Partial Lagrangian relaxation for General Quadratic Programming

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Main results (to appear in 4OR)

1. A complete characterization of constant quadratic functions over an affine variety $\Omega = \{x \in \Re^n : Ax = b\}$.

2. How to convexify the objective function of a general quadratic programming problem (Pb) by using the linear constraints.

3. Formulation as a semidefinite program of the partial Lagrangian relaxation of (Pb) where the linear constraints are not relaxed.

4. Comparison of two semidefinite relaxations made from two sets of null quadratic functions over an affine variety.

General Quadratic Programming

$$\text{(Pb)} \min_x f(x) = x^t Q x + c^t x \text{ s.t.} \begin{cases} x^t B_i x + d_i^t x = e_i & i \in I^= \\ x^t B_i x + d_i^t x \le e_i & i \in I^\le \\ Ax = b \end{cases}$$

◇ Boolean quadratic problems can be formulated as (Pb) by considering $x_i^2 = x_i$ for all i in $\{1, \dots, n\}$

◇ Standard semidefinite relaxations of 0-1 quadratic programs are nothing but particular instances of Lagrangian duality [Poljak et al,1995] [Lemaréchal,2003]

◇ For the Boolean case, the *supremum* of the corresponding augmented Lagrangian function is equal to the value of the partial Lagrangian dual (the linear constraints are not relaxed) [Lemaréchal, Oustry, 2001]

Characterization of constant quadratic functions over $\Omega = \{x \in \Re^n : \ Ax = b\}$

Let $F(x) = x^t Hx + g^t x$ be a quadratic function that takes a constant value over Ω. (underlying idea: add redundant constraints)

For any $x \in \Omega$, $u \in Ker(A) = \{u \in \Re^n : \ Au = 0\}$, and $\lambda \in \Re$
$F(x + \lambda u) = F(x) = F(x) + \lambda^2 u^t Hu + 2\lambda u^t Hx + \lambda g^t u$
i.e. $F(x + \lambda u)$ does not depend on λ.

Necessary and sufficient conditions on $F(x)$:

$$\begin{aligned} u^t Hu &= 0 \ \forall u \in Ker(A) && \textbf{[A]} \\ 2u^t Hx + g^t u &= 0 \ \forall u \in Ker(A) \forall x \in \Omega && \textbf{[B]} \end{aligned}$$

Characterization of constant quadratic functions over $\Omega = \{x \in \Re^n : \ Ax = b\}$

Lemma. $q(u) = u^t H u$ *is a null quadratic form over* $Ker(A)$ *if and only if* $q(u) = u^t \left(A^t W^t + WA \right) u$, *where* W *is any* $n \times p$*-matrix.*

Sketch proof. choose a "good" basis for the quadratic form, write and simplify $P^t H P$, then obtain $H = (P^t)^{-1} P^t H P P^{-1}$.
$P = \left[\begin{array}{cc} A^t & B \end{array} \right]$, the $n - p$ columns of B are a basis of $Ker(A)$.

Theorem. $F(x) = x^t H x + g^t x$ *is a constant quadratic function over* $\{x : Ax = b\}$ *if and only if* $F(x) = x^t \left(A^t W^t + WA \right) x + \left(A^t \alpha - 2Wb \right)^t x$, *where* W *is a* $n \times p$*-matrix and* α *is a* p*-vector.*

Any constant quadratic function over $\Omega = \{x \in \Re^n : \; Ax = b\}$ can be obtained by setting particular values to W and α in

$$F(x) = x^t \left(A^t W^t + WA\right) x + \left(A^t \alpha - 2Wb\right)^t x$$

◇ Product member by variable
$W = \frac{1}{2} E^{ij}$ and $\alpha = 0$ where $E^{ij}_{ij} = 1$ otherwise $E^{kl}_{ij} = 0$.
$F(x) = x_i a_j^t x - b_j x_i = 0 \; \forall x \in \Omega$

◇ Product member by member
$W = \frac{1}{2} A^t V$ with $V = \frac{1}{2}\left(E^{ij} + E^{ji}\right)$ $F(x) = x^t a_i a_j^t x + x^t A^t (\alpha - Vb)$
$\alpha = Vb \Rightarrow x^t a_i a_j^t x = b_i b_j \; \forall x \in \Omega$

◇ Penality term
$W = \frac{1}{2} A^t$ and $\alpha = -b \Rightarrow F(x) = x^t A^t A x - 2 x^t A^t b$
$(Ax - b)^2 = 0 \; \forall x \in \Omega$

Convexifying the objective function of (Pb)

Consequence of Debreu's lemma [Lemaréchal, Oustry, 2001]:
If Q is positive definite over $Ker(A)$ then there exists a matrix V such that $Q + A^tVA \succcurlyeq 0$ (not true when Q is not definite)

We prove: *If Q is positive semidefinite over $Ker(A)$, there exists W such that* $Q + A^tW^t + WA \succcurlyeq 0$ *over* $\Re^n$

Theorem. *Let A and Q be respectively a $p \times n$ matrix and a $n \times n$ symmetric matrix. If Q is positive semidefinite over $Ker(A)$ then there exists a linear combination of $q_{ij}(x) = x_i(a_j^t x - b_j)$ for $i \in \{1, ..., n\}$ and $j \in \{1, ..., p\}$ that convexifies the quadratic form x^tQx over $\Re^n$. (null functions over $\Omega = \{x \in \Re^n : \; Ax = b\}$)*

Remark: such a combination can be obtained in $O(pn^2)$ time (constructive proof)

Sketch proof (induction on p)

Choose $H = U^tW^t + WU$ with $UU^t = I$ and $U = S^tA$.
H decomposes into the sum $H = \sum_{i=1}^{p} H_i$, where $H_i = u_i\omega_i^t + \omega_i u_i^t$.
Let B be a matrix whose columns are the vectors of an orthonormal basis of $Ker(A)$.

$$L_i = \left\{ y = \sum_{j=i+1}^{p} z_j u_j + Bz \,:\, z_j \in \Re \; \forall j \in \{i+1, ..., p\}\,,\, z \in \Re^{n-p} \right\}$$

In particular $L_p = Ker(A)$ and $L_0 = \Re^n$.

Lemma. *Let* $1 \leq i \leq p$ *and* Q *a* $n \times n$ *symmetric matrix, if* $Q + \sum_{j=i+1}^{p} H_j \succcurlyeq 0$ *over* L_i *then there exists* ω_i *such that* $Q + \sum_{j=i+1}^{p} H_j + u_i\omega_i^t + \omega_i u_i^t \succcurlyeq 0$ *over* L_{i-1} .

Using convexification in the Lagrangian Approach

$$\text{(P)} \min_x x^t Q x + c^t x \text{ s.t. } x \in \Omega = \{x : Ax = b\}$$

Lemma. *If (P) has a solution and* $F(x)$ *(a constant quadratic function over* Ω*) convexifies the objective function of (P) then there exists* λ *such that*
$\min_{x \text{ s.t. } Ax=b} x^t Q x + c^t x + F(x)$
$= \min_x x^t Q x + c^t x + F(x) + \lambda^t (Ax - b)$.

This convexification process transforms the *constrained* problem (P) into an *unconstrained* one

(DP): Partial Lagrangian dual problem of (Pb)

(DP)

$$\max_{\mu} \min_{x \text{ s.t. } Ax=b} x^t \left(Q + \sum_{i\in I} \mu_i B_i\right) x + \left(c + \sum_{i\in I} \mu_i d_i\right)^t x - \sum_{i\in I} \mu_i e_i$$

Partial Lagrangian dual problem of (Pb) where the linear equality constraints are not relaxed.

$\mathfrak{J} = \left\{ f_j(x) = x^t C_j x + q_j^t x + \alpha_j \;:\; j \in J \right\}$:
A set of null quadratic functions over Ω

(Pb)$_{\mathfrak{J}}$: add the redundant constraints $f_j(x) = 0$ to (Pb).

(DP)$_{\mathfrak{J}}$: partial Lagrangian dual of (Pb)$_{\mathfrak{J}}$ where the constraints $Ax = b$ are not relaxed.
(DP)$_{\mathfrak{J}}$ $\Leftrightarrow$ (DP) because $f_j(x) = 0$ for $x \in \Omega$ and $\forall j \in J$.

(DT)$_{\mathfrak{J}}$: total Lagrangian dual of (Pb)$_{\mathfrak{J}}$ where all the constraints are relaxed:

$$\max_{\mu,\nu,\lambda} \min_{x} x^t Q(\mu) x + c^t(\mu) x - e(\mu) + \sum_{j \in J} \nu_j f_j(x) + \lambda^t (Ax - b)$$

(DT)$_{\mathfrak{J}}$ $\leq$ (DP)$_{\mathfrak{J}}$

Lemma. (convexification) *Let μ^* be a solution of (DP). If there exists ν^* such that $\sum_{j \in J} \nu_j^* f_j(x)$ convexifies $x^t Q(\mu^*) x + c^t(\mu^*) x$, then (DT)$_{\mathfrak{J}}$ $\Leftrightarrow$ (DP).*

Recall that we have: $\mathfrak{P} = \left\{ x_i a_j^t x - b_j x_i \; : \; \forall i \in \{1, ..., n\} \; \forall j \in \{1, ..., p\} \right\}$
Null quadratic functions over Ω which convexify $x^t \left(Q + \sum_{i \in I} \mu_i B_i \right) x$.

Thus the convexification Lemma implies: (DT)$_{\mathfrak{P}}$ $\Leftrightarrow$ (DP)

Remark. This is not the case with: $\mathfrak{C} = \left\{ (Ax - b)^t (Ax - b) \right\}$

Semidefinite formulation of (DT)$_{\mathfrak{I}}$

The semidefinite dual of (DT)$_{\mathfrak{I}}$ is [Lemaréchal, Oustry 2001]:

$$(SDP)_{\mathfrak{I}} \qquad \min_{X \succcurlyeq xx^t} Q \bullet X + c^t x \text{ s.t. } \begin{cases} B_i \bullet X + d_i^t x = e_i & i \in I^= \\ B_i \bullet X + d_i^t x \leq e_i & i \in I^\leq \\ Ax = b & \\ C_j \bullet X + q_j^t x + \alpha_j = 0 & j \in J \end{cases}$$

Thus (DT)$_{\mathfrak{P}}$ $\Leftrightarrow$ (DP) $\Leftrightarrow$ (SDP)$_{\mathfrak{P}}$

$\mathfrak{C} = \left\{(Ax-b)^t(Ax-b)\right\}$ $\mathfrak{P} = \left\{x_i a_j^t x - b_j x_i \;:\; \forall i \in \{1,...,n\}\ \forall j \in \{1,...,p\}\right\}$

Proposition. $(SDP)_{\mathfrak{C}}$ *and* $(SDP)_{\mathfrak{P}}$ *are equivalent.*

$$(SDP)_{\mathfrak{C}} \qquad \min_{X \succcurlyeq xx^t} Q \bullet X + c^t x \text{ s.t. } \begin{cases} B_i \bullet X + d_i^t x = e_i & i \in I^{=} \\ B_i \bullet X + d_i^t x \leq e_i & i \in I^{\leq} \\ (Ax = b) & \\ A^t A \bullet X - 2b^t Ax + b^2 = 0 & \end{cases}$$

$$(SDP)_{\mathfrak{P}} \qquad \min_{X \succcurlyeq xx^t} Q \bullet X + c^t x \text{ s.t. } \begin{cases} B_i \bullet X + d_i^t x = e_i & i \in I^{=} \\ B_i \bullet X + d_i^t x \leq e_i & i \in I^{\leq} \\ Ax = b & \\ \sum_{k=1}^n A_{jk} X_{ki} - b_j x_i = 0 & i \in \{1,...,n\} \\ & j \in \{1,...,p\} \end{cases}$$

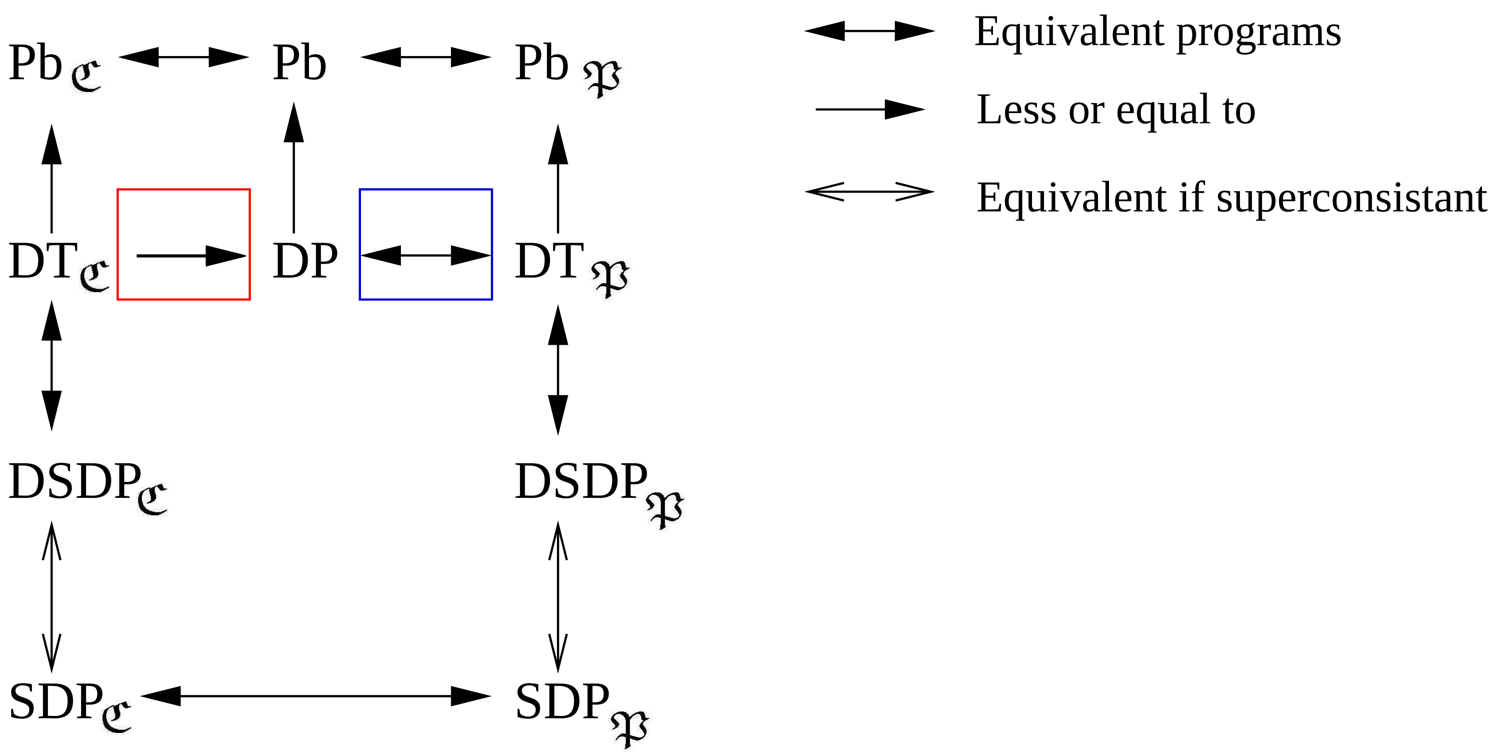
Sketch proof

(X, x) feasible of $(SDP)_{\mathfrak{P}}$.
For each j multiply $\sum_k A_{jk}X_{ki} - b_j x_i = 0$ by A_{ji}, then sum up them all over j and i: $A^tA \bullet X - 2b^tAx + b^2 = 0$

(X, x) feasible for $(SDP)_{\mathfrak{C}}$.
$A^tA \bullet \left(X - xx^t\right) + (Ax - b)^2 = 0.$
$A^tA \bullet \left(X - xx^t\right) = 0 \Rightarrow A^tA\left(X - xx^t\right) = 0.$
$\forall r, i \in \{1, ..., n\}\ \sum_{k=1}^n \sum_{j=1}^p A_{jr}A_{jk}X_{ki} = x_i \sum_{k=1}^n \sum_{j=1}^p A_{jr}A_{jk}x_k$
$\Rightarrow \forall r, i \in \{1, ..., n\}\ \sum_{j=1}^p A_{jr}\left(\sum_{k=1}^n A_{jk}X_{ki} - b_j x_i\right) = 0.$
This a linear combination of the p rows of A.
$rank(A) = p \Rightarrow \sum_{k=1}^n A_{jk}X_{ki} - b_j x_i = 0\ \forall j$ and $\forall i$.

$$\mathfrak{P} = \left\{ x_i a_j^t x - b_j x_i \ : \ \forall i \in \{1, ..., n\} \ \forall j \in \{1, ..., p\} \right\}$$

$$\mathfrak{C} = \left\{ (Ax - b)^t \, (Ax - b) \right\}$$



Conclusion

$\diamondsuit$ A complete characterization of constant quadratic functions over $\{x \in \Re^n : Ax = b\}$.

$\diamondsuit$ For general quadratic programs we have (SDP)$_{\mathfrak{P}}$=(DT)$_{\mathfrak{P}}$ =(DP).

$\diamondsuit$ For Boolean problems (DT)$_{\mathfrak{C}}$ =(DP) but the supremum is not always reached: consequences for some SDP solvers.

$\diamondsuit$ Better design of semidefinite relaxations of quadratic programs.

$\diamondsuit$ Among the $p \times n$ constraints of (SDP)$_{\mathfrak{P}}$, some may be not active, and thus it would be interesting to foresee which constraints are useful for a given problem.