MULTIDIMENSIONAL DATA ANALYSIS
AND QUANTIFICATION OF CATEGORICAL VARIABLES

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Quantification is a powerful tool for dealing
with qualitative variables in data analysis :
it is both an efficient way for description
and prediction with nominal data and a concep-
tual framework for presenting many techniques.
This paper attempts to be a survey of this
topic.

1. USE AND DEFINITION OF QUANTIFICATION

By "quantification" (the words "scaling" or "scoring" are also
often used) we mean a transformation of one or several categori-
cal variables into numerical ones. Before describing the way to
achieve a quantification (and the problems it involves) it is
necessary to argue about the rationale of this approach.

1) Quantification : a device or a fundamental method ?

a) Numerical methods for data which are not numerical.

The main consequence of quantifying qualitative variables is
to allow the use of usual statistical techniques such as prin-
cipal components analysis, multiple regression or discriminant
analysis for instance, by coming down to the usual situation
of numerical variables.

That convenience was during a long time the only justification
of the quantification techniques, what may seem a somewhat
poor argument from a theoretical point of view : isn't it by
lack of imagination or laziness that one does not develop
methods which would be fitted to the qualitative nature of
data ?

Let us notice here that the quantification is a way of proces-
sing variables of different kinds (numerical and categorical)

by giving the same part to each variable. Let us suppose that we have a first set of m numerical variables $x_1...x_n$ and a second set of f qualitative variables x_1, x_2, ... x_m and that we want to make a descriptive data analysis of all the (m+p) variables by means of a component-like analysis ? There are actually four possibilities, the first two giving non symmetrical parts to the two sets :

- Perform a principal component analysis with the x_j j=1,2,...m and use the x_k as additional variables by representing the categories of each x_k by the averages of the individuals which belongs to them . We have here a representation of the x_k in the space of the individuals.

- Perform a multiple correspondence analysis of the x_k and use the x_j as additional variables by computing the correlation coefficient of the x_j with the principal components. The representation of the x_j is here in the space of variables.

- Split into categories the numerical variables and perform a multiple correspondence analysis with (m+p) qualitative variables.

- Quantify the x_k into x_k^x and perform a principal components analysis with (m+p) numerical variables.

This last possibility is what we proposed above ; the third one seems different but we will see that it is also a quantification technique.

b) Multivariate analysis as quantification techniques.

Actually many classical methods which are dealing with categorical variables may be considered as quantification techniques.

For instance analysis of variance or covariance are performing quantification of the nominal "factors of variability" when estimating the effects upon the dependent variable (in the case of the no-interaction model). Canonical discriminant analysis is merely a regression of the quantified group-variable upon a set of predictors with an "optimal" quantification i.e. giving the largest determination coefficient among all possible quantifications (see for instance Mc Donald and Cailliez-Pages).

Furthermore any multidimensional scaling method for qualitative variables (such as correspondence analysis) which gives

coordinates for the categories of a set of qualitative variables is in fact a multidimensional quantification technique : the coordinates along an axis are numerical values (scores) to be assigned to a qualitative variable.

c) An approximation of non-linear analysis.

Multidimensional data analysis is mostly a set of methods using linear algebra and the vector-space structure of the sets of variables and individuals : these methods can only place in prominent position linear relationships between variables what is well fitted to the case of multinormal distribution which is exceptional in most applications.

In most cases, the study of linear relationships is not sufficient and a non-linear analysis is actually necessary.

The quantification of qualitative variables provides an approximation of non linear analysis : let us transform a numerical variable x into a qualitative one X by simply splitting into classes the set of its values and then let us quantify X into x^x.

The set of all possible quantifications of X is an approximation, by piecewise constant functions, of the set of all functions of x f (x).

As the study of linear relationships between any functions f(x) and g(y) of two variables x and y is nothing else than the study of non-linear relationships between x and y, we therefore get a tool for non linear multivariate analysis (Masson). In fact it is possible to define less simple transformation of variables to get non-linearity, such as transformation by spline-functions of higher order (splitting and quantifying is equivalent to splines of degree zero) (Van Rijckevorsel and De Leeuw, Lafaye de Michaux), but it is slightly more complicated (spline functions are a kind of regular functions, very popular in the field of interpolation).

As Dauxois and Pousse quoted it, these techniques are not purely non-linear but only semi-linear in the following sense : if we use transformed variables x^x instead of raw variables x_j in a multidimensional analysis we will deal with linear combinations of the x_j^x but not with any functions of the x_j.

2) How to quantify ?

If we accept quantification as a justifiable practice we have
however to make clear some points such as : among all possi-
ble quantifications of a variable how much must we choose and
according to which criteria ?

a) Quantification and type of variables

When a qualitative variable X is purely nominal (without any
structure upon the set of its categories) a quantification is
a transformation of X into a discrete numerical variable :
we will assign the same numerical value, say a_i, to all indi-
viduals belonging to the ith category of X.

If the variable X is ordinal (there is a total natural order
for its categories such as level of education for instance),
people often suggest to use only quantifications respecting
the order of the categories : the numerical values to be
assigned to the m ordered categories of X must be such as
$a_1 \leq a_2 \leq \ldots \leq a_m$. Some authors (Nishisato mainly) consider
the more general situation of a partial order of the cate-
gories.

Quantification under order restrictions leads to a more
sophisticated theory than quantification without constraints,
using convex cones instead of vector subspaces (see Barlow
and al. or Tenenhaus) and to more difficult computations. But
we get essentially a one-dimensional quantification as we
will see below.

Apart from these difficulties we may wonder if it is actually
necessary to respect order constraints : for instance in a
prediction problem where an explanatory variable is ordinal
and the variable to be predicted is numerical,quantification
with order restrictions comes down to postulate the existence
of a monotonic relationship. Do we have to introduce such
an a priori before having studied the relationship ? It may
be more interesting to make an analysis without order restric-
tions and see afterwards if we get a quantification which
respects the ordinal nature of the categories. If not, it
will be a proof of a non monotonic relationship provided there
is no sampling errors. Of course if we have strong reasons
for a monotonic relationship we need to use this information.

Conversely if it is the dependent variable which is ordinal
we must respect its nature, as in the situation where we have
to describe the relationships between several ordinal variables.

In most cases the quantification of a qualitative variable
involves the assignment of a single numerical value to each
category. F.W. Young and his team have attracted notice to
the difference between the underlying process and the measure-
ment level : for instance a phenomenon may be continuous
(wave-length for colour perception) and the observation
discrete (colour). So why quantify a category by a single
value ? A more general quantification implies that a category
may be represented by an interval of values, the different
intervals being a partition of an interval of $\mathbb{R}$.

For ordinal measurements associated to continuous process
there is, in addition a constraint to order for the intervals.

We may notice that we are looking in fact for a quantifica-
tion of observations more than of categories. Apart of the
techniques proposed by Young, De Leeuw and Takane, quantifi-
cation with intervals is not widely used and we will not
refer to it in the remaining of this paper.

b) Mathematical formulation of quantification.

Let X be a qualitative variable with m categories and E the
set of its categories.

If Ω is the usual universe, X is a mapping of Ω onto E.
A quantification of X is defined by a mapping a of E onto $\mathbb{R}$,
the quantified variable being a o X :

$$\Omega \xrightarrow{\quad X \quad} E = (1, 2, \ldots, m)$$

aoX, a, $\mathbb{R}$
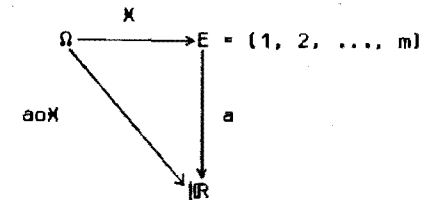
The variable aoX takes only at most m values $a_1, a_2, \ldots a_m$
corresponding to the m categories of X.

If we introduce the m following indicator variables of the
categories Π_j j = 1,2,....,m such that :

$$\Pi_j(\omega) = \begin{cases} 1 \text{ if } X(\omega) = j \\ 0 \text{ if not} \end{cases} \quad \text{for } \omega \in \Omega$$

we have the elementary but capital result : the quantified

variable aoX is nothing else than the linear combination of the indicator variables defined by the scores a_j :

$$\boxed{aoX = \sum_{j=1}^{m} a_j \Pi_j}$$

Therefore the quantification of variables belongs to the field of linear multidimensional analysis.

If there is no restriction upon the scores a_j (purely nominal variables) the set of numerical variables which are a quantification of X is a closed subset of dimension m of $L^2(\Omega)$, the vector subspace spanned by the Π_j.

If X is an ordinal variable with the natural order upon its categories a quantification of X must verify $a_1 \leq a_2 \leq .. \leq a_m$.

This set of constraints may be written as follows with the non negative numbers $b_0, b_1, ..., b_m$

$$\begin{cases} a_1 = b_1 - b_0 \\ a_2 = b_1 + b_2 - b_0 \\ a_m = b_1 + b_2 + .. + b_m - b_0 \end{cases}$$

The quantified variable a . X is then equal to :

$$\sum_{j=1}^{m} a_j \Pi_j = \sum_{j=1}^{m} (b_1 + b_2 + .. + b_j - b_0) \Pi_j$$

$$= \sum_{j=0}^{m} b_j \tilde{\Pi}_j \quad \text{with } b_j > 0$$

where $\tilde{\Pi}_j = \sum_{i \geq j} \Pi_i$ and $\tilde{\Pi}_0 = -1$.

The Π_j are the "indicator variables" of the order (Bouroche, Dupont-Gatelmand, Tenenhaus) in the sense where :

$$\tilde{\Pi}_j (\omega) = \begin{cases} 0 \text{ if } X(\omega) < j \\ 1 \text{ if } X(\omega) \geq j \end{cases} \quad j = 1, 2, ..., m.$$

The set of all possible quantifications of X with the order restriction is thus the polyedric convex cone $\mathbb{C}$ spanned by the variables $\tilde{\Pi}_j$.

$$C = \{ x^x \mid x^x = \sum b_j \tilde{\Pi}_j \quad b_j \geq 0 \}$$

If the variable X has been observed upon n individuals we have the following matrix representation of the data and of the quantification :

for a purely nominal variable X may be represented as the matrix with n rows and m columns of the indicator variables.

$$X = \begin{pmatrix} 0100 \\ 1000 \\ .... \\ .... \end{pmatrix}$$ A numerical variable $\underline{x}^x$ obtained by a quantification of X

is expressed by $\underline{x}^x = X \underline{a}$ where $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ is the vector of

the scores of the categories

The set of all quantified variables is W, the subspace of $\mathbb{R}^n$ of dimension m defined by :

$$W = \{ \underline{x}^x \mid \underline{x}^x = X\underline{a} \quad \underline{a} \in \mathbb{R}^m \}$$

For an ordinal variable X with 3 categories we have for instance for the following five individuals and $a_1 \leq a_2 \leq a_3$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 - b_0 \\ b_1 + b_2 - b_0 \\ b_1 + b_2 + b_3 - b_0 \\ b_1 - b_0 \\ b_1 + b_2 - b_0 \end{pmatrix}$$

The set of all possible $\underline{x}^x$ is the cone $\mathbb{C}$ defined by

$$\mathbb{C} = \{ \underline{x}^x \mid \underline{x}^x = \tilde{X} \underline{b} \quad b_j \geq 0 \}$$

Frequently one has to deal only with zero-mean variables : if $\underline{1}$ is the constant variable which components are all equal to 1, the set of all possible $\underline{x}$ reduce to $W \cap \underline{1}^\perp$ or $I \cap \underline{1}^\perp$ where $\underline{1}^\perp$ is the vector subspace orthogonal for $\underline{1}$ (i.e. made of zero-mean variables).

For nominal variables the equivalence between a quantification and a linear combination of indicator variables shows that the study of relationships between a set of quantified variables comes down to canonical analysis problems for it is nothing else than the study of linear relations between sets of numerical variables (taking only values 0 or 1).

c) Uni or multidimensional quantification ?

As the set of all possible quantifications of a qualitative variable is not of dimension one, it may be necessary to use several quantifications in order to describe a single quali-tative variable. Thus it is not obvious that a nominal varia-ble may be reduced to a single dimension ; correspondance analysis of contingency tables is an example of such a situation : we get two sequences of quantifications for taking into account the dependency between the two cross-classified variables.

In many cases, however, the complexity of a multidimensional quantification may be considered as a drawback and the practitioner would like to satisfy himself with a single quantification but this leads to some difficulties, as we will see below, except for two situations. The first one is that of prediction problems where there is only one best possible quantification and the second one is that of ordinal variables in descriptive studies : multidimensional quanti-fications are generally obtained by search of orthogonal basis (uncorrelated variables). But, if we have a total order upon the categories of X, the admissible quantifications of zero mean belong to the cone $\varepsilon \cap \underline{1}^\perp$: when a variable x belongs to this cone, the variables which are orthogonal to x do not belong to the cone and do not fill the order restrictions.

d) "Optimal" quantification.

In order to quantify qualitative variables we thus have to respect the nature of the variables but there remains an infinite number of possible quantifications : quantification is meaningful only if we have a precise aim which consists generally in the maximisation of some criterium of fit. For

instance if we deal only with two nominal variables, it seems natural that the two quantified variables should be maximally correlated what allows the best prediction of one by the other in the least-square sense.

In the same way if we have to predict one variable (qualita-tive or not) by several variables which may also be qualita-tive or not, there is a natural criterium of quantification: the maximisation of the square correlation between the depen-dent (possibly quantified) and a linear combination of the (possibly quantified) explanatory ones.

But if we have to quantify simultaneously more than two nominal variables without an external dependent variable, there is no unique criterium and there will be many "optimal" quantifications as we will see in the next part.

II. SIMULTANEOUS QUANTIFICATION OF SEVERAL QUALITATIVE VARIABLES

1) The case of two variables

Analyzing contingency tables by means of a quantification of the margins is very classical and leads to an unique solu-tion, whatever is the criterium, that one gives by correspon-dence analysis or by canonical analysis.

Without discussing about rather uninteresting questions of anteriority, it seems instructive to remind briefly cf the different points of view appeared along the years.

For more details the reader may refer to publications of Benzecri or De Leeuw.

In 1935 Hirschfeld, remarking that the analysis of the corre-lation between two variables gives the best results if the two lines of regression are linear, settles and solves the following problem :

"Given a discontinuous distribution, is it always possible to introduce new values for the variates such as both regressions are linear ?"

He then proves that for an array of size (m,p), there exists min ((p-1) ; (m-1)) orthogonal solutions and gives the equa-lity between the Pearson's $\emptyset^2$ and the sum of squares of correlation between the pairs of quantified variables.

This result remained apparently ignored during more than
twenty years until the work of Lancaster (1958) who generali-
zed it to continuous variables and made the synthesis of the
papers of Fisher (1940) Maung (1941) and Williams (1952).

Fisher introduced the quantification in order to apply his
linear discriminant function when the predictor is itself
nominal and settled the equivalence with Hotelling's canoni-
cal analysis. His algorithm to get the simultaneous quanti-
fication is interesting by its closeness to Hirschfeld's
problem and to the alternate least squares method of Young,
De Leeuw and Takane : in a first step we give an arbitrary
quantification to one variable, then the second variable is
quantified according to the means of the first in each of its
categories. The first variable is now quantified again
according to this principle of "reciprocal averaging" (Hill),
apart from a factor of normalization to avoid degenerate
solution, and so on until convergence (due to the fact that
at the optimum the two quantified variables have both linear
regressions).

Formally the solution is given by the canonical analysis of
the two sets of indicator variables X_1 and X_2 : the quantified
variables are the canonical variables and the optimal scores
are given by the eigenvectors of the products of the two
arrays of conditional frequencies.

2) The case of p nominal variables

The simultaneous quantification of more than two nominal
variables have as many solutions as criteria, on the contrary
of the case where $p=2$ where all criteria were equivalent.
This is due to the fact that there does not exist a single
measure of correlation between more than two variables.

However there are two main kinds of quantifying p variables,
leading both to relative simple computations. The first comes
down to an eigenvector problem and provides multidimensional
quantifications of the variables, the second looking for a
single quantification of each variable such as we get an
optimal representation of the set of individuals upon a
subspace of fixed dimension.

We will need the following matrices :

$X = (X_1 \mid X_2 \mid \ldots \quad X_p)$ disjunctive array with n rows and
$\Sigma\, m_i$ columns of the indicator variables of the categories of
all the X_i.

$$X^* = \begin{pmatrix} x_1^* & x_2^* & \ldots & x_p^* \\ \mid & \mid & & \mid \end{pmatrix} \text{ array of size (n,p) of the quantified}$$
variables.

The so-called Burt's matrix :

$$B = X'X = \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1p} \\ C_{21} & C_{22} & \ldots & C_{2p} \\ \vdots & & & \\ C_{p1} & \ldots\ldots & C_{pp} \end{pmatrix} \quad \text{where } C_{ij} \text{ is the contingency}$$
table between X_i and X_j, i.e. $C_{ij} = X_i'\, X_j$.

B is a square-symmetrical matrix of size Σm_i.

D is the diagonal matrix with the same diagonal terms as B.

a) From Guttman to Benzecri ; or the long story of multiple
   correspondence analysis.

Following L.L. Guttman, various authors (Horst, Lord, Bock,
Nishisato, de Leeuw ...) have introduced the problem accor-
ding to the following formulation : "Find the quantification
of X_1, X_2, ... X_p such as the measures of the p quantified
variables x_1^*, x_2^*, .. x_p^* be as homogeneous as possible for
any individual and as scattered as possible between indivi-
duals". This is a very clear criterium from the classical
point of view of factor analysis which comes down to an opti-
mal analysis of variance : we have to maximize the dispersion
of the means of each row of X^* in relation to the total
variance of X^*.

If the x_j^* are of zero mean, we have :

$$\sum_{i=1}^{n} \sum_{j=1}^{p} (x_j^*(i))^2 = p \sum_{i=1}^{n} (\bar{x}^*(i))^2 + \sum_{i=1}^{n} \sum_{j=1}^{p} (x_j^*(i)-\bar{x}^*(i))^2$$

where $\bar{x}^*(i) = \dfrac{1}{p} \sum_{j=1}^{p} x_j^*(i)$.

As $x_j^* = X_j\, a_j$, if a is the supervector of all scores $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix}$

we may write again this analysis of variance equation
in matrix notations, the lefthand side is equal to $a'Da$

and $\Sigma((\bar{x}^x(1))^2$ as $\frac{1}{2}$ $\underline{a}' B\underline{a}$. Thus the problem is the maximisa-
$\qquad p$

tion of $\eta^2 = \frac{a'Ba}{p\ a'Da}$

The solution is given by the eigenvector $\underline{a}$ related to the
first non extraneous eigenvalue λ_1 of $\frac{1}{p} D^{-1}B$.

This is the same eigen-equation as that of correspondence
analysis of X and that of Carroll's generalized canonical
analysis of the p sets of indicator variables (see Bouroche-
Saporta-Tenenhaus). Furthermore, as Guttman quoted it, the
variable $\bar{x}^x$ is even the first principal component of the x_j^x,
hence the name of "principal component of scale". In other
words we have a quantification x^x of each X such as the
first eigenvalue of the correlation matrix of the x^x is maxi-
mal. (For an historical survey and other criteria leading
to the same result see Tenenhaus 1981).

To the other eigenvalues of $\frac{1}{p} D^{-1}B$ correspond different prin-
cipal components each one corresponding to a different quan-
tification of the X's : so we get actually a multidimensional
quantification of the nominal variables, but the quantifica-
tions of various orders of a variable are non uncorrelated ;
we have only the following mean and weak orthogonality pro-
perties between quantification of order k and l :

$$\text{cov} \ (\sum_{j=1}^{p} x_j^x (k) \ , \ \sum_{j=1}^{p} x_j^x {}^{(l)})=0$$

$$\sum_{j=1}^{p} \text{cov} (x_j^x (k), x_j^x {}^{(l)}) = 0$$

(see Saporta 1979).

There are many computer programs performing that kind of
quantification, the best known being MULTM by Lebart-Morineau-
Tabard, HOMALS by De Leeuw and Van Rijckevorsel and OPSCAL
by Nishisato.

b) Obtaining a single quantification.

The problem here may be considered as a modification of the
"principal component of scale" property of the above method:

we look for a quantification of the variables such as we get
the best principal component analysis of the quantified varia-
bles i.e. that the sum of the variances of the first k prin-
cipal components be maximised.

If k = 1 we obtain again the solution of multiple correspon-
dence analysis i.e. the principal component of scale asso-
ciated to the first eigenvalue of $\frac{1}{p} D^{-1}B$. If k > 1. The
problem may be formalised simply in geometrical terms : we
try to force the set of individuals to be in a k-dimensional
subspace.

As we have no longer an eigenvector problem if k > 1, the
solution is obtained by iterative algorithms. The three best
known computer programs use alternating least squares methods
and are chronologically PRINCIPALS (Young, De Leeuw, Takane),
PRINQUAL (Tenenhaus), PRINCALS (De Leeuw, Van Rijckevorsel).

PRINQUAL processes only nominal variables while the two others
(PRINCALS supersedes PRINCIPALS) accept ordinal variables
and missing data (for PRINCALS).

Compared to the first method of quantification (Guttman-
Hayashi-Benzecri..) this approach has the following advan-
tages :

. the configuration of the individuals upon the k-dimensional
  subspace is optimal in the meaning of explained variance
  while the principal components of scale do not fulfill
  this condition. The first component of scale is the first
  principal component of the quantified variables, but the
  following components are not principal components for the
  same quantification.

. It is possible to compute correlation coefficient between
  quantified variables and principal components for instance,
  for we have here representation of the variables itself
  and not only of the categories.

. The method is well fit to ordinal variables for, as we have
  already seen, a single quantification is necessary. Conver-
  sely there are some drawbacks :

. The most important being in our sense that the quantifica-
  tion changes with the number of desired principal components
  and that the configuration of individuals is not the same
  while in the first method the coordinates of the indivi-
  duals along the first k principal axis remain unaltered
  when we need further axis.

. If k increases, the solution becpmes undetermined for k = p

. If there are two variables the second method does not come
down to canonical analysis as it would have been desirable.

3) Further considerations

a) Extension to an infinite number of variables

The first method of quantification has been generalized
(Saporta-Oeville 1979) to nominal stochastic processes i.e.
in the situation where one knows the evolution of a categori-
cal variable (for instance hierarchical status or place of
living) through the time between two instants O and T.
The method comes down basically to a quantification of the
nominal process X_t into a numerical one X_t such as the
Karhunen-Loeve decomposition (P.C.A. of processes) of the
standardized process X_t/δ_t gives a maximal first eigenvalue
for the covariance operator.

b) Mixture of nominal and numerical variables.

It is very easy to generalize P.C.A. to that situation by
quantifying the nominal variables.

Suppose that we have a set of numerical variables $y_1...y_q$
and p nominal variables X_1, X_2, ... X_p. To obtain the
best P.C.A. of all variables in the sense of the first method
of quantification (λ_1 maximal) we just have to perform a
P.C.A. upon the array : $(Y \mid X_1 \mid X_2 \mid ... \mid X_p)$

$$\text{with the metric M} = \begin{pmatrix} s_1^2 & & & \\ 0 & \ddots s_q^2 & 0 & 0 \\ \hline 0 & X_1' X_1 & 0 \\ \hline 0 & 0 & X_p' X_p \end{pmatrix}^{-1}$$

Again in the first method of quantification the following
components will be associated to other quantifications of
the X_s which may be uninteresting. PRINCIPALS, PRINQUAL and
PRINCIPALS give single quantification and accept numerical
variables.

III. QUANTIFICATION WITH DEPENDENT VARIABLES (Optimal scaling
for prediction)

The problem is now to quantify qualitative variables in order
to predict an external variable or criterium which may itself
be nominal or ordinal.

Let X_1, X_2, ... X_p be the explanatory variables and Y the
criterium, the general problem is thus to obtain a quantifi-
cation x_1^*, x_2^*, ... x_p^* and y^* such that the square multiple
correlation between y^* and the x_j^* be maximised.

Differences between algorithms are due essentially to the
number and the nature of variables.

We will briefly give a few indications about the main situa-
tions.

1) Numerical criterium : regression with qualitative predictors

If all the explanatory variables are nominal we have here a
particular case of the linear model which can be described
as the estimation of effects in an analysis of variance with
p factors of variability without interaction.

Mathematically this comes down to the multiple regression of
the dependent variable y over the set of indicator variables
of all the categories of the X s.

As the sum of indicator variables of the categories of each
X is equal to 1, the problem is not of full rank and there
is an infinite number of equivalent quantifications leading
to the same prediction.

To get a solution we need linear constraints : the most usual
being that the quantified variables should be of zero-mean.
It is worth noting here that solution with zero-mean quanti-
fication may be obtained by regressing y onto all the
($\Sigma m_i - p$) principal components of the multiple correspondence
analysis of the X's. An approximate solution is then given
by the regression of y onto the first k principal components
(or onto the k components best correlated with y).

If one or several explanatory variables are ordinal we get
a problem of monotonic regression to obtain quantifications
respecting the order.

For a single ordinal X, the Kruskal's algorithm is well known.
For several variables computer programs ADDALS and MORALS
(Young, De Leeuw, Takane) provide good solutions using an
alternating least squares algorithm (quantification, regres-
sion, new quantification and so on).

2) Nominal criterium, discrimination with qualitative predictors

This situation is very common in many applications such as
credit-scoring or risk-evaluation where we have to predict
the belonging of an individual to some group (good or bad
behaviour for instance) with qualitative variables.

Like in regression, we cannot use without caution the classi-
cal procedures of discriminant analysis with indicator varia-
bles for they are linearly dependent and one usually need
constraints of zero-mean quantification.

Let us remark that as early as in 1952, Hayashi programmed
a method for nominal discrimination with that constraint.

It is possible to use MORALS and ADDALS for both accept a
nominal criterium.

Another computer program DISQUAL (Saporta 1977) performs a
quasi-optimal quantification of purely nominal variables.
Its principle consists in selecting a subset of principal
components of the correspondence analysis of the X's (in
other words a multidimensional quantification of the X's)
and then in a linear discriminant analysis upon that numeri-
cal components. Each linear discriminant function is a
linear combination of the multidimensional quantifications
of X's and gives thus an unique quantification (discriminant
scores) of the X's.

The reader will find in appendix an example of application
of DISQUAL.

3) Ordinal criterium

In that situation the quantification of the criterium must
satisfy the order constraints.

The case of nominal predictors was first solved by Kruskal
with his algorithm called MONANOVA.

Computer programs ADDALS and MORALS are also well fitted to
this kind of problem and accept predictors of ordinal kind.

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APPENDIX

<u>An example of discrimination with nominal predictors by DISQUAL :
risk estimation of accident for young drivers</u>

5814 customers of less than 25 years or having their driving
licence from less than 2 years have been selected from the files
of an important insurance company.

3849 were good drivers, i.e. without accident during the past
three years.

2325 were bad drivers, i.e. responsible of three or more acci-
dents during the same three years or of more than two the first
year.

The true proportion of bad drivers was 4%, but in order to get
good estimates this category was overrepresented in the sample.

After primary statistical investigations 11 explanatory variables
were selected

FRAC kind of payment (yearly, each 6 months, each 3 months)
AG   age at the date of licence (7 categories)
AN   number of years since the licence (less than 2 years or
      more than 2 years)
SEXE (male, female)
MATR matrimonial status (bachelor or other)
ZONT area of tariff (3 categories)
USAG type of use of the car (9 categories)
CLGR tariff group (5 categories)
RAFR special payment for a policy without deduction (yes, no)
CRED car bought with a credit or not
CAPD life insured or not.

Among these variables some may present interactions concerning
the category of drivers (good or bad) and it is necessary to
create crossed variables before a linear (additive formula)
discrimination.

A systematic study of interactions by means of a log-linear model
showed that there was a significant interaction between AG and
AN. It was thus necessary to create a new variable AGAN,repla-
cing the two preceding ones according to the following diagram :

AG	AN < 2	> 2
18 19	0	1
20 21	2	3
22 25	4	
26 29	5	

There is now 10 predictors with a total number of categories 36.
A multiple correspondence analysis will provide 36-10 = 26 non-trivial components.

Here is the output of DISQUAL

Results of diagonalisation

Non trivial eigenvalues :

1.897 1.529 1.312 1.126 1.119 1.103 1.092 1.046 1.026 1.022
1.013 1.002 0.484 0.981 0.960 0.915 0.933 0.932 0.912 0.902
0.860 0.756 0.703 0.678 0.615 0.541

Number of factors to be selected (amount of accounted variance superior to 1%) = 26.

Selection of factors

Rank	Number	Relative discriminant power	% of variance accounted
1	1	0.624	0.073
2	8	0.101	0.029
3	2	0.077	0.059
4	10	0.035	0.043
5	11	0.032	0.043
6	6	0.019	0.026
7	12	0.018	0.042
8	4	0.014	0.050
9	16	0.013	0.040
10	5	0.011	0.024
11	15	0.010	0.035
12	26	0.010	0.039
13	22	0.010	0.039
14	24	0.005	0.038
15	7	0.005	0.027

Number of selected factors	15
Cumulated % of discriminant power	98.22
Cumulated % of variance	60.75

Scoring rule

FRAC	0	3.68 (yearly)
	1	0.50 (each 6 months)
	2	-4.75 (each 3 months)
AGAN	0	-1.73
	1	4.40
	2	-1.25
	3	3.51
	4	0.40
	5	-4.83
SEXE	0	-1.56 (male)
	1	5.50 (female)
MATR	0	-1.01 (bachelor)
	1	2.53 (other)
ZONT	0	3.24
	1	-1.06
	2	-3.85

```
USAG    0    -7.05
        1     0.57
        2     1.72
        3     0.91
        4    -1.82
        5     0.60
        6    -8.64
        7     0.99
        8    -7.68

CLGR    0     3.88
        1    -1.48
        2    -4.58
        3    -7.20
        4    -6.10

RAFR    0    -6.80 (yes)
        1    10.20 (no)

CREO    0     0.09 (yes)
        1    -2.75 (no)

CAPO    0     0.17 (yes)
        1    -4.35 (no)
```

Threshold value -4.27.

We use the scoring rules as follows :

Let us take a male driver who wants to pay each 3 months if he
is bachelor and of category 2 of variable AGAN (20 or 21 years
old and licence from less than 2 years) living in ZONT number 0
for the USAG category 5, the CLGR category 3, taking a no-
deduction policy (RAFR = 0) with a car bought with a credit and
without a life insurance. His score will be

```
-4.75   -1.25   -1.56   -1.01   +3.24   +0.60
-7.20   -6.80   +0.09   -4.35 =-12.26
```

As this score is less than the threshold value -4.27, he will be
classified into the "bad drivers" population.

The efficiency of this scoring rule is described by the following
table obtained by classifying the individuals of the sample :

true group	classified in	
	good	bad
good	2830	659
bad	518	1807

There is an amount of about 80% of correct classifications
and 78% of bad drivers are detected.