On hypochordal graphs

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Abstract

We introduce graphs called *hypochordal*: for any path of length 2, there exists a chord or another path of length 2 between its two endpoints. We show that such graphs are 2-vertex-connected and moreover in the case of an edge or a vertex deletion, the distance between any pair of nonadjacent vertices remains unchanged.

We give properties of hypochordal graphs, then we study the class of minimum hypochordal graphs and finally we give some complexity results for classical combinatorial problems.

Keywords: graph, tree, vertex-connected, $\mathcal{NP}$-complete

1 Introduction

Reliability in networks is a large field of research due to both practical and/or theoretical considerations (see for instance the books [9, 10]). In the literature, the problems of link failures and of node failures are usually considerred separately; that is either the nodes [8] or the links are assumed to be perfectly reliable. Here we want to construct networks where a link or a node can be defective. In addition, for the sake of robustness we want to preserve the length of shortest paths in the case of one failure. For this purpose, we introduce particular graphs called *hypochordal*: for any path of length 2, there exists a chord or another path of length 2 between its two endpoints.

We will show that networks with hypochordal topology are robust and reliable regarding links as well as nodes since the distance between any pair of nonadjacent nodes remains unchanged.

This paper is organized as follow: first we give the definition of hypochordal graphs and their characterisation in terms of distance conservation. Then we give a characterisation of minimum hypochordal graphs, which are those having the minimum number of edges for a fixed order. A study of the complexity of classical combinatorial problems comes next. We finish with a conclusion and perspectives.

2 Definition and motivations

Here we only consider simple undirected graphs. Let $G = (V, E)$ be such a graph with $n(G) = n = |V|$ vertices and $m(G) = m = |E|$ edges. Except when

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mentioned, the graphs will be considered connected and $n \geq 3$ (the connected graphs with $n \leq 2$ vertices being trivially hypochordal). The distance between two vertices x and y is the minimum number of edges of a path $[x, \ldots, y]$ and will be denoted by $d(x, y)$. For $x \in V$, $N_i^x = \{y \in V, d(x, y) = i\}$ is the set of vertices of G at distance i from x. The set of neighbours of x is $N(x) = N_1^x$. We say that y is a twin of x if $N(y) = N(x)$ (x and y are nonadjacent). We denote by $\delta(x)$ the degree of the vertex x and by $\delta(G)$ the minimum degree of a vertex of G. P_k (respectively C_k) is a path (respectively a cycle) of k vertices.

Let S and S' be disjoint sets of vertices of G. We denote by $S - S'$ the relation existing between the sets S and S' if they satisfy $\forall x \in S, \forall y \in S', [x, y] \in E$. For all other definitions, refer to [1].

2.1 Definition and properties of hypochordal graphs

We now formalise the definition of hypochordal graphs.

Definition 1. *A graph $G = (V, E)$ is hypochordal if for every triple of vertices u, v, y such that $[u, y, v]$ is a P_3, we have $[u, v] \in E$ or there exists $z \neq y$ such that $[u, z, v]$ is a P_3.*

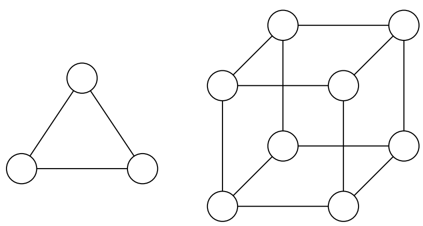


Figure 1: Examples of connected hypochordal graphs – C_3 and the cube

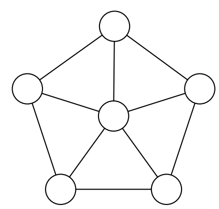


Figure 2: A non perfect hypochordal graph – the 5-wheel

We have the following equivalences.

Proposition 1. *Let $G = (V, E)$ be a connected graph. Then the following definitions are equivalent:*

1. *G is hypochordal;*
2. *every P_3 is included in a C_3 or a C_4;*
3. *$\forall u, v \in V, u \neq v, |N(u) \cap N(v)| = 1 \Rightarrow [u, v] \in E$;*
4. *the distance between any pair of nonadjacent vertices is unchanged by the deletion of any third vertex;*
5. *the distance between any pair of nonadjacent vertices is unchanged by the deletion of any edge;*
6. *for any pair of nonadjacent vertices, there exist two vertex-disjoint shortest paths between them.*

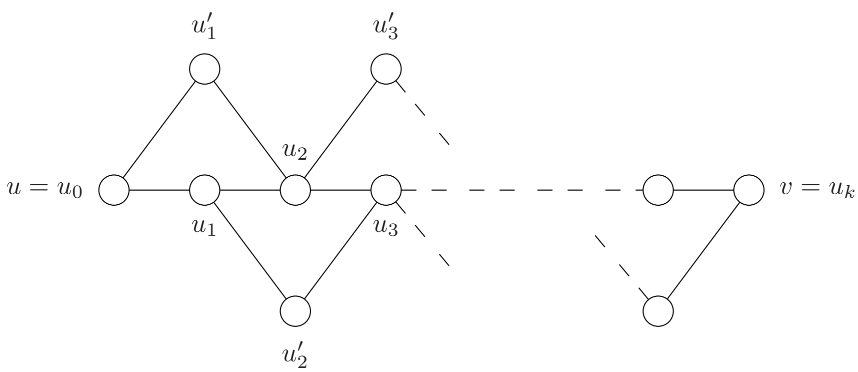
While the equivalences between 1, 2 and 3 are immediate, the equivalences with $4, 5$ and 6 need a little proof:

Proof. We show that $1 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$.

($1 \Rightarrow 4$) Let G be a hypochordal graph, u, v a pair of nonadjacent vertices and y a third vertex, $y \neq u$, $y \neq v$. We denote by $\mu = [u = u_0, u_1, \ldots, u_k = v]$ a shortest path between u and v.
If $y \notin \mu$, then the distance between u and v is unchanged by the deletion of y.
If $y = u_j$ then $[u_{j-1}, u_j = y, u_{j+1}]$ is a P_3, and $[u_{j-1}, u_{j+1}] \notin E$ since μ is a shortest path. So there exists $z \neq y$ such that $[u_{j-1}, z, u_{j+1}]$ is a P_3 and $[u = u_0, \ldots, u'_j = z, \ldots, u_k = v]$ is another shortest path between u and v.

($4 \Rightarrow 5$) Assume that the distance between any pair of nonadjacent vertices is unchanged by the deletion of any third vertex and consider the deletion of an edge $[x, y] \in E$. Since the distances greater than 2 are unchanged by the deletion of y, these distances are unchanged by the deletion of $[x, y]$.

($5 \Rightarrow 6$) Let u and v be nonadjacent vertices linked by a shortest path $\mu = [u = u_0, u_1, \ldots, u_k = v]$. For any $u_i \in \mu, (u_i \neq u, v)$, u_{i-1} and u_{i+1} are at distance two, otherwise μ is not minimal.
From 5, the deletion of $[u_{i-1}, u_i]$ does not change the distance between u_{i-1} and u_{i+1} thus, there exists $u'_i, (u'_i \neq u_i)$ such that $[u_{i-1}, u'_i]$ and $[u'_i, u_{i+1}]$ belong to G. Then, there exist two vertex-disjoint shortest path between u and v: $\mu_1 = [u = u_0, u_1, u'_2, \ldots, u_{2i-1}, u'_{2i}, \ldots, u_k = v]$ and $\mu_2 = [u = u_0, u'_1, u_2, \ldots, u'_{2i-1}, u_{2i}, \ldots, u_k = v]$.



($6 \Rightarrow 1$) Let u and v be nonadjacent vertices such that $[u, y, v]$ is a P_3. Since there exist two vertex-disjoint shortest paths between u and v, there exists z such that $[u, z, v]$ is a P_3 and $z \neq y$. So G is hypochordal. □

Remark 1. *Let $[u, v] \in E$ be an edge of G. Since G is connected and $n \geq 3$, $[u, v]$ belongs to a P_3. G being hypochordal, $[u, v]$ belongs to a C_3 or a C_4. Hence, the deletion of the edge $[u, v]$ increases the distance between u and v by 1 or 2.*

Hypochordal graphs must be studied in depth since they do not fit with most of the "pleasant" properties such as perfectness or heredity. Hypochordal graphs are not necessarily perfect, see Figure 2.

As defined in [2], a property Π is called *hereditary* if it is closed under taking induced subgraphs. In other words, a graph property Π is hereditary if it is closed under removal of vertices. It is clear that the hypochordal property is not hereditary. As an example, consider the 5-wheel of Figure 2, it has a C_5 as an induced subgraph, which is not hypochordal.

Following the definition given in [7], page 9, a property Π is *monotone* if adding edges to a graph with property Π produces a graph satisfying Π. Thus the hypochordal property is not monotone since the cube is no longer hypochordal when adding a chord to a C_4.

2.2 Relations with 2-vertex-connected and chordal graphs

Proposition 1.6 states that hypochordal graphs are the graphs where there exist two vertex-disjoint shortest paths between any pair of nonadjacent vertices. Hence, if G is hypochordal then G is 2-vertex-connected.

There is no inclusion relation between the classes of 2-vertex-connected chordal graphs and hypochordal graphs. The graph of Figure 3 is a 2-vertex-connected interval graph hence it is chordal; but is not hypochordal. Complete graphs K_n are both chordal and hypochordal. Bipartite complete graphs K_{n_1,n_2} for $n_1, n_2 \geq 2$ and hypercubes are hypochordal and not chordal. Thus we are interested in comparing hypochordal graphs to 2-vertex-connected graphs and to 2-vertex-connected chordal graphs:

- Let G be a 2-vertex-connected graph, the deletion of a vertex (respectively an edge) can increase the distance between the vertices by up to $n-4$ (respectively $n-2$). Just consider C_n, $n \geq 4$.

- Let G be a 2-vertex-connected chordal graph, the deletion of any edge of G increases by at most one the distance between any pair of vertices, but the deletion of a vertex of G may increase the distance between two vertices by an arbitrary long value. Consider the graph given in Figure 3, the vertices u and v and the deletion of x.

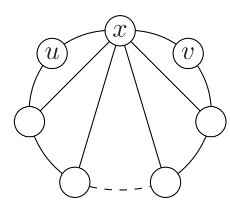


Figure 3: A 2-vertex-connected chordal graph with arbitrary big distance increase

Thereby when interested in networks where a link (edge) or a node (vertex) failure induces small changes in the distance between any pair of nodes, one can consider hypochordal graphs topology. Moreover in the next section we will show that hypochordal graphs can have a number of edges less than two times the number of vertices.

3 Minimum hypochordal graph

Here we characterise the set of connected hypochordal graphs of order n with a minimum number of edges. We call them *minimum hypochordal graphs*. We first introduce some graph partitions. Then we bound the number of edges of such graphs. Finally we characterise the structure of minimum hypochordal graphs.

3.1 Partitioning hypochordal graphs

Definition 2. *A graph $H = (\mathcal{B}, \mathcal{F})$ is a* partition *of G if*

- *each vertex B_i of H is a subset of vertices of G, called a bag, $B_i \subset V$;*
- *the bags realise a partition of V: $\bigcup_i B_i = V$, $B_i \cap B_j = \emptyset, i \neq j$;*
- *if two adjacent vertices of G are in two distinct bags A and B then $A - B$ in G, and the bags A and B are adjacent in H.*

That is, to each edge $[B_1, B_2]$ of H corresponds the complete bipartite subgraph of G induced by the vertices of $B_1 \cup B_2$.

Note that G is a partition of G, thus a partition of G always exists. Note also that there is no condition concerning the adjacency of the vertices inside a bag. Hence the graph with a single vertex is a partition of any graph (H has a single bag equal to V).

Definition 3.

– *a* tree-partition *T of G is a partition such that T is a tree;*

– *a* stable-partition *H of G is a partition such that every bag is a stable set;*

– *a* 2-stable-partition *H of G is a stable-partition such that every bag is of size less than or equal to 2: $\forall B_i \in \mathcal{B}, |B_i| \leq 2$.*

Proposition 2. *If G has a partition $H = (\mathcal{B}, \mathcal{F})$ with $|\mathcal{B}| \geq 2$, H connected and $\forall B_i \in \mathcal{B}, |B_i| \geq 2$, then G is hypochordal.*

Proof. Let u, v be two distinct vertices of G. They can either be in the same bag or not.

- Case u and v are in the same bag A: The partition being connected with at least two bags, there exist a bag B adjacent to A. Hence $A - B$ and $N(u) \cap N(v) \supseteq B$. Thus $|N(u) \cap N(v)| \geq |B| \geq 2$.
- Case u and v are in distinct bags A and B: Suppose that $|N(u) \cap N(v)| = 1$, $N(u) \cap N(v) = \{w\}$; w is either in the bag A (or symmetrically B) or in a third bag C.
  - Case $w \in A$: Since v and w are adjacent in G and belong to two distinct bags A and B of H, we have $A - B$. Hence $[u, v] \in E$.
  - Case $w \in C$: Using the same argument as above, we have $A - C$ and $B - C$. Thus $N(u) \cap N(v) \supseteq C$ and therefore $|N(u) \cap N(v)| \geq |C| \geq 2$, a contradiction.

$\square$

Proposition 3. *If G is 2-vertex-connected and has a tree-partition $T = (\mathcal{B}, \mathcal{F})$ with $|\mathcal{B}| \geq 3$, then G is hypochordal.*

Proof. Let $A \in \mathcal{B}$ be a bag of T such that $|N(A)| \geq 2$. The subgraph of G induced by the vertices of $V \setminus A$ is disconnected. Since G is 2-vertex-connected, $|A| \geq 2$. Hence the bags of size 1 can only be leaves of T.

In the case where $|\mathcal{B}| = 3$, with both leaves of size 1, we create another tree-partition T' of G by aggregating the two bags of size 1 into a single bag of size 2. Proposition 2 shows that G is hypochordal.

In any other case, we remove the bags of size 1 of T, the remaining graph is called T'. From Proposition 2, the subgraph G' of G induced by the vertices of the bags of T' is hypochordal. Let u, v be two vertices of G, with at least one common neighbour w:

- Case u, v are vertices of G': G' being hypochordal, $|N(u) \cap N(v)| = 1$ implies $[u, v] \in E$.
- Case u, v are both in bags of size 1: u and v are in bags that are leaves of T so w belongs to a bag A which is not a leaf of T. $N(u) = N(v) = A$, with $|A| \geq 2$.
- Case u is in a bag of size 1 and v belongs to G': w can either be in the same bag as v or in another bag of T.
  - If v and w are in the same bag, since u and w are adjacent in G, we have $[u, v] \in E$.
  - If w is in a bag A and v in a bag B of T, then $N(u) \cap N(v) = N(u) = A$ with $|A| \geq 2$.

□

Definition 4. *Let $G = (V, E)$ be a graph, the graph $2G = (U, F)$ is as follows: $U = V \times \{1, 2\}$; any vertex v of G has two corresponding vertices v_1 and v_2 in $2G$; if $[u, v] \in E$ then $[u_i, v_j] \in F, i, j \in \{1, 2\}$.*

See Figure 4 for an example.

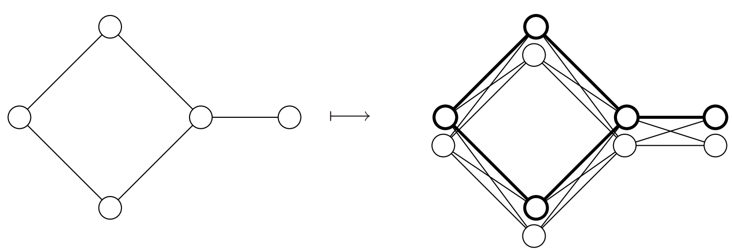


Figure 4: A graph G and the corresponding graph $2G$

Note that G is a 2-stable-partition of $2G$.

Proposition 4. *For any connected graph G with at least two vertices, the graph $2G$ is hypochordal.*

Proof. G is a partition of $2G$ with at least two bags, each bag being of size 2. We conclude using Proposition 2. □

Remark 2. *From Proposition 4, we can deduce that for any k, there are hypochordal graphs with diameter k and there also exist hypochordal graphs with a girth of size k (consider $2C_k$), thus hypochordal graphs cannot be characterised by a finite set of forbidden induced subgraphs.*

We define a transformation $G \mapsto \widetilde{2G}$ similar to Definition 4. In fact, it is the same transformation except for pendant vertices which are not duplicated. See Figure 5 for an example.

Definition 5. *Let $G = (V, E)$ be a graph. Let $V_1 \subset V$ be the set of vertices of G with degree 1 and $V_2 = V \setminus V_1$. The graph $\widetilde{2G} = (U, F)$ is as follows: $U = V_1 \cup V_2 \times \{1, 2\}$; any vertex $u \in V_1$ corresponds to a vertex u_1 of $\widetilde{2G}$; any vertex $v \in V_2$ has two corresponding vertices v_1 and v_2 in $\widetilde{2G}$; if $[u, v] \in E$ then $[u_i, v_j] \in F, i, j \in \{1, 2\}$.*

Note again that G is a 2-stable-partition of $\widetilde{2G}$

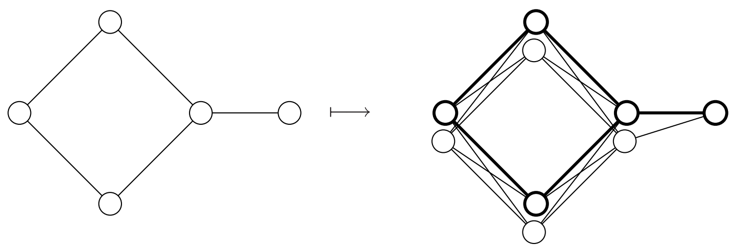


Figure 5: A graph G and the corresponding graph $\widetilde{2G}$

Proposition 5. *For any connected graph G with at least three vertices, the graph $\widetilde{2G}$ is hypochordal.*

Proof. Let G' be the subgraph of G induced by the vertices with degree greater than or equal to 2. Since $n(G) \geq 3$ and G is connected, $n(G') \geq 1$.
If $n(G') = 1$, then G is the star $K_{1,n(G)-1}$. Thus $\widetilde{2G}$ is the bipartite complete graph $K_{2,n(G)-1}$ which is hypochordal since $n(G) - 1 \geq 2$.
If $n(G') \geq 2$, then, from Proposition 4, the induced subgraph $2G'$ of $\widetilde{2G}$ is hypochordal. Using the same argument as in the proof of Proposition 3, we show that $\widetilde{2G}$ is hypochordal. □

Proposition 6. *A graph G with $n \geq 5$ is a $\widetilde{2H}$ if and only if $\delta(G) \geq 2$ and G has a connected 2-stable-partition with at least one bag of size 2 and every vertex with degree greater than or equal to 3 is in a bag of size 2.*

Proof. ($\Rightarrow$) If $G = \widetilde{2H}$ and $n \geq 5$, then H is a connected 2-stable-partition of G with at least 3 bags. Hence there is at least one bag of size 2. The vertices of G in bags of size 1 have degree 2 in G.
($\Leftarrow$) Let K be the 2-stable-partition of G satisfying the conditions of Proposition 6. Thus every pair of vertices in bags of size 2 have the same degree.

Since vertices with degree greater than or equal to 3 are in a bag of size 2 and $\delta(G) \geq 2$, bags of size 1 can only contains vertices of G with degree 2. The neighbours of any vertex x with degree 2 in G are the two vertices of a bag of size 2: by contradiction, suppose that x has its two neighbours in two different bags, necessarily these bags are of size 1 and the two neighbours of x have degree 2. This argument implies that G has a connected component which is a cycle. As G is connected, G is a cycle where every vertex is a bag of K which is impossible since there is at least one bag of size 2 and $n \geq 5$.
So any bag of size 1 in K has a single adjacent bag, which is necessarily of size 2. We construct a new 2-stable-partition of G, K' such that if a vertex x of G with degree 2 is in a bag A of size 2 of K, then in K', A is replaced with two bags of size 1, each one containing a single vertex with degree 2. This way, every vertex with degree 2 is in a bag of size 1 of K', every vertex with degree greater than or equal to 3 is in a stable bag of size 2 of K'. Since K' is a 2-stable-partition, the two vertices of a same bag of size 2 have exactly the same neighbours. Hence $G = \widetilde{2K'}$. □

Lemma 1. *Let H be a connected graph with $n(H) \geq 3$, $\widetilde{2H}$ satisfies $m(\widetilde{2H}) = 2 \times n(\widetilde{2H}) - 4$ if and only if H is a tree.*

Proof. Let n_1 be the number of pendant vertices of H and n_2 be the number of vertices with degree greater than or equal to 2. $\widetilde{2H}$ has $n(\widetilde{2H}) = n_1 + 2n_2$ vertices. Now let us count the number of edges of $\widetilde{2H}$: any edge in H with an endpoint with degree 1 corresponds to two edges of $\widetilde{2H}$, the other edges of H correspond to four edges of $\widetilde{2H}$ (see Figure 5), hence $m(\widetilde{2H}) = 2n_1 + 4(m(H) - n_1) = 4m(H) - 2n_1$ edges.

We have the following:

$$\begin{array}{rrcl} & m(\widetilde{2H}) & = & 2n(\widetilde{2H}) - 4 \\ \Leftrightarrow & 4m(H) - 2n_1 & = & 2(n_1 + 2n_2) - 4 \\ \Leftrightarrow & m(H) & = & n_1 + n_2 - 1 \\ \Leftrightarrow & m(H) & = & n(H) - 1 \\ \Leftrightarrow & H & \text{is} & \text{a tree.} \end{array}$$

□

3.2 How many edges?

For $n = 2$ (respectively $n = 3$), there is a unique minimum connected hypochordal graph which is K_2 (respectively K_3).
Now we consider $n \geq 4$.

Lemma 2. *Let G be a connected hypochordal graph and x be a vertex of G, then $\forall i \geq 2$ and $\forall v \in N_i^x$, $|N(v) \cap N_{i-1}^x| \geq 2$.*

Proof. Let $\mu = [x = v_0, \ldots, v_{i-1}, v_i = v]$ be a shortest path from x to v. G being hypochordal, $d(x, v) = i$ in the subgraph $G \setminus \{v_{i-1}\}$. So in G we have $|N(v) \cap N_{i-1}^x| \geq 2$. □

Lemma 3. *Let G be a connected hypochordal graph with $n \geq 4$, we have $m \geq 2n - 4$. Moreover if $m = 2n - 4$ then $\delta(G) \leq 3$.*

Proof. Since G is hypochordal and $n \geq 4$, we have $\delta(G) \geq 2$.

- Case $\delta(G) = 2$: let x be a vertex with minimum degree. From Lemma 2, for each vertex $v \in N_i^x, i \geq 2$ there are at least two edges $[v, w_1]$ and $[v, w_2]$ with $w_1, w_2 \in N_{i-1}^x$. Moreover x has two neighbours in N_1^x so we have $m \geq 2 \times |\cup_{i\geq 2} N_i^x| + |N_1^x| = 2(n-3) + 2 = 2n - 4$.

- Case $\delta(G) = 3$: when $n = 4$, $G = K_4$ for which $m = 6 > 2n - 4$. We consider $n \geq 5$. Let x be a vertex with minimum degree and y a vertex at maximum distance k from x. We use the same token as above plus the fact that y has degree at least 3. So we have $m \geq 2 \times (|\cup_{i\geq 2} N_i^x| - 1) + |N_1^x| + \delta(G) = 2(n-5) + 3 + 3 = 2n - 4$.

- Case $\delta(G) \geq 4$: we have $m = \frac{1}{2} \sum_{v \in V} \delta(v) \geq \frac{1}{2} \sum_{v \in V} \delta(G) \geq 2n > 2n - 4$.

It follows immediately that if $m = 2n - 4$, we have $2 \leq \delta(G) \leq 3$ □

Lemma 4. *Let G be a connected hypochordal graph with $m = 2n-4$ and x be a vertex of G with minimum degree $\delta(x) = 2$: $\forall i \geq 2$, $\forall u \in N_i^x, |N(u) \cap N_{i-1}^x| = 2$ and $\forall i \geq 0, N_i^x$ is a stable set.*

Proof. Let us consider the proof of Lemma 3. Since $m = 2n - 4$, every vertex in $N_i^x, i \geq 2$ has exactly two neighbours in N_{i-1}^x and there is no edge $[u, v]$ with $u, v \in N_i^x$. □

Lemma 5. *Let G be a connected hypochordal graph with $m = 2n-4$ and x be a vertex of G with minimum degree $\delta(x) = 3$, let k be the maximum distance from a vertex to x: $\forall\ 2 \leq i \leq k-1$, $\forall u \in N_i^x, |N(u) \cap N_{i-1}^x| = 2$ and $\forall\ 0 \leq i \leq k-1, N_i^x$ is a stable set. Moreover $|N_k^x| \leq 2$.*

Proof. Let us consider the proof of Lemma 3. K_4 does not satisfies $m = 2n - 4$, so $n \geq 5$. For each vertex $v \in N_i^x, 2 \leq i < k$, there are exactly two edges $[v, w_1]$ and $[v, w_2]$ with $w_1, w_2 \in N_{i-1}^x$ and there is no edge $[u, v]$ with $u, v \in N_i^x$.
If $|N_k^x| \geq 3$, using the counting argument of Lemma 3, then $m \geq 2 \times |\cup_{i\geq 2} N_i^x| + |N_1^x| + \frac{1}{2}|N_k^x| = 2(n-4) + 3 + \frac{1}{2}|N_k^x| > 2n - 4$. Hence $m = 2n - 4$ implies that $|N_k^x| \leq 2$. □

We use these lemmata to prove the following.

Theorem 1. *A minimum connected hypochordal graph G with $n \geq 4$ is such that $m = 2n-4$, G is bipartite and $\delta(G) = 2$ or 3. Moreover there exists a unique minimum hypochordal graph with $\delta(G) = 3$ which is the cube. Furthermore there exists an infinite family of minimum hypochordal graphs with $\delta(G) = 2$.*

Proof.

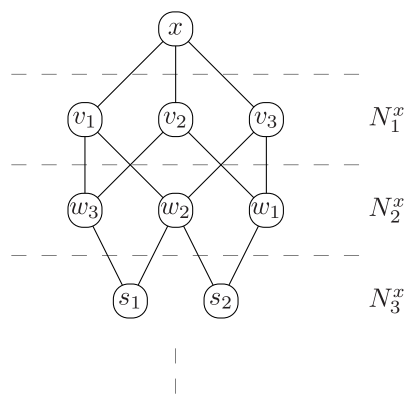
- Case $\delta(G) = 2$: if $G = K_{2,n-2}$ we have $m = 2n - 4$ and G is hypochordal. For $m = 2n - 4$, any hypochordal graph is necessarily bipartite because we have seen in the proof of Lemma 4 that every edge $[u, v]$ of G is such that u (or v) $\in N_i^x$ and v (or u) $\in N_{i-1}^x, i \geq 1$.

- Case $\delta(G) = 3$: the cube satisfies $m = 2n - 4$ and is hypochordal.
  Let x be a vertex with minimum degree and k be the maximum distance from a vertex to x.

  First we show that $k \geq 3$. From Lemma 5 we know that $|N_k^x| \leq 2$.
  If $|N_k^x| = 1$, let $N_k^x = \{y\}$, then $k = 2$ is impossible: $N(x) = N(y) = N_1^x$

and N_1^x is a stable set so the vertices of N_1^x would have degree 2.
If $|N_k^x| = 2$, let $N_k^x = \{y, z\}$. Necessarily $[y, z] \in E$: by contradiction, assume that $[y, z] \notin E$, since $\delta(G) = 3$, both y and z have at least three neighbours in N_{k-1}^x. Hence using the same arguments as in the proof of Lemma 3, we have $m \geq 2|\cup_{i=2}^{k-1} N_i^x| + |N(x)| + 2\delta(G) = 2(n-6)+9 = 2n-3$. So $[y, z]$ is the unique edge with its two endpoints in the same $N_i^x, i \geq 1$. Let v be a neighbour of y in N_{k-1}^x, the existence of the $P_3[z, y, v]$ implies the existence of the edge $[v, z]$: Thus $N(y) \cap N_{k-1}^x = N(z) \cap N_{k-1}^x$. We have $N(y) \cap N_{k-1}^x = N_{k-1}^x$: by contradiction, assume there exists $u \in N_{k-1}^x \setminus N(y)$, $N(u) \subset N_{k-2}^x$, then from Lemma 5, $|N(u)| = 2$ whereas $\delta(G) = 3$. Hence we have $|N_{k-1}^x| = 2$. $k = 2$ is impossible since $N_{k-1}^x = N_1^x$ and $|N_1^x| = 3$.

Hence $k \geq 3$. Let $v_1, v_2, v_3 \in N_1^x$ be the three neighbours of x. Since G is hypochordal and N_1^x is a stable set, v_2 and v_3 must have a common neighbour $w_1 \in N_2^x$. From Lemma 5, $N(w_1) \cap N_1^x = \{v_2, v_3\}$. By symmetry, there exists a vertex $w_2 \neq w_1$ such that $N(w_2) \cap N_1^x = \{v_1, v_3\}$; and there exists another vertex $w_3 \neq w_1, w_3 \neq w_2$ such that $N(w_3) \cap N_1^x = \{v_1, v_2\}$. Hence $N_2^x = \{w_1, w_2, w_3\}$



Suppose that $|N_3^x| \geq 2$: since $|N_2^x| = 3, \delta(G) = 3$ and from Lemma 5, $\forall s \in N_3^x, |N(s) \cap N_2^x| = 2$, there exists a pair $s_1, s_2 \in N_3^x$ such that $N(s_1) \cap N_2^x = \{w_2, w_3\}$ and $N(s_2) \cap N_2^x = \{w_1, w_2\}$. Now the vertices v_1 and s_2 have a unique common neighbour w_2, which is a contradiction since G is hypochordal.

Hence $|N_3^x| = 1$. Since hypochordal graphs have no separating vertex, $k = 3$ and G is the cube.

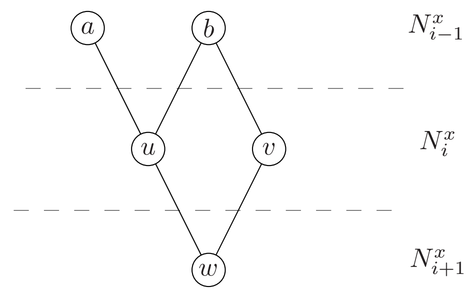
□

3.3 Shape of minimum hypochordal graphs

Theorem 1 states that the sole minimum hypochordal graph with $\delta(G) = 3$ is the cube and there are no minimum hypochordal graphs for $\delta(G) \geq 4$. Hence in this section, we consider minimum hypochordal graphs with $\delta(G) = 2$ (there exists an infinite number of such graphs). We recall that these graphs satisfy $m = 2n - 4$, they are bipartite, the sets N_i^x are stable and any vertex v at distance i from x has exactly two neighbours at distance $(i-1)$ from x.

Lemma 6. *Let G be a minimum connected hypochordal graph with $n \geq 4$ and $\delta(G) = 2$, and x be a vertex with minimum degree. Let u and v be two vertices in N_i^x, $i \geq 2$. If $N(u) \cap N(v) \cap N_{i+1}^x \neq \emptyset$ then $|N(u) \cap N(v) \cap N_{i-1}^x| = 2$.*

Proof. By Lemma 4, we know that both u and v have two neighbours in N_{i-1}^x. Let $w \in N_{i+1}^x$ be a common neighbour of u and v. Suppose that u has a neighbour a in N_{i-1}^x which is not a neighbour of v.



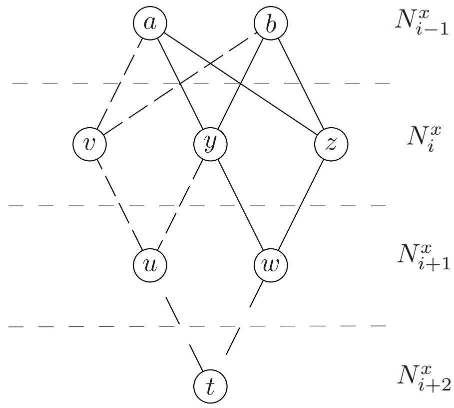
w has exactly two neighbours in N_i^x which are u and v. $[a, u, w]$ is a P_3 but neither $[a, w]$ nor $[a, v]$ are edges of G. This contradicts the fact that G is hypochordal. $\square$

Lemma 7. *Let G be a minimum connected hypochordal graph with $\delta(G) = 2$ and $n \geq 5$. For every vertex y such that $\delta(y) \geq 3$, y has a twin vertex z. Furthermore z is unique.*

Proof. Let x and y be two vertices such that $\delta(x) = 2$ and $\delta(y) \geq 3$. Thus $y \in N_i^x, i \geq 1$.

If $y \in N_1^x$, let $N_1^x = \{y, z\}$. We show that z is the twin of y. From Lemma 4, $[y, z] \notin E$ and every vertex $w \in N_2^x$ has two neighbours in N_1^x which are necessarily y and z. So $N(y) = N(z) = \{x\} \cup N_2^x$ and z is the unique twin of y.

We consider now $y \in N_i^x, i \geq 2$. From Lemma 4, $|N(y) \cap N_{i-1}^x| = 2$; let a and b be the two neighbours of y in N_{i-1}^x. Since $\delta(y) \geq 3$, y has a neighbour $w \in N_{i+1}^x$. Now $|N(w) \cap N_i^x| = 2$; let z be such that in $N(w) \cap N_i^x = \{y, z\}$.
We show that $N(z) = N(y)$. G being minimum hypochordal, we know that $N(y) \subset N_{i-1}^x \cap N_{i+1}^x$. Since $N(y) \cap N(z) \cap N_{i+1}^x \neq \emptyset$, Lemma 4 and Lemma 6 ensure that $N(y) \cap N_{i-1}^x = N(z) \cap N_{i-1}^x = \{a, b\}$. Suppose there exists $u \in N_{i+1}^x \cap N(y) \setminus N(z)$. Let $v \neq z$ be such that $N(u) \cap N_i^x = \{y, v\}$. If $\delta(u) = 2$ or $\delta(w) = 2$, $N(u) \cap N(w) = \{y\}$; since $[u, w] \notin E$ and $[v, w] \notin E$, this is impossible. Hence $\delta(u) \geq 3$ and $\delta(w) \geq 3$.



Since $[u, w] \notin E$ and $N(u) \cap N(w) \neq \emptyset$, we have $|N(u) \cap N(w)| \geq 2$; so there exists $t \in N(u) \cap N(w) \cap N^x_{i+2}$; from Lemma 4 we have $N(t) \cap N^x_{i+1} = \{u, w\}$, thus $N(z) \cap N(t) = \{w\}$, a contradiction since $[z, t] \notin E$.

Now we show that y has exactly one twin z. By contradiction, suppose y has two twins z and z'. So $N(y) = N(z) = N(z')$ and $\delta(y) = \delta(z) = \delta(z') \geq 3$. Then there exists $t \in N(y) \cap N^x_{i+1}$ but $|N(t) \cap N^x_i| \geq 3$, which contradicts Lemma 4. □

Theorem 2. *G is a minimum connected hypochordal graph with $n \geq 4$ and $\delta(G) = 2$ if and only if $G = \widetilde{2T}$ for an appropriate tree T with $n(T) \geq 3$.*

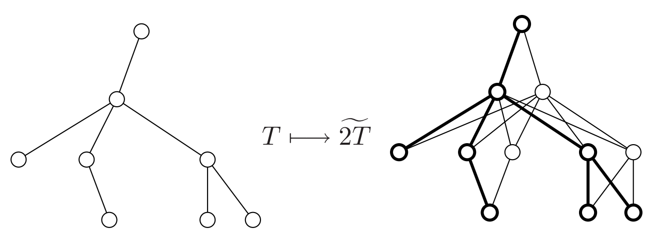


Figure 6: A tree T and the corresponding graph $\widetilde{2T}$

Proof. $(\Rightarrow)$

- Case $n = 4$: The only minimum hypochordal graph G is C_4 and $G = \widetilde{2P_3}$;
- Case $n \geq 5$: Due to Lemma 7, every vertex y with $\delta(y) \geq 3$ has a unique twin vertex z. We make H a 2-stable-partition of G as follows: each vertex w, $\delta(w) = 2$ forms a bag; each vertex y, $\delta(y) \geq 3$ with its twin z form a bag. Given the characterisation of Proposition 6, $G = \widetilde{2H}$. Moreover G satisfies $m = 2n - 4$ which means from Lemma 1 that H is a tree.

($\Leftarrow$) From Proposition 5, $\widetilde{2T}$ is hypochordal and from Lemma 1, $m(\widetilde{2T}) = 2n(\widetilde{2T}) - 4$. Hence $\widetilde{2T}$ is a minimum hypochordal graph. $\square$

Theorem 3. *A graph G with $n \geq 5$ and $\delta(G) = 2$ is a minimum hypochordal graph if and only if G is 2-vertex-connected and there exists a 2-tree-stable-partition T of G.*

Proof.
($\Rightarrow$) If G is a minimum hypochordal graph, G is 2-vertex-connected and from Theorem 2, $G = \widetilde{2T}$ where T is a tree. Hence, T is a 2-tree-stable-partition of G.
($\Leftarrow$) Let G be a graph having T as a 2-tree-stable-partition. Since $n(G) \geq 5$, a 2-tree-stable-partition of G has at least three bags; then due to Proposition 3, G is hypochordal. Hence $m(G) \geq 2n(G) - 4$.

We still need to show that because G has a 2-tree-stable-partition T, $m(G) \leq 2n(G) - 4$:

$n_1(T)$ will denote the number of bags of size 1. T being a tree, its number of edges is $m(T) = n(T) - 1$; let us denote by $m_1(T)$ the number of edges of T which are incident to bags of size 1. We observe that every internal bag of T has size 2 since G is 2-vertex-connected, so $m_1(T) = n_1(T)$.
We have $n(G) = n_1(T) + 2 \times (n(T) - n_1(T)) = 2n(T) - n_1(T)$ and $m(G) = 2m_1(T) + 4[(n(T) - 1) - m_1(T)] = 2(2n(T) - m_1(T)) - 4$. Since $m_1(T) = n_1(T)$, $m(G) = 2n(G) - 4$. $\square$

This characterisation of minimum hypochordal graphs implies that classical combinatorial problems are polynomial in this class: minimum hypochordal graphs are bipartite, hence they are 2-colorable, their maximum clique has size two and the problem of maximum stable set is polynomial; they do not have a hamiltonian cycle, except the cube and $\widetilde{2P_k}, \forall k \geq 3$. In the next section, we consider the same combinatorial problems in the case of (general) hypochordal graphs.

4 Classical combinatorial problems

Since hypochordal graphs are not necessarily perfect and the hypochordal property is neither hereditary nor monotone, there is very few hope that $\mathcal{NP}$-complete problems would become polynomial in hypochordal graphs. This is confirmed by the following results.

4.1 Hamiltonian cycle

We are interested in deciding if a given hypochordal graph is hamiltonian or not.

Theorem 4. *The problem* Hamiltonian Cycle *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. We know that deciding if a 3-regular graph is hamiltonian is $\mathcal{NP}$-complete. We are going to show that this problem reduces to the problem of the existence of a hamiltonian cycle in a hypochordal graph.

The problem of deciding if a hypochordal graph is hamiltonian is in $\mathcal{NP}$.

Let $G = (V, E)$ be a 3-regular graph. We build a hypochordal graph $H = (U, F)$ by the following polynomial transformation. Each vertex a of G becomes a subgraph H_a of H isomorphic to K_6. Each edge $[a, b]$ of G is substituted by a gadget (Figure 7) connected to H_a and H_b. The vertices a_1, a_2 are belonging to H_a and b_1, b_2 to H_b. G being a 3-regular graph, any vertex a has three incident

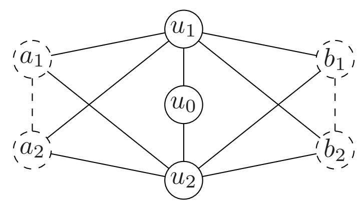


Figure 7: The gadget corresponding to the edge $[a, b]$ of G

edges $[a, b], [a, c]$ and $[a, d]$. You can see the corresponding subgraph of H on Figure 8.

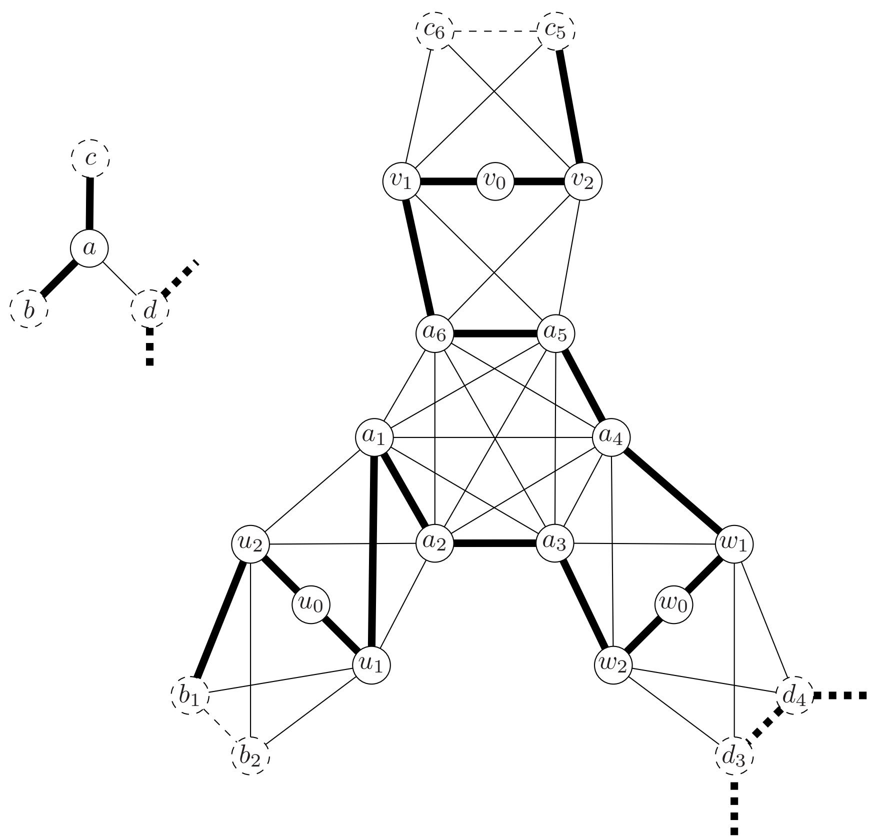


Figure 8: The neighbourhood of a vertex a of G and the corresponding subgraph of H

We should first make sure that the graph H we have build is hypochordal.

We consider the common neighbours of vertices at distance two (see Figure 8). Given the symmetry of H, we need to consider the vertices at distance two from a_1 plus the pair $\{u_1, u_2\}$. The vertices at distance two from a_1 are b_1 (and b_2), u_0 and v_1 (and v_2).

- Vertices a_1 and b_1 have two common neighbours u_1 and u_2;
- vertices a_1 and u_0 have two common neighbours u_1 and u_2;
- vertices a_1 and v_1 have two common neighbours a_5 and a_6;
- vertices u_1 and u_2 have five common neighbours a_1, a_2, b_1, b_2 and u_0.

We show that if G has a hamiltonian cycle then there is a hamiltonian cycle in H. Let $\mathcal{C}_G$ be a hamiltonian cycle of G: from any vertex $v \in V$, $\mathcal{C}_G$ induces a total order $<$ on V.

For an edge $[x, y]$ of G, $[x, y] \in \mathcal{C}_G$, we associate a path in the gadget between H_x and H_y as shown in Figure 9.

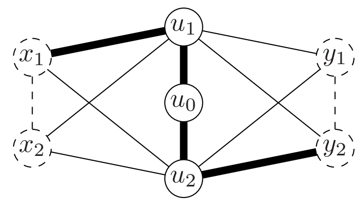


Figure 9: Path in H corresponding to an edge of the hamiltonian cycle of G

For an edge $[x, y]$ of G with $[x, y] \notin \mathcal{C}_G$ and $x < y$, we associate a path of the gadget which does not bridge over H_x and H_y, as shown in Figure 10.

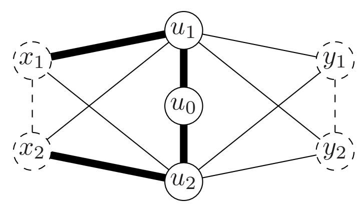


Figure 10: Path corresponding to an edge not in the hamiltonian cycle of G

For any vertex a of G, two of its three incident edges are in $\mathcal{C}$. Assume that b, c, d are the three neighbours of a and $[a, b]$ and $[a, c]$ belong to $\mathcal{C}_G$. We have to connect the corresponding paths through H_a. There are two different cases:

- $a < d$: there are three paths to connect, one between H_a and H_b (with a_1 as end point), one between H_a and H_c (with a_6 as endpoint) and one in the gadget corresponding to the edge $[a, d]$ (with a_3, a_4 as endpoints). We join these three paths with the vertices of H_a that are not in a path yet i.e. a_2 and a_5, see Figure 8.
- $d < a$: there are two paths to connect, one between H_a and H_b (with a_1 as endpoint) and one between H_a and H_c (with a_6 as endpoint). We join these two path with the path $[a_1, a_2, a_3, a_4, a_5, a_6]$.

Let $\mathcal{C}_H$ be the cycle of H obtained this way. For any x of H, x is either a vertex of a gadget or a vertex of a subgraph isomorphic to a K_6. In both cases, it belongs to $\mathcal{C}_H$, thus $\mathcal{C}_H$ is hamiltonian.

We will now show that if H has a hamiltonian cycle then there is a hamiltonian cycle in G. For this, let us look to the different ways to have a hamiltonian path through a gadget. Due to vertex u_0, a hamiltonian cycle can either get in and out the gadget in the same K_6 (as in Figure 10), or get in on one side and out on the other (as in Figure 9). In the second case, for any gadget, we keep the corresponding edge of G. This way, we have a hamiltonian cycle of G. □

4.2 Vertex colouring

Theorem 5. *The problem* Vertex Colouring *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. We recall that the mapping $G \mapsto 2G$ constructs a hypochordal graph. Suppose having a minimum colouring of G, we obtain a colouring of $2G$ by affecting to v_1 and v_2 the same colour as the corresponding vertex v in the colouring of G. Since G is an induced subgraph of $2G$, this colouring of $2G$ is minimum.

Now consider a minimum colouring of $2G$. Since the twin vertices v_1 and v_2 are nonadjacent and have the same neighbours, we can affect them the same colour. Given such a colouring of the vertices of $2G$, we deduce a colouring of G by affecting to v the colour of its corresponding vertices in $2G$. This colouring of G is minimum, otherwise we would obtain a better colouring of $2G$ by affecting to v_1 and v_2 the same colour as the corresponding vertex v in a minimum colouring of G. □

4.3 Maximum clique

Theorem 6. *The problem* Maximum Clique *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. A clique in $2G$ can contain only one of the twin vertices v_1 and v_2. Therefore given a set K of vertices of G and K' a set of vertices of $2G$ where every vertex $v \in K$ corresponds to either v_1 or v_2 in K' ($|K| = |K'|$), K is a maximum clique of G if and only if K' is a maximum clique of $2G$. □

4.4 Maximum stable set

Theorem 7. *The problem* Maximum Stable Set *is $\mathcal{NP}$-complete in hypochordal graphs.*

Proof. A stable set of $2G$ can contain the twin vertices v_1 and v_2 since they are nonadjacent. Let S be a set of vertices of G and $S' = 2S$ the set of corresponding vertices of $2G$, $|S'| = 2 \times |S|$. S is a maximum stable set of G if and only if S' is a maximum stable set of $2G$. □

5 Conclusion

In this paper, we have introduced a new class of graphs which turns out to be interesting for shortest paths routing in networks since the distances between

nonadjacent vertices remain unchanged in case of one vertex or one edge deletion. Then we have characterised the class of minimum hypochordal graphs, attractive when creating a network from scratch since their number of edges is two times minus four their number of vertices. Moreover a minimum hypochordal network is easy to construct from a graph with the vertices corresponding to the sites to connect and the edges corresponding to the possible links between these sites, it is polynomial to find a spanning tree T with bounded diameter [3]; then we build the network $\widetilde{2T}$ with two servers per internal vertex of T which has a bounded diameter to limit the transmission time through this network.
Finally, we have proven that the problems of hamiltonian cycle, vertex colouring, maximum clique and maximum stable set remain $\mathcal{NP}$-complete in hypochordal graphs.
In this paper, we have not tackled the hypochordal recognition issue. This problem is obviously solved by a matrix multiplication and runs at worst in $\mathcal{O}(n^{2.376})$ due to the result of D. Coppersmith and S. Wmograd in [4]. A challenge is to find a $\mathcal{O}(n^2)$ or $\mathcal{O}(m)$ algorithm for the recognition of hypochordal graphs. Moreover a research in progress consists in the minimum edge-completion and edge-deletion problems [6] i.e. making hypochordal an existing graph; some additional constraints like diameter or maximum degree should be taken into account.

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