Tensor-based approaches for learning flexible neural networks

Yassine Zniyed
CRAN, Université de Lorraine, CNRS (Nancy)

Joint work with Konstantin Usevich, Sebastian Miron, David Brie

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What is a tensor ?

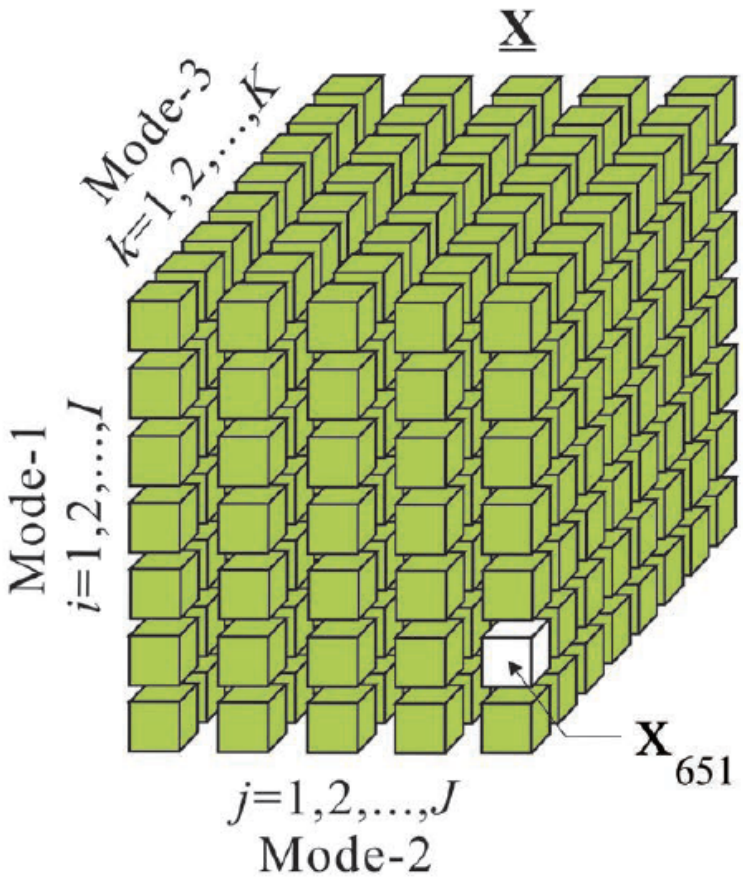


FIGURE – A 3-order tensor $\mathcal{X}$. [Cichocki et al., 2014]

Matrix and tensor rank-one decomposition

- $\circ$ outer product : $[\mathrm{p}_1 \circ \mathrm{p}_2 \circ \mathrm{p}_3]_{i,j,k} = \mathrm{p}_1(i) \cdot \mathrm{p}_2(j) \cdot \mathrm{p}_3(k)$.

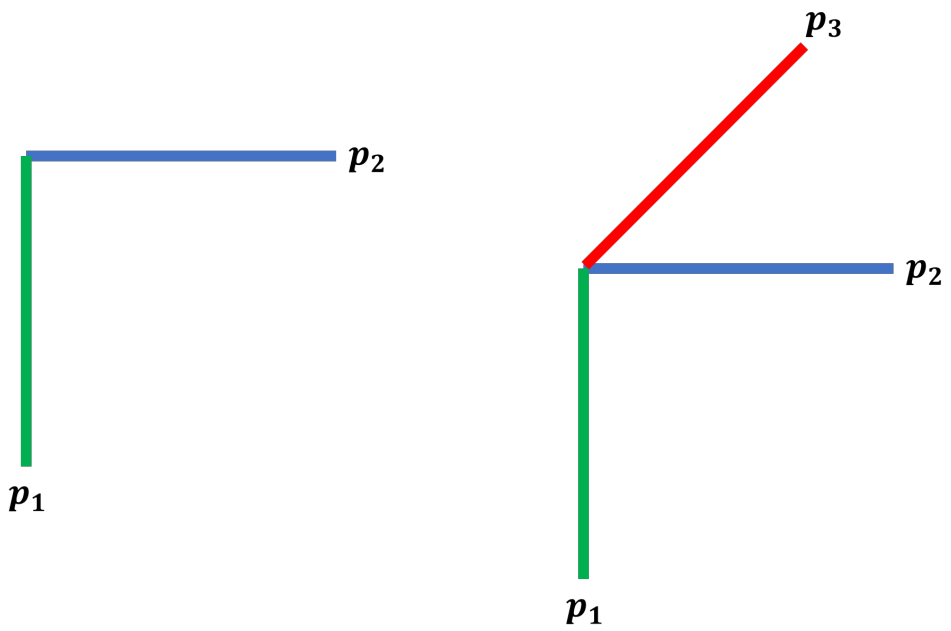


FIGURE – Left : Rank-1 matrix - Right : Rank-1 tensor.

- Matrix rank-1 decomposition : $\mathrm{X} = \mathrm{p}_1 \circ \mathrm{p}_2 = \mathrm{p}_1 \mathrm{p}_2^T$.
- What about rank-R decomposition ?

From matrix to tensor rank decomposition

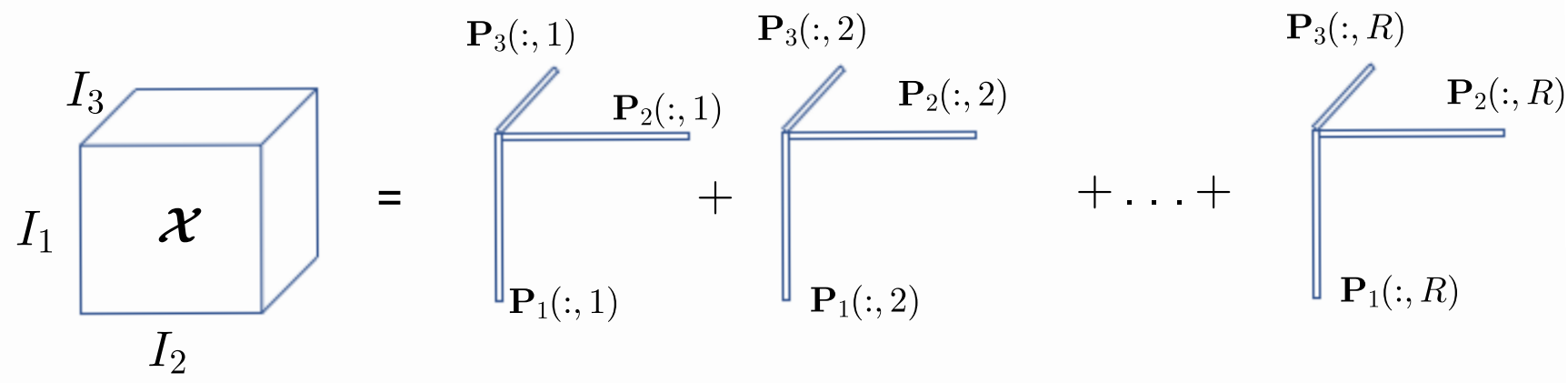


FIGURE – Tensor rank decomposition [Hitchcock, 1927].

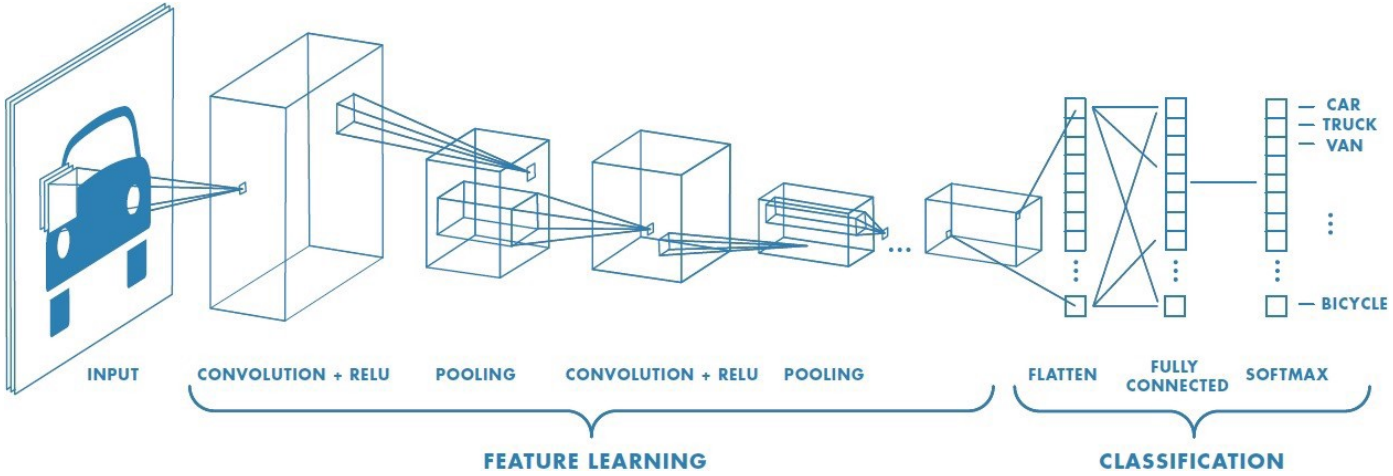
- Canonical polyadic decomposition (CPD) :

$$\mathcal{X} = \sum_{l=1}^{R} \mathrm{P}_1(:,l) \circ \mathrm{P}_2(:,l) \circ \mathrm{P}_3(:,l).$$

- Often unique, if rank R is smaller than a bound.

Overview

Typical feed-forward convolutional network :



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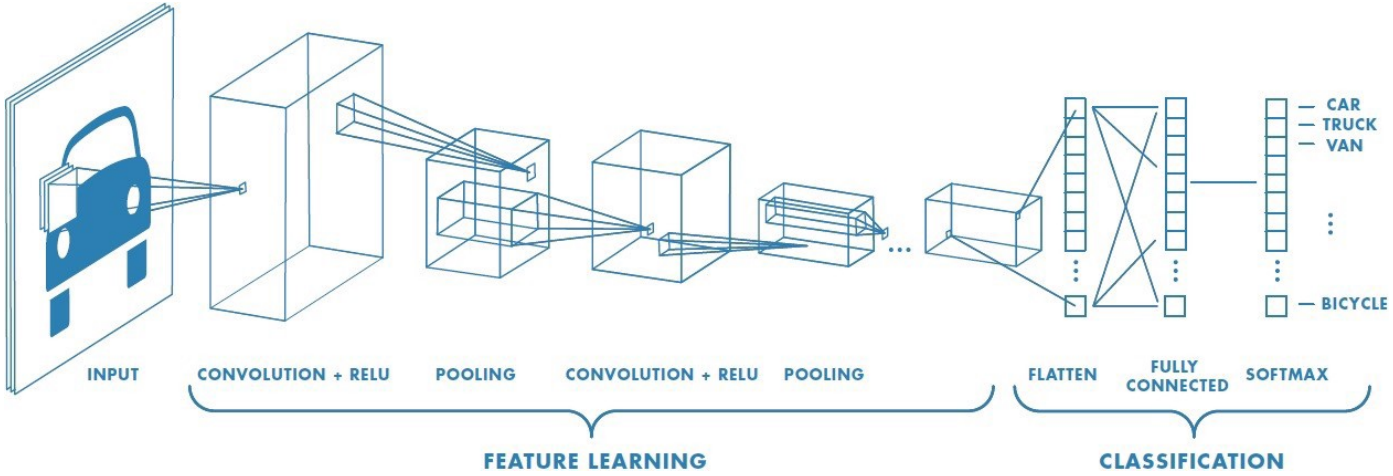
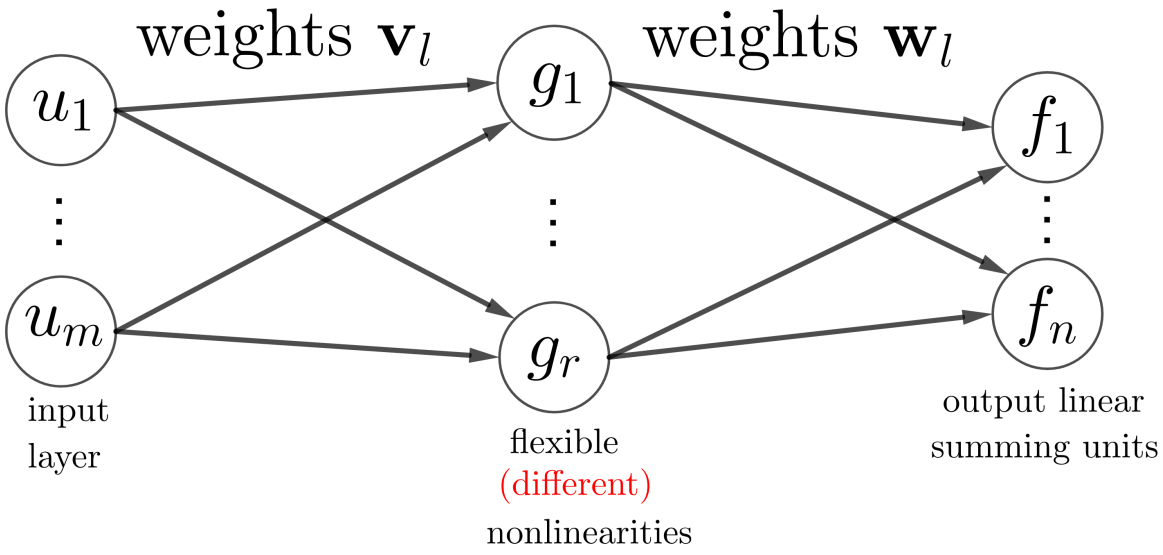


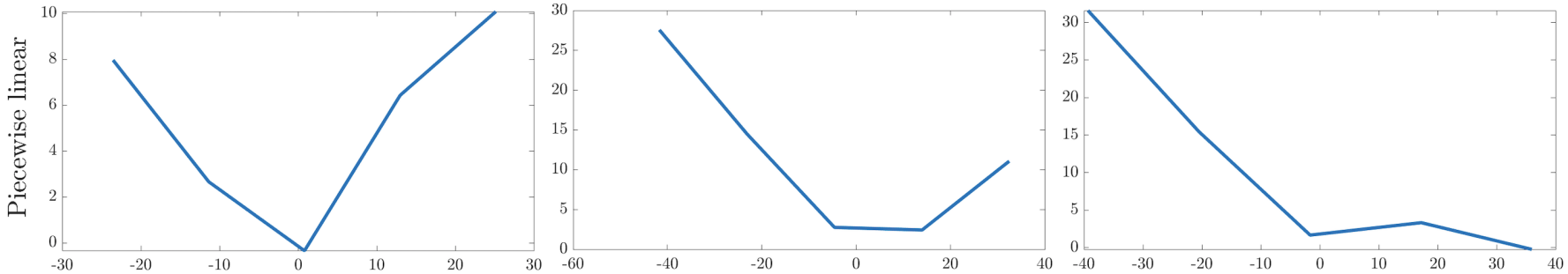
1. Compression of tensors in convolutional layers.
   Lebedev et al., "Speeding-up Convolutional Neural Networks Using Fine-tuned CP-Decomposition", ICLR, 2015.
2. Interpreting tensor decompositions as product-sum units.
   Cohen et al., "On the Expressive Power of Deep Learning : A Tensor Analysis", COLT, 2016
3. Our proposed architecture : flexible activation functions

Main idea

Graphical representation of the basic one-layer flexible NN :



Example :



Some motivation

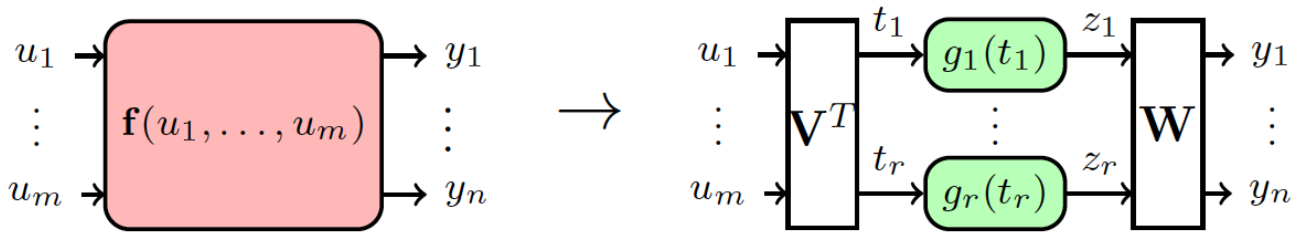
- Kolmogorov-Arnold representation theorem (1957) :
  Any continuous $f : [0,1]^m \to \mathbb{R}$ can be represented as

$$f(u) = \sum_{k=1}^{2m+1} \chi_k \Big(\sum_{j=1}^{m} \psi_{k,j}(u_j)\Big),$$

  with different continuous χ_k, $\psi_{k,j}$.
- Fixed AF with bias $(g(\mathrm{v}_k^T \mathrm{u} + b))$.
- Existing architectures (e.g. polynomial NN) are difficult to train from scratch.

Factorization : a tensor-based approach

- Our framework : compression of pre-trained NN.
- Goal : decompose (approximate) given f by



i.e., $\mathrm{f}(\mathrm{u}) = \mathrm{W}\mathrm{g}(\mathrm{V}^T\mathrm{u})$, with $\mathrm{g}(t_1, \cdots, t_r) = [g_1(t_1) \cdots g_r(t_r)]^T$.

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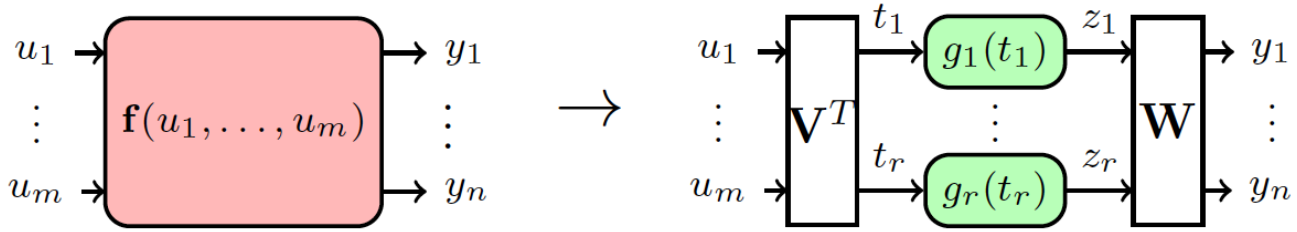

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Borrow ideas from system identification ([Dreesen et al, 2015])

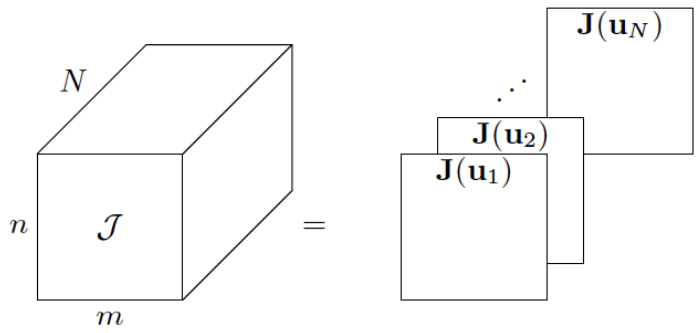
Key idea : by chain rule

$$\mathrm{J_f}(\mathrm{u}) := \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\mathrm{u}) & \cdots & \frac{\partial f_1}{\partial u_m}(\mathrm{u}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1}(\mathrm{u}) & \cdots & \frac{\partial f_n}{\partial u_m}(\mathrm{u}) \end{bmatrix} = \mathrm{W} \cdot diag\left(g_1'(\mathrm{v}_1^T\mathrm{u}) \cdots g_r'(\mathrm{v}_r^T\mathrm{u})\right) \cdot \mathrm{V}^T.$$

Factorization : a tensor-based approach

Algorithm.

1. Evaluate $\mathrm{J_f}(\mathrm{u})$ at N "operating points" $\mathrm{u}_1, \ldots, \mathrm{u}_N \in \mathbb{R}^m$
2. Stack them into a tensor :



3. Joint matrix diagonalization $\leftrightarrow$ CPD

$$\begin{array}{ccc} \mathrm{J_f}(\mathrm{u}^{(1)}) = \mathrm{W}\mathrm{D}^{(1)}\mathrm{V}^\top, & & \\ \vdots & \leftrightarrow & \mathcal{J} = \sum_{l=1}^{r} \mathrm{w}_l \circ \mathrm{v}_l \circ \mathrm{h}_l \\ \mathrm{J_f}(\mathrm{u}^{(N)}) = \mathrm{W}\mathrm{D}^{(N)}\mathrm{V}^\top & & \end{array}$$

4. Retrieve v_l, w_l from factors of the CPD

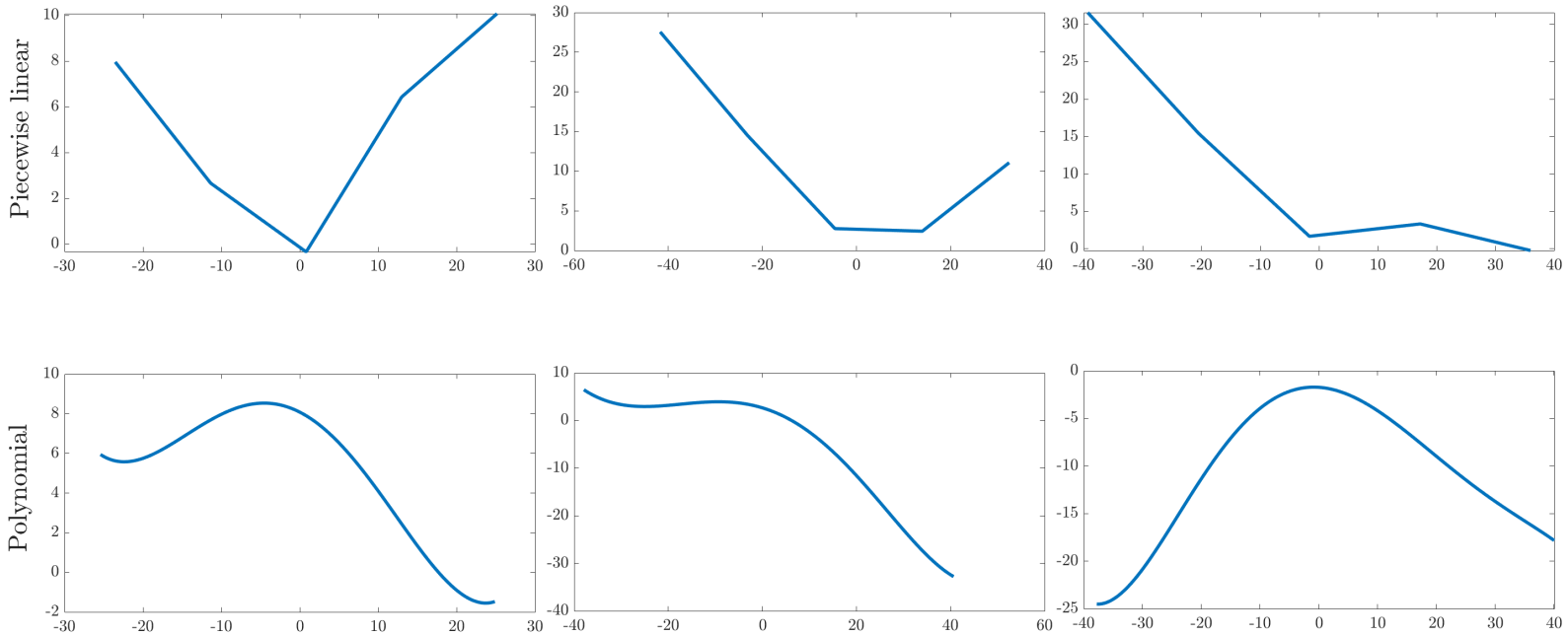
- Issue : assumptions needed to estimate g_l

Basis functions

The AFs are expressed as

$$g_l(t) = c_{0,l} + c_{1,l}\phi_1(t) + \cdots + c_{d,l}\phi_d(t).$$

Examples :



Uniqueness

The flexible NN model function $\mathrm{f} : \mathbb{R}^m \to \mathbb{R}^n$ is expressed as :

$$\mathrm{f}(\mathrm{u}) = \mathrm{w}_1 \cdot g_1(\mathrm{v}_1^T \mathrm{u}) + \cdots + \mathrm{w}_r \cdot g_r(\mathrm{v}_r^T \mathrm{u}). \tag{1}$$

Worth to mention : [Comon, Qi, Usevich, 2017]

- The above decomposition is associated with some interesting uniqueness results, when $\phi_k(t) = t^k$.
- The decomposition (1) is unique, when $d \geq 3$, $m \geq 2$ and r is not too large.
- The decomposition (1) is partially identifiable, when $r > mn$.

Constrained Coupled Matrix-Tensor Factorization (CMTF)

For a base $g_l(t) = c_{0,l} + c_{1,l}\phi_1(t) + \cdots + c_{d,l}\phi_d(t)$, we solve

$$\min_{\mathrm{w}_l,\mathrm{v}_l,\mathrm{h}_l,\mathrm{z}_l} \underbrace{\left\| \mathcal{J} - [|\mathrm{W},\mathrm{V},\mathrm{H}|] \right\|^2}_{\text{Jacobian}} + \lambda \cdot \underbrace{\left\| \mathrm{F} - \mathrm{W}\mathrm{Z}^T \right\|^2}_{\text{function evaluation (data fidelity)}}$$

$$\text{subject to} \quad \mathrm{h}_l = \mathrm{X}_l \cdot \mathrm{c}_l, \quad \mathrm{z}_l = \mathrm{Y}_l \cdot \mathrm{c}_l, \quad \text{with}$$

$$\mathrm{X}_l = \begin{bmatrix} 0 & \phi_1'(\mathrm{v}_l^T\mathrm{u}^{(1)}) & \cdots & \phi_d'(\mathrm{v}_l^T\mathrm{u}^{(1)}) \\ \vdots & \vdots & & \vdots \\ 0 & \phi_1'(\mathrm{v}_l^T\mathrm{u}^{(N)}) & \cdots & \phi_d'(\mathrm{v}_l^T\mathrm{u}^{(N)}) \end{bmatrix}, \quad \mathrm{Y}_l = \begin{bmatrix} 1 & \phi_1(\mathrm{v}_l^T\mathrm{u}^{(1)}) & \cdots & \phi_d(\mathrm{v}_l^T\mathrm{u}^{(1)}) \\ \vdots & \vdots & & \vdots \\ 1 & \phi_1(\mathrm{v}_l^T\mathrm{u}^{(N)}) & \cdots & \phi_d(\mathrm{v}_l^T\mathrm{u}^{(N)}) \end{bmatrix}.$$

- We use an alternating least squares (block-coordinate descent) with projection

Neural networks compression

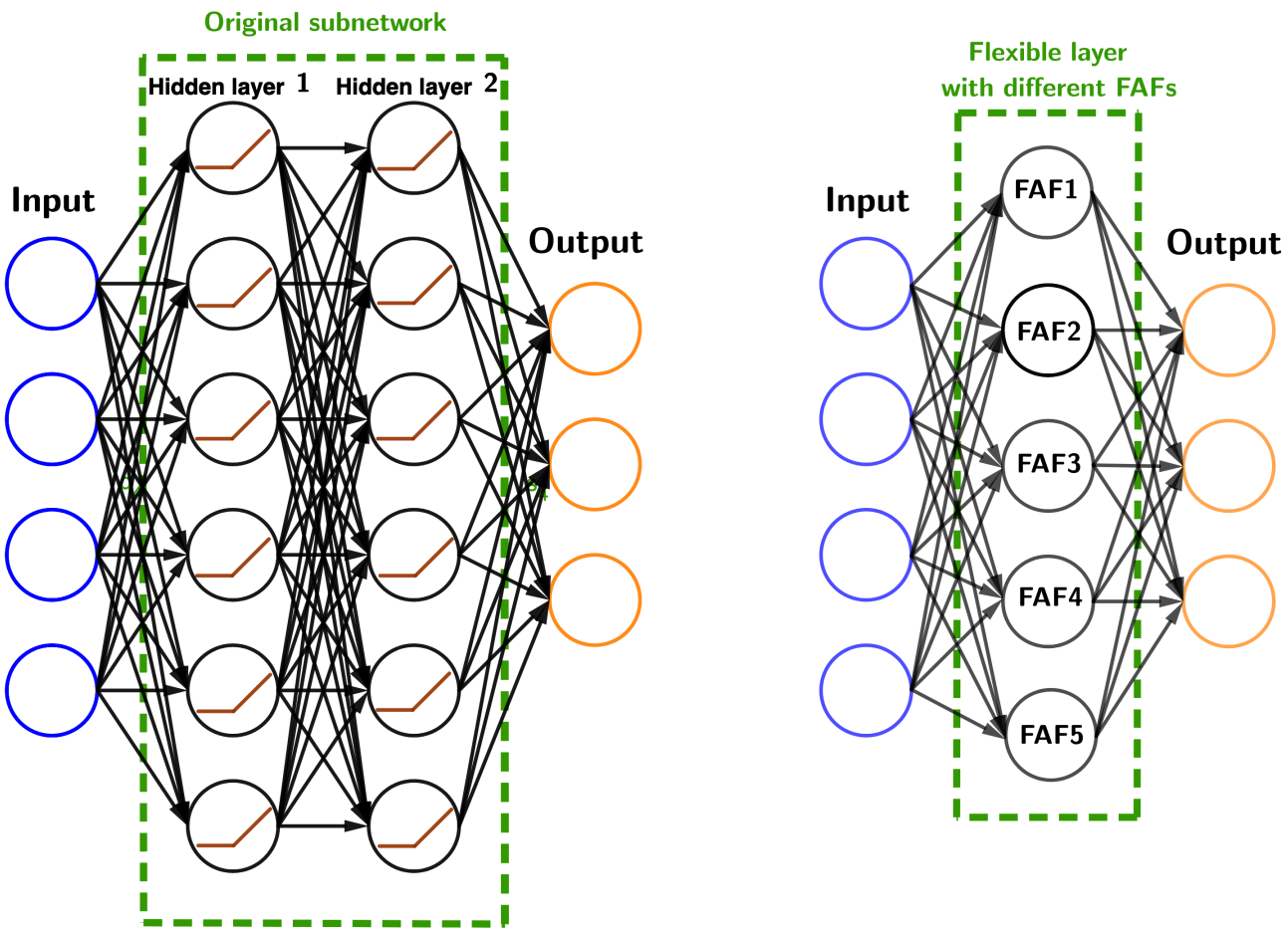
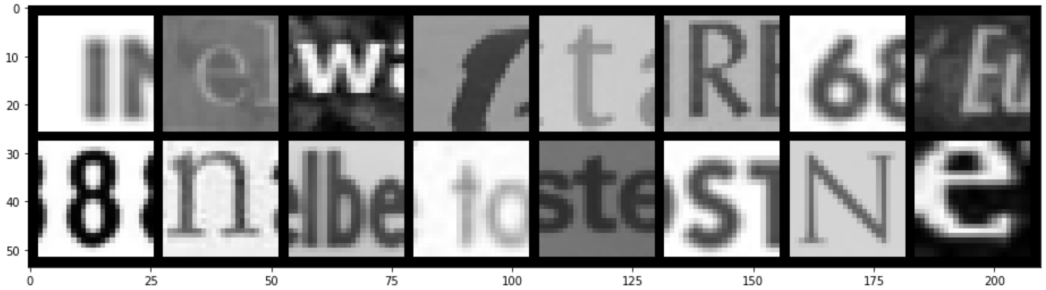


FIGURE – Graphical representation of the compression process : (left) the original pretrained NN with fixed AFs - (right) the approximated NN with flexible AFs.

ICDAR and CharNet

ICDAR 2003 dataset : ≈163000 train, ≈ 5300 test, 36 classes

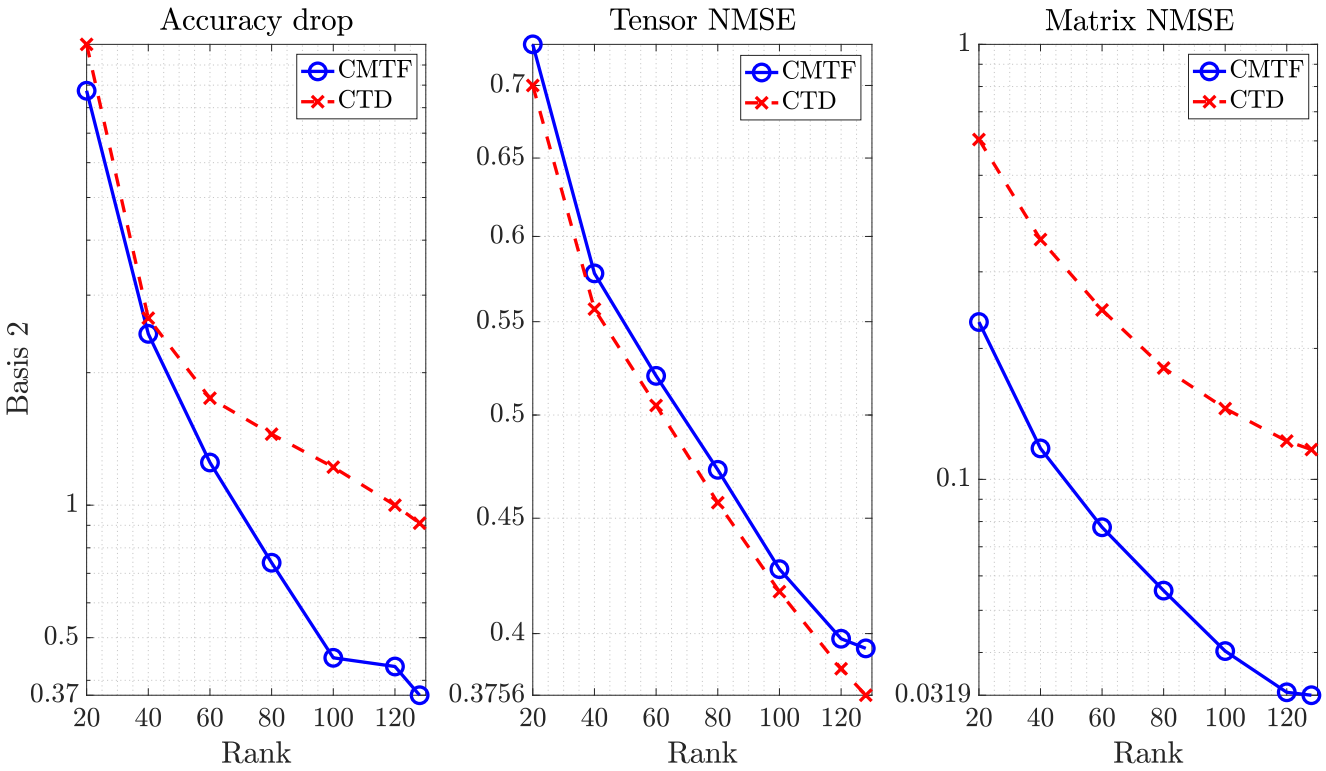


CharNet [Jaderberg et al., 2014] (in MatConvNet from Oxford)

layer	type	filter size	input	output
conv1	convolutional	9x9	1	96
	maxout (/2)		96	48
conv2	convolutional	9x9	48	128
	maxout (/2)		128	64
conv3	convolutional	8x 8	64	512
	maxout (/4)		512	128
fc	fully connected	1x 1	128	192
	maxout (/4)		192	48
softmax	softmax	-	48	36

Some results

- we compress the conv3 layer of CharNet (viewed as fully connected layer $\mathbb{R}^{4096} \to \mathbb{R}^{128}$)
- use only 360 points (10 per class), without fine-tuning
- 4x compression



Thank you !

References :

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- K. Usevich, P. Dreesen and M. Ishteva., *Decoupling multivariate polynomials : interconnections between tensorizations*, JCAM, 2020.
- Y.Zniyed, K. Usevich, S. Miron, D. Brie, *Tensor-based framework for training flexible neural networks*, preprint on arXiv :2106.13542.