \right\rfloor$
- $r' = \alpha' - \left\lfloor \frac{\alpha'}{\beta'} \right\rfloor \beta' = \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) \left(\left\lceil \frac{\alpha}{m^2} \right\rceil - \left\lfloor \frac{1}{m^1} \left\lceil \frac{\alpha}{m^2} \right\rceil \right\rfloor m^1\right)$
- $\mathbf{x}' = \sum_{i\in A} \mathbf{z}^1_i$
- $\mathbf{y}' = \sum_{i\in A}\sum_{j\notin A|[i,j]\in E} \mathbf{f}^3_{ji} + \left(\alpha - \left\lfloor \frac{\alpha}{m^2} \right\rfloor m^2\right) \sum_{i\in A}\sum_{j\notin A|[i,j]\in E} \mathbf{f}^2_{ji}$

Then, $r'\mathbf{x}' + \mathbf{y}' \geq r'(q'+1)$ *is a valid inequality for* $(\mathcal{PON})$.

We choose A such that α (> 0) is not multiple of m^2 and $\left\lceil \frac{\alpha}{m^2} \right\rceil$ is not a multiple of m^1; then: r' is strictly positive.

We could obtain other valid inequalities by using different combinations of the flow constraints. We shall see in Section 5 that these cuts are efficient for solving the problem $(\mathcal{PON})$.

4 Graph reductions

The graph $G = (V, E)$ defined in section 2 has generally a great number of nodes (several thousands) but a very low density. In this section, we propose some reductions inspired by those used for the Steiner tree problem (see [9]). They are applied recursively to G. These reductions reduce drastically the number of nodes and make easier the use of integer linear programming to solve the problem, as it will be seen in Section 5.

An edge $[v, w]$ of E is simply note vw, $\Gamma(v)$ denotes the set of nodes adjacent to v. In the following, if we "remove" a node v with $\Gamma(v) = \{w\}$ then we remove the edge vw and we set $a_w = a_w + a_v$. If we "remove" a node $v \notin C$ with $\Gamma(v) = \{t, w\}$ then we replace the edges tv and vw by an only edge tw with capacity $b_{tw} = min(b_{tv}, b_{vw})$.

Let v be such that $\Gamma(v) = \{t\}$: if $v \notin C$, it is clear that v can be removed from the graph.

We assume now that $\Gamma(v) = \{t, w\}$ and $v \notin C$. Considering a solution $\mathcal{S}^o$ with splitters on v, we are going to show that we can obtain a better (or equal) solution $\mathcal{S}^*$ by "moving" these splitters to t or to w. Doing so, we prove that there is an optimal solution without splitters in v and thus, v can be removed from the graph. "Moving" a splitter from v (in $\mathcal{S}^o$) to t (in $\mathcal{S}^*$) means that in $\mathcal{S}^*$ there is one less splitter in v and one more in t and so the cost relative to the splitters is the same in $\mathcal{S}^o$ as in $\mathcal{S}^*$; the fibers connected to this splitter are the same but in $\mathcal{S}^*$ they originate (or end) at t while in $\mathcal{S}^o$ they originate (or end) at v, all the other fibers remain unchanged (see Figure 3), so the cost of the fibers in $\mathcal{S}^*$ differs from the one of $\mathcal{S}^o$ only for the cost relative to the fibers connected to the moved splitters. We note $\Delta_{\mathcal{S}} = cost(\mathcal{S}^o) - cost(\mathcal{S}^*)$. In addition, the transformation from $\mathcal{S}^o$ to $\mathcal{S}^*$ is possible only if the capacity constraint on vt is verified (the flow remains unchanged on all the other edges). A splitter can be moved from v to w in a similar way.

Remark: in the following, we only consider solutions such that the number of level k splitters is (locally) minimized at each node, i.e. it is the minimum number necessary to produce the level $k+1$ fibers originated at this node. This property is obviously verified by any optimum solution.

We explain first in which direction (t or w) we move level 2 splitters. We define some more variables:

φ^3_{vt} (resp. φ^3_{vw}) is the total number of level 3 fibers produced by all the level 2 splitters in v and routed to t (resp. w). In $\mathcal{S}^o$, from the previous remark, we have: $z^{o2}_v = \left\lceil \frac{\varphi^3_{vt}+\varphi^3_{vw}}{m^2} \right\rceil$. $\Delta_{\mathcal{S}} = \Delta^2_{vt}$ (resp. Δ^2_{vw}) is the variation of the cost when a level 2 splitter is moved from v to t (resp. w). It takes into account the modifications of the level 2 fibers that connects the splitter and of all the level 3 fibers produced by this splitter. Recall that $d^k_{vt} = l_{vt}\gamma^k$. We assume that $\gamma^i \leq 2\gamma^j$ for all $(i, j) \in \{1, 2, 3\}^2$, which is compliant with reality as explained in the beginning of Section 2; this implies $\gamma^i \leq \gamma^j + \gamma^k$ for all $(i, j, k) \in \{1, 2, 3\}^3$.

Proposition 7. *Let v be such that $\Gamma(v) = \{t, w\}$. There is an optimal solution with at most two splitters of level 2 in v and with $\varphi^3_{vt} < m^2$ and $\varphi^3_{vw} < m^2$.*

Proof. We can consider that $\left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor$ level 2 splitters in v produce $m^2 \left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor$ level 3 fibers routed to t and that $\left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor$ level 2 splitters produce $m^2 \left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor$ level 3 fibers routed to w, that is each one of these splitters produce m^2 fibers routed either all to t or all to w. If we move from v to t a splitter producing fibers routed to t, the cost is not increased since:

if the splitter is connected to a level 2 fiber routed from t then $\Delta^2_{vt} = -d^2_{vt} - m^2 d^3_{vt} < 0$

else $\Delta^2_{vt} = d^2_{vt} - m^2 d^3_{vt} = l_{vt}(\gamma^2 - m^2\gamma^3) \leq l_{vt}(\gamma^2 - 2\gamma^3) \leq 0$ (since $m^2 \geq 2$ and $\gamma^2 \leq 2\gamma^3$).

In addition, the capacity constraints are verified since the flow on vt is decreased by at least $m^2 - 1 \geq 1$.

The proof is similar for moving to w a splitter producing m^2 fibers routed to w.

Let $\mathcal{S}^*$ be obtained from $\mathcal{S}^o$ by moving these $\left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor$ splitters to t and $\left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor$ splitters to w: $\mathcal{S}^*$ is a solution at least as good as $\mathcal{S}^o$. The number of level 2 splitters remaining in v is at most two since ${z^*}^2_v = {z^o}^2_v - \left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor - \left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor = \left\lceil \frac{\varphi^3_{vt}+\varphi^3_{vw}}{m^2} \right\rceil - \left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor - \left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor \leq 2$.

They produce $\varphi^3_{vt} - m^2 \left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor < m^2$ level 3 fibers routed to t and $\varphi^3_{vw} - m^2 \left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor < m^2$ level 3 fibers routed to w (plus possibly unused fibers). If $\varphi^3_{vt} - m^2 \left\lfloor \frac{\varphi^3_{vt}}{m^2} \right\rfloor = 0$ and (resp. or) $\varphi^3_{vw} - m^2 \left\lfloor \frac{\varphi^3_{vw}}{m^2} \right\rfloor = 0$ there is 0 (resp. 1) splitter in v.

□

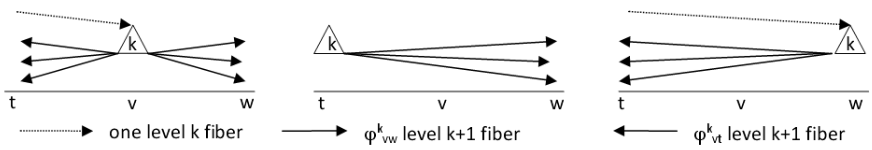

Figure 3: Moving from v to t or to w a splitter connected to t

Now we consider solutions verifying Proposition 7 and we can give the following proposition:

Proposition 8. *Let v be such that $\Gamma(v) = \{t, w\}$. There is an optimal solution with at most one splitter of level 2 in v.*

Proof. If $0 \leq \varphi^3_{vt} + \varphi^3_{vw} \leq m^2$ there is at most one level 2 splitter in v. Consider now a solution $\mathcal{S}^o$ with two level 2 splitters in v: $\varphi^3_{vt} + \varphi^3_{vw} \geq m^2 + 1 \geq 3$, thus $\varphi^3_{vt} \geq 2$ and/or $\varphi^3_{vw} \geq 2$. From Proposition 7, $\varphi^3_{vt} < m^2$ and $\varphi^3_{vw} < m^2$; so we can consider that one of the splitters produces φ^3_{vt} fibers routed to t and the other produces φ^3_{vw} fibers routed to w. If $\varphi^3_{vt} \geq 2$ (resp. $\varphi^3_{vw} \geq 2$), we modify $\mathcal{S}^o$ by moving the splitter producing these fibers to t (resp. to w): the capacity constraints are verified since the flow is decreased by at least $\varphi^3_{vt} - 1 \geq 0$ on vt (resp. $\varphi^3_{vw} - 1 \geq 0$ on vw). The cost is not increased: $\Delta^2_{vt} \leq d^2_{vt} - \varphi^3_{vt} d^3_{vt} = l_{vt}\left(\gamma^2 - \varphi^3_{vt}\gamma^3\right) \leq 0$ since $\varphi^3_{vt} \geq 2$ and $\gamma^2 \leq 2\gamma^3$ (resp. similarly $\Delta^2_{vw} \leq 0$).

□

So we consider now solutions with at most one level 2 splitter in v. For the level 1 splitters, we have:

Proposition 9. *Let v be such that $\Gamma(v) = \{t, w\}$. There is an optimal solution with at most one splitter of level 1 in v.*

Proof. Let $\mathcal{S}^o$ be a solution with $z^{o}{}^2_v \in \{0,1\}$ and $z^{o}{}^1_v \geq 2$. We build first a solution $\mathcal{S}^*$ with $z^*{}^1_v \leq 2$ and then a new one with $z^*{}^1_v \leq 1$. The proof is very similar to the proof for splitters of level 2 and we leave it to the reader. Notice that the fact that $z^{o}{}^2_v \leq 1$ from Proposition 8 makes the proof easier. $\square$

Now, we can consider solutions with $z^1_v \in \{0,1\}$ and $z^2_v \in \{0,1\}$ and we give the last proposition.

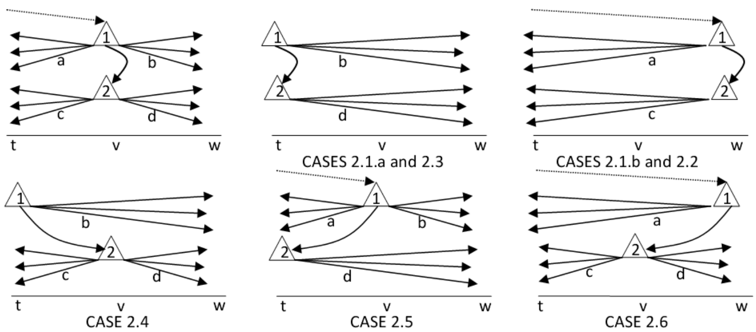

Figure 4: Moving a splitter connected to another splitter at v

Proposition 10. *Let v be such that $\Gamma(v) = \{t, w\}$. There is an optimal solution such that $z^*{}^1_v = z^*{}^2_v = 0$*

Proof. We study two cases in v: in the first one, we can have one splitter or two splitters that are not connected (see Figure 3); in the second one, there are two splitters connected by a level 2 fiber (see Figure 4).

For each case, we propose a new solution $\mathcal{S}^*$ without splitter in v such that the capacity constraints are verified and the cost is not increased.

CASE 1 Consider a level k splitter in v that is not connected to another splitter in v, $k \in \{1,2\}$. We assume, w.l.o.g., that it is connected to t (see Figure 3). If it is moved to t we have $\Delta_{\mathcal{S}} = \Delta^k_{vt} = -d^k_{vt} + \left(\varphi^{k+1}_{vw} - \varphi^{k+1}_{vt}\right) d^{k+1}_{vt} = l_{vt}\left(-\gamma^k + \left(\varphi^{k+1}_{vw} - \varphi^{k+1}_{vt}\right)\gamma^{k+1}\right)$. If it is moved to w we have $\Delta_{\mathcal{S}} = \Delta^k_{vw} = d^k_{vw} + (\varphi^{k+1}_{vt} - \varphi^{k+1}_{vw}) d^{k+1}_{vw} = l_{vw}\left(\gamma^k + (\varphi^{k+1}_{vt} - \varphi^{k+1}_{vw})\gamma^{k+1}\right)$.

1. If $\Delta^k_{vw} \leq 0$ then the splitter is moved to w.

   The cost is not increased. Let us verify the capacity constraint on vw: $\gamma^k + \left(\varphi^{k+1}_{vt} - \varphi^{k+1}_{vw}\right)\gamma^{k+1} \leq 0$ and $\gamma^{k+1} \leq 2\gamma^k \Rightarrow \left(\varphi^{k+1}_{vt} - \varphi^{k+1}_{vw}\right)\gamma^{k+1} \leq -\gamma^k \leq -\frac{1}{2}\gamma^{k+1} \Rightarrow \varphi^{k+1}_{vt} - \varphi^{k+1}_{vw} \leq -\frac{1}{2} \Rightarrow \varphi^{k+1}_{vt} - \varphi^{k+1}_{vw} \leq -1$ since the flows are integral. The flow on vw is increased by $1 + \left(\varphi^{k+1}_{vt} - \varphi^{k+1}_{vw}\right) \leq 0$.

2. If $\Delta^k_{vw} > 0$ then the splitter is moved to t.

   $\Delta^k_{vw} > 0 \Rightarrow \Delta^k_{vt} < 0$ so the cost is decreased. Moreover, $\Delta^k_{vt} < 0 \Rightarrow \left(\varphi^{k+1}_{vw} - \varphi^{k+1}_{vt}\right)\gamma^{k+1} < \gamma^k \Rightarrow \varphi^{k+1}_{vw} - \varphi^{k+1}_{vt} < \frac{\gamma^k}{\gamma^{k+1}}$; since the flows are integral and $\frac{\gamma^k}{\gamma^{k+1}} \leq 2$ we obtain $\varphi^{k+1}_{vw} - \varphi^{k+1}_{vt} \leq 1$. The flow on vt is increased by $-1 + \varphi^{k+1}_{vw} - \varphi^{k+1}_{vt} \leq 0$ so the capacity constraint is verified.

CASE 2 There is a level 1 splitter connected to a level 2 splitter in v. (See Figure 4).

To simplify the notation, we set $a = \varphi^2_{vt}$, $b = \varphi^2_{vw}$, $c = \varphi^3_{vt}$, $d = \varphi^3_{vw}$ and we assume w.l.o.g. that the level 1 splitter is connected to t. Let us study each case:

1. $a + c + 1 = b + d$: splitters are moved either both to t or both to w

   The capacity constraints on vw and vt are verified if we move both splitters either to t or to w.

   If $-\gamma^1 + (b-a)\gamma^2 + (d-c)\gamma^3 \leq 0$ then we move both splitters to t (case 2.1-a), otherwise $\gamma^1 + (a-b)\gamma^2 + (c-d)\gamma^3 \leq 0$ and we move both splitters to w (case 2.1-b); and we have $\Delta_{\mathcal{S}} \leq 0$

   In the following, $a + c + 1 \neq b + d$

2. $a \geq b - 1$ and $c \geq d$: both splitters are moved to t

   $b + d - 1 \leq a + c$ so the capacity constraint on vt is verified.

   $\Delta_{\mathcal{S}}$ has the same sign as $\delta = -\gamma^1 + (b-a)\gamma^2 + (d-c)\gamma^3$.

   If $a \geq b - 1$ and $c \geq d + 1$ then $\delta \leq -\gamma^1 + \gamma^2 - \gamma^3 \leq 0$.

   If $a \geq b$ and $c = d$ then $\delta \leq -\gamma^1 \leq 0$.

3. $a \leq b$ and $c \leq d$ and $a + c \leq b + d - 2$: both splitters are moved to w

   $1 + a + c \leq b + d$ so the capacity constraint on vw is verified.

   $\Delta_{\mathcal{S}}$ has the same sign as $\delta = \gamma^1 + (a-b)\gamma^2 + (c-d)\gamma^3$.

   If $\gamma^2 \leq \gamma^3$ then $\delta \leq \gamma^1 + (a + c - b - d)\gamma^2 \leq \gamma^1 - 2\gamma^2 \leq 0$.

   If $\gamma^3 \leq \gamma^2$ then $\delta \leq \gamma^1 + (a + c - b - d)\gamma^3 \leq \gamma^1 - 2\gamma^3 \leq 0$.

4. $a \geq b + 1$ (and $c \leq d - 1$): the level 1 splitter is moved to t.

   $b + 1 \leq a + 1$ so the capacity constraint on vt is verified.

   $\Delta_{\mathcal{S}}$ has the same sign as $\delta = -\gamma^1 + (b - a + 1)\gamma^2 \leq -\gamma^1 \leq 0$.

   It remains one level 2 splitter in v and we are back to CASE 1: this splitter is moved to t or w.

5. $c \geq d + 2$ (and $a \leq b - 2$): the level 2 splitter is moved to t.

   $d + 1 \leq c$ so the capacity constraint on vt is verified.

   $\Delta_{\mathcal{S}}$ has the same sign as $\delta = \gamma^2 + (d-c)\gamma^3 \leq \gamma^2 - 2\gamma^3 \leq 0$.

   It remains one level 1 splitter in v and we are back to CASE 1: this splitter is moved to t or w.

6. $a \leq b - 3$ (and $c = d + 1$): the level 1 splitter is moved to w.

   $1 + a \leq b$ so the capacity constraint on vw is verified.

   $\Delta_{\mathcal{S}}$ has the same sign as $\delta = \gamma^1 + (a - b + 1)\gamma^2 \leq \gamma^1 - 2\gamma^2 \leq 0$.

   It remains one level 2 splitter in v and we are back to CASE 1: this splitter is moved to t or w.

All the possibilities have been studied since each case for a and b ($a \leq b-3, a = b-2, a = b-1, a = b, a \geq b+1$) has been coupled with each case for c and d ($c \leq d-2, c = d-1, c = d, c = d+1, c \geq d+2$). □

Reduction Let v be a node with exactly two neighbors t and w. If $v \notin C$, then v can be removed. Let v be a node with a single neighbor t, then v can be removed.

Proof. If v with two neighbors has no demand, from Proposition 10 there is a solution without splitter on v, thus v can be removed from the graph as it is explained at the beginning of Section 4. For each vertex v removed, a solution $\mathcal{S}$ in the initial graph (network) is obtained from a solution $\mathcal{S}^*$ in the reduced graph by setting: $f^k_{tv} = f^k_{vw} = f^{*k}_{tw}$ and $f^k_{wv} = f^k_{vt} = f^{*k}_{wt}$ for $k = 1, 2, 3$. This is done recursively, in reverse order from the order considered to reduce the graph. If now we consider a vertex v with a single neighbor t, we are in a special case of the case with 2 neighbors where $\varphi^k_{vw} = 0$ for $k \in \{1, 2, 3\}$ thus there is an optimal solution without splitter in v; the fibers routed on vt are used to satisfy the demand in v: we can add this demand to the demand in t and remove v from the graph. For each vertex v removed, a solution in the initial graph (network) is obtained from a solution $\mathcal{S}^*$ in the reduced graph by setting: $f^3_{tv} = a_v$. □

5 Numerical tests

5.1 Test description

The objective of the performed tests is to assess the efficiency of Branch & Bound approaches for solving the problem, and particularly to measure the benefit obtained through the proposed reduction schemes and by adding the valid inequalities. Tests have been performed on real data from Orange. Two categories of areas have been focused on, they are representative of where FTTH deployments have already started or are impending. We distinguish areas with high density of population (typically corresponding to the largest cities in France) and areas with moderate density of population as small towns. Instances where the existing civil engineering infrastructure and the location of the Clients are represented for both cases in Figures 5 and 6.

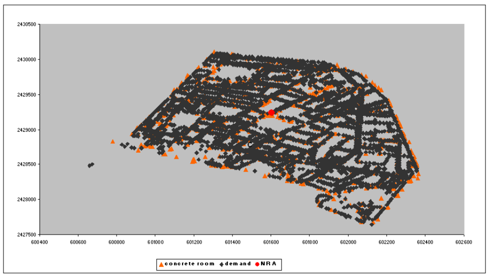

Figure 5: Local area in Paris (capital of France)

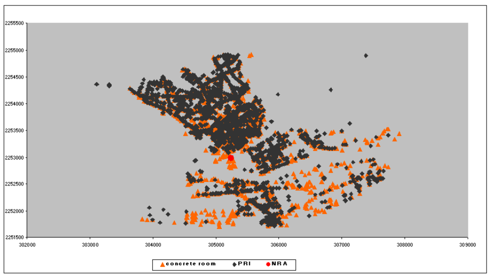

Figure 6: Local area in the surroundings of Nantes (West part of France)

Each instance is characterized by the structure of the existing civil engineering and the demand in fibers. The demand is described through both the number of points of cumulative demand (i.e. the number of Clients), denoted by nb_{Client} and the average demand per Client, denoted by d_{Client}. Description of the test instances is given in Table 1.

Instance	Existing infrastructure		Fiber demand	
	$\|V\|$	$\|E\|$	nb_{Client}	d_{Client}
Data_1	342	375	184	72.7
Data_2	920	1000	570	73.0
Data_3	1072	1163	667	78.2
Data_4	932	951	583	75.9
Data_5	1478	1614	1010	77.2
Data_6	712	772	441	82.6
Data_7	3044	3337	2061	79.0
Data_8	1265	1365	497	26.2
Data_9	2853	3139	1301	25.2
Data_10	844	905	327	21.1
Data_11	2076	2280	973	24.7
Data_12	901	996	347	24.1
Data_13	181	218	46	31.7
Data_14	3276	3639	1652	25.9

Table 1: Test instances features.

Two major observations can be made on this set of real instances. First the indicator d_{Client} clearly enables us to identify instances from local areas of high population density (Data_1 to Data_7) and those from local areas with moderate population density (Data_8 to Data_14): the average ratio between those two categories is 3 (76.9 versus 25.6). Second we note the underlying graphs corresponding to the civil engineering are graphs of low density : the mean value of the ratio $\frac{|E|}{|V|}$ is 1.1 (with a maximum of 1.2) with no significant difference between the two types of areas. This feature is characteristic of access networks that are close to tree graphs.

The architecture considered corresponds to the description given in the previous sections, with two levels of splitters (and three levels of fibers) and linear costs of fibers are compliant with the hypothesis used in the reduction schemes in Section 4. In our tests, splitters of both levels have the same capacity 1:8 (i.e. $m^k = 8, k = 1, 2$).

Finally, let us precise that all mixed integer programs have been solved with the dedicated commercial solver Cplex 11.0, with computation time limit set to 1 hour.

5.2 Numerical results

Several strategies have been tested and the best results have been obtained by designing and performing the algorithm according to the the following steps:

- reduce the graph using preprocessing schemes designed for nodes of degree 1 and 2, as described in Section 4;
- add valid inequalities (described in Section 3) at the root node, with a maximum number of inequalities set to 1000;

- perform a Branch & Bound algorithm on the resulting mixed integer linear program using Cplex.

These results are summarized in Table 2, where the indicator "Size" denotes the number of nodes visited in the Branch & Bound tree during the process. To measure the quality of the solution, the "UB" indicator represents the best (feasible) solution found during the process and "LB" the best lower bound : precisely "LB" is the minimum of the values of the continuous relaxations performed at the leaves of the Branch & Bound tree at the end of the solving process. Classically the indicator "Gap" refers to the relative difference between these two values, Gap= $\frac{UB-LB}{UB}$.

Instance	Size (B&B tree)	UB	LB	Gap (%)
Data_1	616009	80598	80528	0.08
Data_2	876711	257089	256324	0.30
Data_3	954926	302175	301203	0.32
Data_4	702446	265894	265249	0.24
Data_5	598893	452732	451166	0.35
Data_6	11729991	199857	199434	0.21
Data_7	283240	931339.6	922673	0.93
Data_8	538494	158896	156673	1.40
Data_9	209571	383986	377728	1.63
Data_10	662166	88318	87050	1.44
Data_11	291528	281501	276454	1.79
Data_12	633018	106634	105340	1.20
Data_13	1039895	21163	20902	0.57
Data_14	138503	504803	496829	1.58

Table 2: Synthesis of results (obtained in 1 hour).

First observation is that no guarantee of optimality has been obtained within the computation time limit (for no instance); however, observed gaps prove that the solutions are very good (gap $<$ 1% on average), the best gaps being obtained for the smallest instances of each category (namely Data_1 and Data_13, with respectively 0.08% and 0.57%). In addition, the evolution of the gap over time strongly suggests that the CPU time limitation little influenced the results. We can also notice quite significant differences in gaps between the two categories of areas: the average gap observed on the first type of areas is 0.35% whereas the one of areas of moderate population density is 1.37% (ratio approximately equal to 4). This is compliant with what we could expect: high demands, i.e. high values of the data, intuitively favors the quality of continous relaxations.

5.3 Impact of the reduction scheme

The following chart describes the impact of preprocessing on the civil engineering graph. In addition to the size of reduced graphs, we report three additional indicators. Let first $\text{nb}_{removed}$ denote the number of nodes or edges removed (we recall that in the reduction schemes, the number of removed nodes and edges are equal). Then we denote the reduction factors associated to respectively vertices and edges by $\text{red}_V = \frac{|V|-|V^{red}|}{|V|}$ and $\text{red}_E = \frac{|E|-|E^{red}|}{|E|}$ (where V^{red} and E^{red}

respectively denote the number of nodes and edges of the graph obtained after reduction).

Instance	Preprocessed graph			Reduction indicators		
	$\lvert V^{red}\rvert$	$\lvert E^{red}\rvert$	nb_{Client}	$nb_{removed}$	red_V (%)	red_E (%)
Data_1	103	136	57	239	70	64
Data_2	260	340	178	660	72	66
Data_3	288	379	195	784	73	67
Data_4	278	368	176	654	70	64
Data_5	393	529	291	1085	73	67
Data_6	200	260	129	512	72	66
Data_7	808	1101	586	2236	73	67
Data_8	533	633	285	732	58	54
Data_9	1253	1539	719	1600	56	51
Data_10	365	426	200	479	57	53
Data_11	897	1101	534	1179	57	52
Data_12	408	503	215	493	55	49
Data_13	107	144	33	74	41	34
Data_14	1624	1987	922	2125	57	52

Table 3: Impact of graph reduction schemes.

The main observation is the significant impact of our preprocessing schemes on the size of the graphs: the observed mean reduction factors for both vertices and edges are respectively 63% and 58% . These results are compliant with what could be expected since, as we mentionned before, initial graphs (corresponding to existing civil engineering infractructures) are of low density and consequently are likely to have a great proportion of nodes of degree 1 or 2.

5.4 Impact of valid inequalities on the continuous relaxation

The aim of this paragraph is to highlight and quantify the benefit from using the valid inequalities. We compared the continuous relaxation of our mathematical program without valid inequalities (denoted by "no cuts") and the one obtained after adding the valid inequalities (denoted by "with cuts") (once the reduction preprocessing scheme performed). Those values are given in Table 4 as well as the relative gap between them (column "gap"). As seen in the previous section, the final gap between the best solution and the best relaxation are small (0.86% in average over all the tested instances) and thus in order to be more relevent in our assessment, we introduce 2 additionnal indicators : the first one, denoted by gap_{UB} is the % of the gap relative to the best solution found that the valid inequalities enables to close ; gap_{LB} is the same indicator but relatively to the best lower bound. These indicators aim at measuring the percentages of initial gaps (at the root of the Branch & Bound) that are closed thanks to the introduction of valid inequalities in the formulation. We keep previous notations for best solution and best lower bound (columns "UB" and "LB"). Mathematically speaking, we actually define $gap_{UB} = 1 - \frac{UB - relax_{with\ cuts}}{UB - relax_{no\ cuts}}$ and $gap_{LB} = 1 - \frac{LB - relax_{with\ cuts}}{LB - relax_{no\ cuts}}$.

For the sake of clarity, the computation of gap$_{LB}$ is illustrated on Data_1 results (see Figure 7). The same reasoning obviously applies to the one of gap_{UB}.

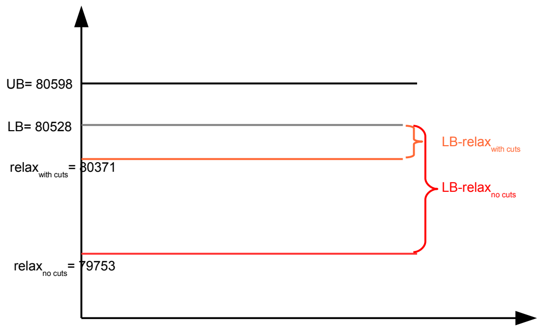

Figure 7: Illustration of indicator gap$_{LB}$ computation

On the example given on Figure 7 (which corresponds to Data_1), the initial gap relative to the indicator gap_{LB} is equal to 775 (= LB - relax$_{no\ cuts}$) if we do not introduce the cuts while it is equal to 157 (= LB - relax$_{with\ cuts}$) with the use of valid inequalities. The ratio between these two quantities, which represents the percentage of the initial gap that remains to be closed in order to reach our best lower bound, is equal to 20.3%; conversely considering gap_{LB}, it means that 79.7% of the initial gap has been closed only by using our valid inequalities.

The results obtained on our set of instances are given in Table 4.

First observation is the quite small relative gap between the two continuous relaxations, equal to 1.5% in average. However, such small absolute gains are compliant with the small absolute final gaps mentionned earlier and must not conceal the strength of these valid inequalities. Therefore, let us stress the fact that the use of valid inequalities enables to close in average 57.7% of the gap between the raw relaxation (i.e. without added valid inequalities) and the best solution and 96.8% of the one between the raw continous relaxation and the best lower bound. These results highlight more strikingly the efficiency of the valid inequalities.

Once again, a more detailed analysis proves that it is worth distinguishing between the two categories of instances as significant differences occur: indeed we have a ratio of 3 between instances corresponding to local areas of high population density (average gap of 0.8) and those corresponding to local areas of moderate population density (average gap of 2.2). This is compliant with our expectations: as mentioned above, the smaller the average demand per Client is, the worst the continuous relaxation is expected to be because the relaxation sets very fractional variables, notably for splitters. And we recall that the aim

Instance	UB	LB	Continuous relaxation			% Gap closed	
			relax$_{no\,cuts}$	relax$_{with\,cuts}$	gap (%)	gap$_{UB}$	gap$_{LB}$
Data_1	80598	80528	79753	80371	0.8	73.1	79.7
Data_2	257089	256324	254210	256231	0.8	70.2	95.6
Data_3	302175	301203	298904	301094	0.7	67.0	95.3
Data_4	265894	265249	262716	265142	0.9	76.3	95.8
Data_5	452732	451166	446894	450731	0.9	65.7	89.8
Data_6	199857	199434	197653	199321	0.8	75.7	93.7
Data_7	931339	922673	914984	922567	0.8	46.4	98.6
Data_8	158896	156673	151955	156481	3.0	65.2	95.9
Data_9	383986	377728	370449	377614	1.9	52.9	98.4
Data_10	88318	87050	84433	86821	2.8	61.5	91.2
Data_11	281501	276454	271180	276352	1.9	50.1	98.1
Data_12	106634	105340	102805	105226	2.4	63.2	95.5
Data_13	21163	20902	20587	20902	1.5	54.7	100.0
Data_14	504803	496829	486279	496694	2.1	56.2	98.7

Table 4: Impact of valid inequalities.

of the valid inequalities is precisely to prevent such behavior. First important conclusion is that these test results confirm that the designed valid inequalities benefit more to instances with low demand per Client (low population density areas), which must be kept in mind considering that FTTH is to be deployed not only in urban areas of moderate population density but also, at term, in rural areas.

6 Conclusion

Deployment of optical access network appears to be of primary importance regarding the associated economic stakes, as well deployment costs as opportunities to gain market shares due to innovative services conveyed by optical fibers.
This paper deals with the optimization of FTTH optical access networks based on a specific Point to Multipoint architecture : the Passive Optical Networks. A mixed integer formulation has been proposed for the decision problem. Then efforts have focused on the design of valid inequalities based on the polyhedral structure of the constraints and reduction of the problem size based on the analysis of optimal solutions. Finally, extensive tests performed on real life instances prove the relevancy of Branch & Bound approaches to solve the designed model, highlighting the benefit taken from the proposed preprocessing and formulation reinforcement strategies. The Orange choice is that clients are served by level three fibers. Nevertheless, the proposed model is general: it can be applied to any network without specific structure and could be easily adapted if the politic of the company is to serve demands at any level of fibers.

A major prospect for this work is undeniably to manage clients' demand uncertainty. Indeed, deregulation occurred in most countries in late 70's, giving birth to highly competitive markets. As a direct consequence, each competitor of the fix access network market has to deploy its own network, not knowing with certainty who their customers will be: the handling of uncertainty of the demand location and size is consequently a key point for realistic successful long-term deployments. Therefore the modeling of this uncertainty and the design of dedicated solving algorithms is a necessary future step for this work.

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