MAXIMUM EDGE DISJOINT PATHS AND MINIMUM UNWEIGHTED MULTICUTS IN GRID GRAPHS

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In this paper, we solve in polynomial time the maximum edge disjoint paths problem and the related minimum unweighted multicut problem in grid graphs.

Let $G = (V, E)$ be an undirected rectilinear grid and let $\mathcal{L}$ be a list of K pairs (source s_k, sink t_k) of terminal vertices of G. Each pair (s_k, t_k) defines a *net*. We assume that sources are on the upper horizontal line, sinks are on the lower horizontal line and all the terminals are distinct. The maximum edge disjoint paths problem (MaxEDP) consists in maximizing the number of nets linked by edge disjoint paths. The related minimum multicut problem is to find a minimum set of edges whose removal separates each pair (s_k, t_k) in an augmented grid (i.e. where each terminal vertex is linked to the graph by a unique edge, e.g. [7]). There exists a duality relationship between the continuous relaxations of linear formulations of these two problems. That implies the *cut condition* : the number of edge disjoint paths is at most the value of any multicut.

Both problems are known to be NP-hard in unrestricted graphs [1], and even in planar graphs ([4], [6]). MaxEDP defined in grids with terminals set at any vertices is also NP-hard [5].

A. Frank gives in ([2], [3]) necessary and sufficient conditions for the existence of K edge disjoint paths when the terminals lay on the upper and lower lines of the grid : there exist K edge disjoint paths if and only if the density is at most the number of horizontal lines and some specific vertices are not terminal vertices. Recall that the density (or the congestion) of a

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grid is the maximum number of nets that cross the space between any two consecutive vertical lines.

In MaxEDP, we seek to maximize the number of nets linked by edge disjoint paths : the main idea of our approach is to remove the minimum number of nets in order to fulfil the most restrictive Frank's conditions. The nets to be removed are selected by solving a particular linear program (LPN), whose constraints matrix is proved to be totally unimodular. If the number of horizontal lines is odd, then all the nets not removed by (LPN) can be linked by edge disjoint paths. Otherwise, one can build from the optimal solution of (LPN) a set of nets linked by edge disjoint paths, whose cardinality is at least the optimal value of MaxEDP minus one. Moreover, even in this last case, one can get the optimal solution of MaxEDP by solving a polynomial number of linear programs.

In addition, we show that the minimum unweighted multicut problem is polynomial time solvable in augmented grids, since a feasible solution whose value is equal to the maximum continuous multiflow can be obtained by solving the dual linear program of (LPN).

Eventually, we propose a specific combinatorial algorithm based on the above ideas to find in O(KlogK) an optimal solution of (LPN).

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