Contributions to multivariate risk models

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Working Papers, Preprints & Technical Reports


Symbol Conventions

In the following we explain symbols and conventions that might be used in this manuscript.

The item symbol − lists elements, aims, goals, etc.

The item symbol ⤹→ indicates obtained theoretical results in a compact stepwise fashion.

The symbol ‡ indicates recent papers (written by other researchers) where our results and models are further investigated.

The symbol * denotes papers written during my Ph.D. thesis.

Sketch of proofs of most important results are given using this font and this font size.

Crucial formulas are presented in this box type.
Preface

This manuscript presents in a synthetic fashion my scientific production developed after my Ph D. thesis, defended in December 2011 at the Université Claude Bernard Lyon 1. The study presented here can be seen as the continuation of the research started during my thesis and enriched by the contact with new themes. This work concerns dependence modeling in risk management within a multivariate framework and papers summarized in this document can be interpreted as a contribution to multivariate risk theory.

In the last decade, much research has been devoted to the construction of risk measures that account both for marginal effects and dependence between risks. The work presented in this document belongs to this large body of literature and it can be seen as a further effort in order to extend risk analysis in a multidimensional setting. Several important challenges exist in this direction: the ability to incorporate important covariates in the model; the dependence between variables is neither linear nor constant; tail dependence, and so on. The multivariate risks literature mainly treats one or more of the following three elements: (1) showing the importance and explaining the usefulness of the multivariate framework, (2) fitting the appropriate multivariate distribution in order to model risks and (3) defining and studying multivariate risk measures. In this manuscript, we present contributions in these directions.

While there is only one way for variables to be mutually independent, mathematically translated by the fact that their joint distribution function is factorized into the product of its marginals, there are an infinite number of ways to introduce dependence in a multivariate model. In this paper, we use for instance the notion of copula, Gaussian processes correlated by a common noise and multivariate fields.

This dissertation is organized around three distinct but interconnected and complementary themes:

- definition and study of new multivariate risk measures;
- proposals and study of distortion/transformation copula models;
- study of level-crossing models in a multivariate framework.

These three topics are investigated independently in three separate chapters. Of course, organization in independent chapters is sometimes artificial and certain transversal notions such as the theory of regular variation or the notion of copula link the three different parts. The order of the chapters is not chronological: some older articles can be found together with more recent ones. Indeed, the goal of this document is to unify, in a spirit of synthesis, my scientific research during the past five years.

The first part of this manuscript extends some results obtained during my thesis (see [E23]). Indeed, Chapter 1 completes, with collaborations with different researchers, the study of multivariate risk measures started during my three years of thesis. After an introduction, highlighting the issues and the relevant scientific literature, we study the problem of constructing risk measures in dimensions greater than one.

We extend classic univariate risk measures to a multidimensional framework, where the risks involved cannot necessarily be aggregated. This is the case, for example, if the latter are heterogeneous or cannot be compared. Moreover, the risk measures introduced verify direct extensions of properties of positive homogeneity, translation invariance and comonotone additivity. We also studied how our measures behave with respect to some risk modifications (such as an increase in marginal risk or an increase in the dependence between risks).
Then, in the same chapter, we present a statistical study for the proposed measures. Different estimators are provided and their consistency is studied. The techniques used to prove the convergence of the considered estimators include techniques derived from Extreme Values Theory and the theory of empirical processes.

Some of these works have been developed during the Ph D. thesis of Raúl Andres Torres and of Fátima Palacios Rodríguez, under my co-supervision. These two theses have been defended respectively on December 19, 2016 and March 20, 2017. Due to its relevance, this co-supervised work has been included in this manuscript (see also papers [E2; E5; E9; E19] in my list of publications).

Chapter 2 proposes distortion/transformation copula models which allow to change the copula in multivariate tails or in the center of multivariate distributions. These modifications can distort an initial copula in order to go beyond for instance the class of Archimedean copulas. By using other transformation models the resulting transformed copula will still be Archimedean. All these possibilities give suitable flexibility in order to model multivariate dependence, which is essential in the construction of risk models. Particular attention is paid in Chapter 2 to the upper and lower tail dependence coefficients associated with transformed copulas.

Chapter 3 aims to investigate theoretical and statistical properties related to some level functionals in the context of stationary random processes and fields. Notice that the proposed level-crossing models can be considered as an elegant way to model risks. Indeed, a variety of phenomena in physical and biological sciences can be mathematically understood by considering the statistical properties of level-crossings of random processes or fields. Notably, a growing number of these phenomena requires a consideration of correlated level-crossings emerging from multiple correlated processes. While many theoretical results have been obtained in the last decades for individual Gaussian level-crossing processes, few results are available for multivariate, jointly correlated threshold crossings. In this chapter we will focus essentially on this latter point. In particular, we present an application in neuroscience by studying up-crossing of multivariate, jointly correlated neuronal voltages. Furthermore, a Gaussianity test is proposed by using again some level functionals of a single realisation of a random field and observations exceeding certain thresholds. This can be a very realistic situation in many real-life applications when the whole trajectory of the considered phenomenon cannot be observed (see, for instance, the neurological application above).

In conclusion, we would like to emphasize that models considered in this manuscript could be adapted in a wide range of situations. Several real-life applications are indeed possible. These models can be used for instance to manage multivariate dependent risks in finance, insurance, neuroscience, hydrology and climatology, as we will remark in different sections of this manuscript. For example, the question of the knowledge of the distribution, in particular the question of the Gaussianity of a multidimensional phenomenon studied in Chapter 3, is a historical and fundamental problem in the statistical literature. This type of information can be crucial in many application problems: oceanography and waves behavior, neurology and spike behavior, insurance and finance, astrophysics, etc.

We conclude all three chapters with some research perspectives as well as some bibliographical references where our models are further studied or used in applications. For the sake of completeness, papers written during my Ph D. thesis and some other works are briefly summarized at the end of this manuscript.

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Avant-propos

Ce mémoire d'habilitation présente la synthèse des travaux que j'ai effectués après ma thèse de doctorat, soutenue en décembre 2011 à l'Université Claude Bernard Lyon 1. Ils s'inscrivent dans la continuité d'un parcours initié avec ma thèse de doctorat et enrichi par l'ouverture sur des nouvelles thématiques. La ligne directrice de ces travaux concerne la modélisation de la dépendance dans la gestion des risques en dimension plus grande que un.

Au cours de la dernière décennie, de nombreuses recherches ont été consacrées à la construction de mesures de risque qui prennent en compte à la fois des effets marginaux et de la dépendance entre les risques. Le travail présenté dans ce document fait partie de cette vaste littérature et il peut être considéré comme une contribution supplémentaire afin d'étendre l'analyse de risque dans le cadre multidimensionnel. Plusieurs défis importants existent dans ce sens: la capacité d'incorporer des covariables dans le modèle; la dépendance entre les variables n'est ni linéaire ni constante; les variables montrent une dépendance dans les extrêmes, etc. La littérature sur les risques multivariés a principalement traité d'au moins un des éléments suivants: (1) montrer l'importance et expliquer l'utilité d'un cadre multivarié, (2) choisir et estimer la distribution multivariée appropriée pour modéliser les risques et (3) définir et étudier des mesures de risque multivariées. Dans ce manuscrit, nous essaierons de fournir une contribution dans ces trois directions.

S'il n'y a qu'une seule façon pour des variables d'être mutuellement indépendantes, traduite mathématiquement par le fait que leur fonction de répartition jointe se factorise en le produit de ses marginales, il y a bien entendu une infinité de manières d'introduire de la dépendance dans un modèle multivarié. Dans ce mémoire, nous utilisons la notion de copule, de processus Gaussiens corrélés par un bruit commun et de champs multivariés.

Les travaux résumés ci-dessous peuvent être interprétés comme une contribution à la construction et à l'analyse de modèles de risques multivariés. Le mémoire s'organise autour de trois thèmes distincts mais interconnectés et complémentaires :

- la définition et l'étude de nouvelles mesures de risques multivariées;
- la proposition et l'étude de modèles de distorsion/transformation de copules;
- l'analyse de modèles de dépassement de niveau dans un cadre multivarié.

Ces trois thèmes sont traités de manière indépendante dans trois chapitres distincts. L'organisation en chapitres indépendants est parfois artificielle et certaines notions transversales comme la théorie des variations régulières ou la notion de copule relient les différentes parties. L'ordre des chapitres proposé n'est pas chronologique : certains articles plus anciens sont regroupés avec des articles plus récents. En effet, l'objectif de ce document est de retracer, dans un esprit de synthèse, le fil rouge de mon parcours scientifique tout au long de ces derniers cinq ans.

La première partie de ce mémoire d'habilitation prolonge des résultats obtenus dans ma thèse de doctorat (voir [E23]). Elle vient compléter, suite à des collaborations avec différents chercheurs, l'étude des mesures de risques multivariées que j'ai pu démarrer durant mes trois ans de thèse. Après une introduction rappelant les enjeux et la littérature scientifique, nous étudions le problème de la construction de mesures de risque en dimension plus grande que un.

Nous avons considéré des extensions de mesures univariées classiques à un cadre multidimensionnel où les risques ne peuvent pas nécessairement être agrégés. C'est le cas par exemple lorsque
ces derniers sont de nature hétérogène et ne peuvent pas être comparés. Par ailleurs, les mesures que nous avons introduites vérifient des extensions directes des propriétés d’homogénéité positive, d’invariance par translation et d’additivité comonotope. Nous avons également étudié le comportement de nos mesures par rapport à des perturbations de risque, comme une augmentation d’un risque marginal ou une augmentation du degré de dépendance entre les risques.

Ensuite, dans le même chapitre, nous présentons une étude statistique des mesures proposées précédemment. Différents estimateurs sont proposés et leur consistance est démontrée. Les techniques utilisées pour prouver la convergence des estimateurs considérés sont notamment les techniques issues de la théorie des valeurs extrêmes et des processus empiriques. Certains de ces travaux ont été développés dans le cadre de la thèse de Raúl Andrés Torres et de Fátima Palacios Rodríguez, sous ma co-direction. Ces deux thèses ont été soutenues respectivement le 19 décembre 2016 et le 20 mars 2017. Il me semble intéressant, dans le cadre de l’habilitation à diriger des recherches, de mettre en valeur ce travail de co-encadrement doctoral (voir aussi les articles [E2; E5; E9; E19] dans ma liste de publications).

Le deuxième chapitre propose des modifications de structure de dépendance qui peuvent changer la copule dans la queue, ou bien dans le centre de la distribution multivariée. Ces modifications peuvent distordre la copule en lui permettant de quitter la classe des copules Archimédiennes, ou bien d’y rester. Dans le premier cas nous parlerons de distorsions et dans le second de transformations. Toutes ces possibilités donnent une flexibilité à la modélisation de la dépendance multivariée, essentielle dans la construction des modèles de risques. Une attention particulière est portée dans ce chapitre aux coefficients de dépendance extrême associés à la copule transformée.

Le troisième chapitre centré autour de modèles de dépassement, illustre des propriétés statistiques liées à des fonctionnelles de niveau dans le cadre de processus et champs stationnaires. Les modèles de dépassement (level-crossing models) peuvent être considérés comme une élégante modélisation pour l’analyse des risques. En effet, certains phénomènes dans les sciences physiques et biologiques peuvent être mathématiquement compris en considérant les propriétés statistiques des dépassements de niveau de processus ou de champs aléatoires. Bien que de nombreux résultats théoriques ont été obtenus au cours des dernières décennies pour les dépassements de niveau des processus univariés, à ce jour, peu de résultats sont disponibles pour les processus multivariés et conjointement corrélés. Dans ce chapitre, nous nous concentrons essentiellement sur ce dernier point. En particulier, dans ce chapitre, nous présentons une application en neuroscience qui traite de l’étude des dépassements des signaux neuronaux multivariés et conjointement corrélés. Toujours à partir de fonctionnelles de niveau, un test de normalité pour les champs stationnaires est proposé en utilisant les observations dépassant un certain seuil.

En conclusion, nous souhaitons souligner que les modèles examinés dans ce document pourraient être adaptés à un large éventail de situations. Plusieurs applications de la vie réelle sont en effet possibles. Ces modèles peuvent être utilisés par exemple afin de modéliser les risques multivariés en finance, en assurance, en neuroscience, en hydrologie et en climatologie, comme nous le verrons dans les différentes sections de ce manuscrit. À titre d’exemple, la connaissance de la distribution d’un phénomène multidimensionnel, en particulier la question de sa normalité étudiée dans le Chapitre 3, est un problème historique et fondamental dans la littérature statistique. Ces informations peuvent s’avérer cruciales dans de nombreux problèmes d’application:

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océanographie et comportement des vagues, neurologie et comportement des “spikes”, assurance et finance, astrophysique, etc.

De plus dans le Chapitre 3, nous utilisons les informations issues de certaines fonctionnelles de niveau d’une réalisation unique d’un champ aléatoire, afin d’en déduire des informations sur l’ensemble de la distribution du champ. Cette situation peut s’avérer réaliste dans de nombreuses applications de la vie réelle lorsque l’ensemble de la trajectoire d’un phénomène ne peut s’observer (cf. l’exemple des spikes neuronaux, ci-dessus).

Tous les chapitres se terminent par quelques perspectives de recherche ainsi que par quelques références bibliographiques dans lesquelles nos modèles ont été ultérieurement étudiés ou utilisés, suite à la publication de nos travaux. Par souci d’exhaustivité, les travaux de thèse ainsi que quelques papiers complémentaires, sont brièvement résumés à la fin de ce mémoire.
This chapter is based on papers [E19], [E18]*, [E16]*, [E15], [E13], [E11], [E10], [E9], [E8], [E5] and [E2] in my list of publications. First, we introduce a (not exhaustive) literature on multivariate risk measures. Then, we devote a section to our upper and lower-orthant multivariate risk measures and we study different theoretical properties (see Section 1.2). We propose several inferential methodologies in order to estimate these measures (see Sections 1.3 and 1.4). We conclude with some research perspectives on this topic and with an overview of recent papers (written by other researchers) where our measures are further investigated.

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### 1.1 Introduction

During the last decades, researchers joined efforts to properly compare, quantify and manage risk. Different risk measures have been proposed in the one-dimensional framework, and an axiomatic approach was developed to characterize the “fairest” risk measure (see Artzner et al. [10]). Traditionally, risk measures are thought of as mappings from a set of real-valued random variables to the real numbers. In this univariate setting, one of the most popular measure is the Value-at-Risk (VaR). This measure in hydrology and environmental sciences is also called *return level* related to a certain *return period* (RP). This quantity represents the magnitude of an event that occurs at a given time and at a given site. More precisely, the return level is
the quantile \( x_p \) which expresses the magnitude of the event that is exceeded with a probability equal to \( p \), with \( p = 1/T \) (\( T \) is called the return period).

The RP is a frequently used concept which allows us to identify dangerous events and find the means for rational decision-making and risk assessment (see, e.g., Singh et al. [179]). The notion of design quantile related with RP is also of great importance. The design quantile represents the “value of the variable(s) characterizing the event associated with a given RP” (see, e.g., Salvadori et al. [167]).

Remark that environmental risks frequently involve several variables that are often correlated. For instance, a flood can be described by the volume, the peak and the duration (see, e.g., Chebana and Ouarda [49], Pappad`a et al. [155]). For this reason, it is crucial to evaluate risks in a multivariate setting. Indeed, a univariate frequency analysis can not provide a complete assessment of the probability of occurrence (see, e.g., Yue and Rasmussen [198], Chebana and Ouarda [49]) and might lead to an over- or underestimation of the risk associated with a given event (see, e.g., Salvadori and De Michele [164], Salvadori et al. [166]). Then, the sole tool provided by the RP may be insufficient to identify a design realization (see e.g., Serfling [175], Vandenberghe et al. [193]).

Another interesting example is given by financial/insurance risks. As illustrated by the recent financial turmoil, financial risks are strongly interconnected and the evaluation of an individual risk may strongly be affected by the degree of dependence amongst all risks. Consequently, risk quantification in multivariate financial framework has recently been the subject of great interest (see, for instance, Chavez-Demoulin et al. [47], Embrechts and Hofert [80], Embrechts and Hofert [78]).

Remark that, in the univariate case, the notion of quantile is usually defined without ambiguity. Conversely, the notion of multivariate return level or multivariate quantile is not univalent (see e.g., Vandenberghe et al. [193]) and several definitions can be found in the recent literature. We refer to Serfling [175] for a large review on multivariate quantiles. Unsurprisingly, the main difficulty is the fact that multivariate vector preorders are, in general, partial preorders. Furthermore, since different combinations of probabilities may produce the same return period, a multivariate return level is inherently ambiguous.

For instance, what can be considered, in a multidimensional portfolios setting, as the analogous of a “worst case” scenario and a related “tail distribution”? This is why several definitions of quantile-based risk measures are possible in a higher dimensions, see for instance, the geometric quantiles (see, e.g., Chaouch et al. [40]) or definitions by using depth functions (see, e.g., Chauvigny et al. [45], Chebana and Ouarda [48]).

In the same spirit, Embrechts and Puccetti [84], Nappo and Spizzichino [146]), Prékopa [156] use the notion of multivariate quantile curve which is defined as the boundary of the upper-level set of a distribution function or the lower-level set of a survival distribution function. Then, events that have equal probability of exceedance define iso-hyper-surfaces. Among their many advantages, it appears that multivariate quantile curves are easily interpretable and probability-based (see, e.g., Chebana and Ouarda [49]). In the literature multivariate quantile curves are also called critical layers (see, e.g., Salvadori et al. [165]) and both these designations will be used in this manuscript.

Another well known univariate risk measure is the Conditional-Tail-Expectation (CTE) that is the expectation of the loss \( X \) in the tail. Contrary to VaR, it is a coherent risk measure according to the axiomatic properties introduced by Artzner et al. [10] (see Fermanian and Scaillet [92], Embrechts and Wang [87]; the reader is also referred to Embrechts et al. [86] for a discussion about the choice of CTE with respect to VaR in banking regulation). As discussed before in the VaR case, in the recent literature, several generalizations of the classic univariate
CTE have been proposed, mainly using as conditioning events the total risk or some univariate extreme risk in the portfolio. This kind of measures are suitable to model problems of capital allocation in a portfolio of dependent risks. More precisely, let \( d \geq 2 \) and \( \mathbf{X} = (X_1, \ldots, X_d) \) be a risk vector and \( S = X_1 + \cdots + X_d \) be the total risk of this portfolio, \( X_{(1)} = \min\{X_1, \ldots, X_d\} \) and \( X_{(d)} = \max\{X_1, \ldots, X_d\} \) be some extreme risks. One can define CTE\(_{\text{sum}}\)(\( \mathbf{X} \)) = \( \mathbb{E}[\mathbf{X} \mid S > \text{VaR}^-_S(\alpha)] \), CTE\(_{\text{min}}\)(\( \mathbf{X} \)) = \( \mathbb{E}[\mathbf{X} \mid X_{(1)} > \text{VaR}^-_{X_{(1)}}(\alpha)] \), CTE\(_{\text{max}}\)(\( \mathbf{X} \)) = \( \mathbb{E}[\mathbf{X} \mid X_{(d)} > \text{VaR}^-_{X_{(d)}}(\alpha)] \) (see, e.g., Cai and Li [35], Bargès et al. [19], Landsman and Valdez [129], Tasche [184]).

Another problem of interest is the construction of systemic risk measures. As we will see in Section 1.2, Marginal Expected Shortfall (MES) and Conditional Value-at-Risk (CoVaR) have been defined to detect which firms in the economy are the more vulnerable in case of a global financial distress (see, e.g., Acharya et al. [2], Brownlees and Engle [32], Cai et al. [36], Adrian and Brunnermeier [5], Mainik and Schaanning [135]).

In this chapter, we introduce and study some alternative extensions of the classic univariate VaR (or return level), CTE and CoVaR presented above, in a multivariate setting based on a “distributional approach”. The proposed measures are based on the Embrechts and Puccetti [84]’s definitions of multivariate quantiles but they quantify multivariate risks in a more parsimonious and synthetic way. Indeed, multivariate quantile curves are iso-hyper-surfaces and thus quantify a vector of risks with an infinite number of points. Conversely, we define our lower-orthant (resp. upper-orthant) risk measures at risk level \( \alpha \) by conditioning to the fact of the underlying vector of risks \( \mathbf{X} \) stands in the \( \alpha \)-level set of its distribution function (resp. in the \( (1 - \alpha) \)-level set of its survival distribution function). Then, our extensions are real-valued vectors with the same dimension as the considered portfolio of risks. This feature can be relevant from an operational point of view.

Furthermore, contrary to many existing generalizations, such as those presented above, our measures do not use an arbitrary real-valued aggregate transformation (sum, min, max, ... see for instance Fougeres and Mercadier [94], Embrechts and Puccetti [85], Alink et al. [7] in the Archimedean copula case and Alink et al. [8] in the non-Archimedean case). For this reason, these measures may be useful for some applications where risks are intrinsically heterogeneous in nature, where risks cannot be expressed under the same numéraire or when one has to deal with non-monetary risks. Indeed, using an aggregate procedure between risks can be inappropriate to measure risks with heterogeneous characteristics. For instance, in a hydro-meteorological context, to be meaningful, multivariate risk measures must take into account the possible dependence between several heterogenous variables of interest. These variables can be of different nature (e.g. precipitation, temperature, discharge, ...), prohibiting the aggregation of the various components as the sum, the maximum, ... Recall the flood example presented above, described by a set of three heterogeneous correlated random variables. Moreover, as we will see in Section 1.2, as opposed to measures presented above, our measures satisfy the Artzner et al. [10]’s invariance properties.

As regard the risk measures estimation in the context of extreme losses, there are specific estimation approaches in the literature including block maxima, peaks-over-threshold and point processes. A thorough literature search will guide efforts in the most appropriate direction (see, e.g., de Haan and Ferreira [58], Embrechts et al. [82], Reiss and Thomas [158]). The problem of consistent estimation of quantile based risk-measures (VaR and CTE, see above) has received attention in literature essentially in the univariate case (e.g. see Brazauskas et al. [30]; Necir et al. [147]; Ahn and Shyamalkumar [6]).

Due to a number of theoretical and practical reasons, the estimation of multivariate risk-measures has been treated less extensively than the univariate case. In the recent literature, some efforts have been done to provide a consistent estimation of CTE\(_{\text{sum}}\)(\( \mathbf{X} \)), CTE\(_{\text{min}}\)(\( \mathbf{X} \)) and
Definition 1.2.1 (Critical layers and level sets)
Let \( F \) be a cumulative distribution function associated with an univariate risk \( X \) and \( F \) its associated survival function. From the usual definition in the univariate setting, the Value-at-Risk is the minimum amount of the loss which accumulates a probability \( \alpha \) to the left tail and \( 1 - \alpha \) to the right tail, i.e., \( \text{VaR}_\alpha(X) := \inf \{ x \in \mathbb{R} : F(x) \geq \alpha \} \). Notice that the classic univariate VaR above can be viewed as the boundary of the set \( \{ x \in \mathbb{R} : F(x) \geq \alpha \} \) or, similarly, the boundary of the set \( \{ x \in \mathbb{R} : F(x) \leq 1 - \alpha \} \).

In [E16]* we extend this idea in higher dimensions, keeping in mind that the two previous sets are different in general as soon as \( d \geq 2 \). Then, the proposed multivariate measures are constructed from the boundary of level sets of multivariate distribution functions (resp. of multivariate survival distribution functions).

Let \( I = \{1, \ldots, d\} \). We now consider an absolutely-continuous random vector \( X = (X_1, \ldots, X_d) \) (with respect to Lebesgue measure \( \lambda \) on \( \mathbb{R}^d \)) with partially increasing multivariate distribution function\(^1\) \( F \) and such that \( \mathbb{E}(X_i) < \infty \), for \( i \in I \). These conditions will be called in this chapter regularity conditions.

Starting from considerations above, we recall below the definition of critical layers, also called multivariate quantile curves (see Definition 1.2.1) and then we introduce our multivariate generalizations of the VaR measure (see Definition 1.2.2).

Definition 1.2.1 (Critical layers and level sets) Let \( X = (X_1, \ldots, X_d) \) be a random risk vector with joint distribution function \( F \) satisfying the regularity conditions. Let \( \overline{F} \) be the associated survival distribution function. For \( \alpha \in (0,1) \) and \( d \geq 2 \), we define
\[
\underline{L}(\alpha) = \{ x \in \mathbb{R}^d : F(x) \geq \alpha \},
\]
and
\[
\overline{L}(\alpha) = \{ x \in \mathbb{R}^d : F(x) \leq 1 - \alpha \}.
\]
We denote \( \partial \underline{L}(\alpha) \) and \( \partial \overline{L}(\alpha) \) the associated critical layers, i.e., the boundary of the set \( \underline{L}(\alpha) \) in (1.1) and of \( \overline{L}(\alpha) \) in (1.2) respectively.

\(^1\) A function \( F(x_1, \ldots, x_d) \) is partially strictly increasing on \( \mathbb{R}^d \setminus \{(0, \ldots, 0)\} \) if for any \( i \in I \), the function of one variable \( g_i(\cdot) = F(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_d) \) is strictly increasing. About properties of partially increasing multivariate distribution functions we refer the interested reader to Rossi (1973), Tibiletti (1991).
1.2. Upper/lower-orthant multivariate risk measures \([E9; E10; E13; E16]*; E18]*\)

**Definition 1.2.2 (Multivariate upper and lower-orthant VaR, \([E16]*\))** Consider a random vector \(X = (X_1, \ldots, X_d)\) with distribution function \(F\) satisfying the regularity conditions. Let \(\overline{F}\) be the associated survival distribution function. Let \(\partial L(\alpha)\) and \(\partial \overline{L}(\alpha)\) be the critical layers introduced in Definition 1.2.1. For \(\alpha \in (0, 1)\), we define:

the multidimensional upper-orthant VaR at probability level \(\alpha\) by

\[
\text{VaR}_\alpha(X) = \left( \begin{array}{c} \mathbb{E}[X_1 | X \in \partial L(\alpha)] \\ \vdots \\ \mathbb{E}[X_d | X \in \partial L(\alpha)] \end{array} \right) = \left( \begin{array}{c} \mathbb{E}[X_1 | \overline{F}(X) = 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \overline{F}(X) = 1 - \alpha] \end{array} \right),
\]

the multidimensional lower-orthant VaR at probability level \(\alpha\) by

\[
\text{VaR}_\alpha(X) = \left( \begin{array}{c} \mathbb{E}[X_1 | X \in \partial L(\alpha)] \\ \vdots \\ \mathbb{E}[X_d | X \in \partial L(\alpha)] \end{array} \right) = \left( \begin{array}{c} \mathbb{E}[X_1 | F(X) = \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(X) = \alpha] \end{array} \right),
\]

We remark that our multivariate VaRs are a more parsimonious and synthetic version of the Embrechts and Puccetti (2006)'s measures (see Definition 17 in Embrechts and Puccetti [84]; see also Nappo and Spizzichino [146] and Tibiletti [189]). In particular in our propositions, instead of considering the whole hyperspace \(\partial L(\alpha)\) (or \(\partial \overline{L}(\alpha)\)), we only focus on the particular point in \(\mathbb{R}^d\) that matches the conditional expectation of \(X\) given that \(X\) stands in these sets. The latter feature could be relevant on practical grounds.

Several recent studies (see, e.g., Acerbi and Tasche [1], Artzner et al. [10]) recommend to use the Conditional-Tail-Expectation risk measure\(^2\) as a coherent alternative to the VaR. Furthermore, the CTE accounts for the severity of the failure, not only the chance of failure. In the univariate setting, since the sets \(\{X \geq \text{VaR}_\alpha(X)\}\), \(\{F(X) \geq \alpha\}\) and \(\{\overline{F}(X) \leq 1 - \alpha\}\) correspond to the same event, the univariate CTE can be defined as

\[
\text{CTE}_\alpha(X) := \mathbb{E}[X | X \geq \text{VaR}_\alpha(X)] = \mathbb{E}[X | F(X) \geq \alpha] = \mathbb{E}[X | \overline{F}(X) \leq 1 - \alpha].
\]

Then, keeping in mind this trivial result, we introduce below our upper- and lower-orthant multivariate Conditional-Tail-Expectation.

**Definition 1.2.3 (Multivariate upper and lower-orthant CTE, \([E13]*\))** Consider a random vector \(X = (X_1, \ldots, X_d)\) with distribution function \(F\) satisfying the regularity conditions. Let \(\overline{F}\) be the associated survival distribution function. Let \(L(\alpha)\) and \(\overline{L}(\alpha)\) be the level sets introduced in Definition 1.2.1. For \(\alpha \in (0, 1)\), we define:

the multidimensional upper-orthant CTE at probability level \(\alpha\) by

\[
\text{CTE}_\alpha(X) = \left( \begin{array}{c} \mathbb{E}[X_1 | X \in L(\alpha)] \\ \vdots \\ \mathbb{E}[X_d | X \in L(\alpha)] \end{array} \right) = \left( \begin{array}{c} \mathbb{E}[X_1 | \overline{F}(X) \leq 1 - \alpha] \\ \vdots \\ \mathbb{E}[X_d | \overline{F}(X) \leq 1 - \alpha] \end{array} \right),
\]

the multidimensional lower-orthant CTE at probability level \(\alpha\) by

\[
\text{CTE}_\alpha(X) = \left( \begin{array}{c} \mathbb{E}[X_1 | X \in L(\alpha)] \\ \vdots \\ \mathbb{E}[X_d | X \in L(\alpha)] \end{array} \right) = \left( \begin{array}{c} \mathbb{E}[X_1 | F(X) \geq \alpha] \\ \vdots \\ \mathbb{E}[X_d | F(X) \geq \alpha] \end{array} \right).
\]

\(^2\)As far as continuous distribution functions are considered, the CTE measure is coherent in the sense of Artzner's axioms and it coincides with the Expected Shortfall or Tail VaR.
Notice that the lower-orthant CTE in Definition 1.2.3 was firstly introduced in [E18]* in the bivariate case (see also Section “Papers written during my Ph.D. thesis” at the end of this manuscript).

We can write, for \( i \in I \), the following relations,

\[
\text{CTE}_i^\alpha(X) = \frac{1}{1 - K(\alpha)} \int_0^1 \text{VaR}^i_\gamma(X) K'(\gamma) d\gamma, \quad \text{CTE}_c^\alpha(X) = \frac{1}{K(1 - \alpha)} \int_0^1 \text{VaR}_c^i(X) \tilde{K}'(1 - \gamma) d\gamma,
\]

where \( K \) is the Kendall distribution of \( X \), i.e., \( K(x) = \mathbb{P}(F(X) \leq x) \), and \( \tilde{K}(x) = \mathbb{P}(\tilde{X}(X) \leq x) \), for all \( x \in (0, 1) \) and \( K' \) and \( \tilde{K}' \) the associated densities (see, e.g., Nelsen et al. [149]). Like in the univariate case, our multidimensional Conditional-Tail-Expectations contain a safety loading in order to cover the fluctuations of loss experience (see Corollary 2.5 in [E13] and Section 2.4.3.3 in Denuit et al. [65]).

Another interesting topic in risk management is the construction of systemic risk measures. Let \( S = X_1 + \ldots + X_d \) the aggregate financial system and \( X_j \) the risk exposure of company \( j \), for \( j \in I \). Then, the Conditional Value-at-Risk (shortly, CoVaR) associated with company \( j \) can be defined by \( \text{CoVaR}_j^\alpha(X) = \text{VaR}_\alpha(S|X_j \geq \text{VaR}_\alpha(X_j)) \), (see Girardi and Ergün [103], Mainik and Schaanning [135]). A similar definition of the CoVaR measure was originally introduced by Adrian and Brunnermeier [5]. The notion of Marginal Expected Shortfall (MES) for the systemic risk, introduced in Girardi and Ergün [103], is closely related to CoVaR. It is defined as \( \text{MES}(X_j|S) = \mathbb{E}[X_j|S > \text{VaR}_\alpha(S)] \), for \( j \in I \) (see also Mainik and Schaanning [135], Cai et al. [36]). As remarked by Mainik and Schaanning [135], with MES, the conditioning random variable is the system, and the target random variable is a part of the system. In the original work on CoVaR by Adrian and Brunnermeier [5], it is the opposite case.

In [E9], following the same multivariate quantile settings as in Definitions 1.2.2 and 1.2.3 we introduce new multivariate generalizations of CoVaR.

**Definition 1.2.4 (Multivariate upper and lower-orthant CoVaR, [E9])** Consider a random vector \( X = (X_1, \ldots, X_d) \) with distribution function \( F \) satisfying the regularity conditions. Let \( \tilde{F} \) be the associated survival distribution function. Let \( \omega = (\omega_1, \ldots, \omega_d) \) a marginal risk vector with \( \omega_i \in (0, 1) \), for \( i \in I \). Let \( \partial L(\alpha) \) and \( \partial L'(\alpha) \) be the critical layers introduced in Definition 1.2.1. For \( \alpha \in (0, 1) \), we define:

the multivariate upper-orthant CoVaR at probability level \( \alpha \) and with marginal risk vector \( \omega \) by

\[
\text{CoVaR}^\omega_{\alpha, \omega}(X) = \begin{pmatrix} \text{VaR}_{\omega_1}(X_1|X \in \partial L(\alpha)) \\ \vdots \\ \text{VaR}_{\omega_d}(X_d|X \in \partial L(\alpha)) \end{pmatrix} = \begin{pmatrix} \text{VaR}_{\omega_1}(X_1|F(X) = 1 - \alpha) \\ \vdots \\ \text{VaR}_{\omega_d}(X_d|F(X) = 1 - \alpha) \end{pmatrix},
\]

(1.3)

the multivariate lower-orthant CoVaR at probability level \( \alpha \) and with marginal risk vector \( \omega \) by

\[
\text{CoVaR}^\omega_{\alpha, \omega}(X) = \begin{pmatrix} \text{VaR}_{\omega_1}(X_1|X \in \partial L(\alpha)) \\ \vdots \\ \text{VaR}_{\omega_d}(X_d|X \in \partial L(\alpha)) \end{pmatrix} = \begin{pmatrix} \text{VaR}_{\omega_1}(X_1|F(X) = \alpha) \\ \vdots \\ \text{VaR}_{\omega_d}(X_d|F(X) = \alpha) \end{pmatrix},
\]

(1.4)

**Remark 1** We prove that our CoVaRs are the minimizers of suitable expected losses (see property (P7) in [E9]). They therefore verify the *elicitability* property (see Gneiting [104] and Bellini and Bignozzi [21]). The relevance of this property in connection with a natural methodology to perform backtesting is discussed, for instance, in Embrechts and Hofert [80]. Ziegel [199] studied the connections between elicitability and coherence properties of risk measures.
Some other multivariate risk measures

In the financial econometrics literature, we are often interested in analyzing the behavior of a univariate return measure $Y$ (average return, skewness, . . .) with respect to a set of $d$ risk factors $X$ (volatility or variance, kurtosis, . . .). Furthermore, in climatology, one may be interested in how climate change over the years might affect high temperatures. Multivariate examples include the study of rainfall with covariate $Y$ given by the geographical spatial location. In this sense Daouia et al. [55] deal with the problem of estimating quantiles when covariate information is available. A general methodology for modeling loss data depending on covariates is developed in Chavez-Demoulin et al. [46]. In this paper the parameters of the frequency and severity distributions of the losses may depend on covariates.

In [E10] we study the behavior of a covariate $Y$ on the level sets of a $d$-dimensional vector of risk-factors $X$. More precisely, in analogy with measures introduced in Definitions 1.2.2, 1.2.3 and 1.2.4, we define the multivariate Covariate-Conditional-Tail-Expectation (CCTE)

$$
\text{CCTE}_\alpha(X, Y) := \mathbb{E}[Y \mid X \in L(\alpha)],
$$

(1.5)

where $\alpha \in (0, 1)$. In [E10] we discuss a real application of the CCTE measure in the evaluation of the mean overtopping discharge ($Y$) conditionally to the fact that the sea variable conditions (significant wave height, the still water level and the wave period), belong to the joint risk area $L(\alpha)$. A plug-in estimator for the CCTE measure in (1.5) is proposed in [E10] (see Section 1.4.1 below for a syntactic discussion).

For practical purposes in environmental risk management, Salvadori et al. [165] and Salvadori et al. [168] introduce a new multivariate risk measure, called Component-wise Excess (CE) design realization, based on critical layers in Definition 1.2.1. They define the CE design realization as

$$
\delta_{\text{CE}}(\alpha) = \arg \max_{x \in \partial L(\alpha)} \mathbb{P}(X \in [x, \infty)).
$$

(1.6)

Salvadori et al. [168] provide practical guidelines for coastal and off-shore engineering by using the risk measure in (1.6). In [E2] we provide the explicit expression of $\delta_{\text{CE}}$ in the Archimedean copula setting. Furthermore, measure in (1.6) is estimated by using Extreme Value Theory techniques and the asymptotic normality of the proposed estimator is studied (see Section 1.4.3 below for a syntactic discussion).

In Section 1.2.1 below, we analyse the multivariate risk measures introduced in Definitions 1.2.2, 1.2.3 and 1.2.4 in several directions. In particular, several characterizations are provided in terms of the copula structure and stochastic orderings of the marginal distributions.

1.2.1 Axiomatic risk measures properties

In this section, we show that many properties satisfied by the univariate classic risk measures (VaR and CTE) expand to the proposed multivariate versions under some conditions. In particular, our measures satisfy the positive homogeneity and the translation invariance property which are parts of the classic axiomatic properties of Artzner et al. [10]. Some inferential results for our risk measures in Definitions 1.2.2, 1.2.3 and 1.2.4 and Equations (1.5) and (1.6) are postponed to Section 1.4.

**Proposition 1.2.1 (Positive homogeneity and translation invariance)** Consider a random vector $X$ with a distribution function satisfying the regularity conditions. Let $\omega$ be a vector in $[0, 1]^d$ and $\alpha \in (0, 1)$. Then, multivariate VaR, CTE and CoVaR in Definitions 1.2.2, 1.2.3 and 1.2.4 satisfy the following properties:

Positive Homogeneity: $\forall \ c \in \mathbb{R}_+^d$, 
Proposition 1.2.2 Consider a comonotonic non-negative random vector $X$. Let $\omega$ be a vector in $[0,1]^d$ and $\alpha \in (0,1)$. Therefore, for $i \in I$,

$$\text{VaR}_{\alpha}^i(X) = \text{CoVaR}_{\alpha,\omega}^i(X) = \text{VaR}_{\alpha}(X_i) = \overline{\text{CoVaR}}_{\alpha,\omega}^i(X)_i,$$

$$\text{CTE}_{\alpha}^i(X) = \text{CTE}_{\alpha}(X_i) = \overline{\text{CTE}}_{\alpha,\omega}^i(X)_i.$$

Proof of Proposition 1.2.1 is given in Propositions 2.5 [E9] and [E16]* and Proposition 2.7 in [E13].

We now show, as in the univariate setting, an additivity property of our measures in the case of $\pi$-comonotonic\(^3\) couple $(X,Y)$ of $d$-dimensional random vectors (see Puccetti and Scarsini [157]).

Proposition 1.2.3 Let $(X,Y)$ be a $\pi$-comonotonic couple of random vectors. Therefore, for $\omega \in [0,1]^d$ and $\alpha \in (0,1),$

$$\text{VaR}_{\alpha}(X + Y) = \text{VaR}_{\alpha}(X) + \text{VaR}_{\alpha}(Y) \quad \text{and} \quad \overline{\text{VaR}}_{\alpha}(X + Y) = \overline{\text{VaR}}_{\alpha}(X) + \overline{\text{VaR}}_{\alpha}(Y),$$

$$\text{CTE}_{\alpha}(X + Y) = \text{CTE}_{\alpha}(X) + \text{CTE}_{\alpha}(Y) \quad \text{and} \quad \overline{\text{CTE}}_{\alpha}(X + Y) = \overline{\text{CTE}}_{\alpha}(X) + \overline{\text{CTE}}_{\alpha}(Y),$$

\(^3\)Definition ($\pi$-comonotonicity) A couple $(X,Y)$ of $d$-dimensional random vectors is said to be $\pi$-comonotonic if there exists a $d$-dimensional random vector $Z = (Z_1, \ldots, Z_d)$ and non-decreasing functions $f_1, \ldots, f_d, g_1, \ldots, g_d$ such that $(X,Y) \overset{d}{=} ((f_1(Z_1), \ldots, f_d(Z_d)), (g_1(Z_1), \ldots, g_d(Z_d))).
\[ \text{CoVaR}_{\omega}(X + Y) = \text{CoVaR}_{\omega}(X) + \text{CoVaR}_{\omega}(Y) \]

\[ \text{CTE}_{\omega}(X + Y) = \text{CTE}_{\omega}(X) + \text{CTE}_{\omega}(Y). \]

Subadditivity of the multivariate CTE for independent random vectors with independent components is given in Proposition 2.4 in [E13]. This result is consistent with existing subadditivity property for univariate CTE measure.

### 1.2.2 Archimedean copulas case

Multivariate risk measures introduced in Definitions 1.2.2, 1.2.3 and 1.2.4 can be computed analytically for any \( d \)-dimensional random vector with an Archimedean copula \( C \), i.e.,

\[ C(u_1, \ldots, u_d) = \phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d), \quad \text{for } u_1, \ldots, u_d \in (0, 1]. \]  \hspace{1cm} (1.7)

where \( \phi \) is a real function \( \phi : \mathbb{R}^+ \rightarrow [0, 1] \), called the generator of the copula, and where \( \psi \) is the generalized inverse function of \( \phi \), i.e. \( \psi(u) = \inf \{ x \in \mathbb{R}^+ : \phi(x) \leq u \} \), for \( u \in [0, 1] \). The generator \( \phi \) is continuous, decreasing and convex function, with \( \lim_{x \rightarrow +\infty} \phi(x) = 0 \) and \( \phi(0) = 1 \) (see, e.g., Definition 2 in McNeil and Nešlehová [141]).

Archimedean copulas are widely used in the probabilistic literature (see e.g. Nelsen [148], Joe [120], McNeil and Nešlehová [141]) as well as in the literature concerning fuzzy logics (see e.g. Baczyński [15], Dolati et al. [68]).

**Proposition 1.2.4** Let \( X \) be a \( d \)-dimensional random vector with marginal distributions \( F_1, \ldots, F_d \). Assume that the dependence structure of \( X \) is given by an Archimedean copula as in (1.7) and \( \alpha \in (0, 1) \). Then, for any \( i \in I \),

\[ \text{VaR}_{\alpha}^i(X) = \mathbb{E} \left[ F_i^{-1} \left( \phi(S_i \phi^{-1}(\alpha)) \right) \right], \]

\[ \text{CoVaR}_{\alpha,\omega}^i(X) = \text{VaR}_{\omega^i} \left[ F_i^{-1} \left( \phi(S_i \phi^{-1}(\alpha)) \right) \right], \]

where \( \omega \in [0, 1]^d \) and \( S_i \) is a random variable with Beta\( (1, d - 1) \) distribution.

Let \( X \) be a \( d \)-dimensional random vector with marginal survival distributions \( F_1, \ldots, F_d \). Assume that the survival copula of \( X \) is an Archimedean copula as in (1.7) and \( \alpha \in (0, 1) \). Then, for any \( i \in I \),

\[ \text{VaR}_{\alpha}^i(X) = \mathbb{E} \left[ F_i^{-1} \left( \phi(S_i \phi^{-1}(1 - \alpha)) \right) \right], \]

\[ \text{CoVaR}_{\alpha,\omega}^i(X) = \text{VaR}_{\omega^i} \left[ F_i^{-1} \left( \phi(S_i \phi^{-1}(1 - \alpha)) \right) \right]. \]

Similar representations for \( \text{CTE} \) and \( \overline{\text{CTE}} \) are given in Equations (16) and (17) in [E13].

Proof of Proposition 1.2.4 follows from the McNeil and Nešlehová’s stochastic representation of Archimedean copulas (see McNeil and Nešlehová [141]). More precisely, the following relations hold: \( \{ U \mid C(U) = \alpha \} \overset{d}{=} (\phi(S_1 \phi^{-1}(\alpha)), \ldots, \phi(S_d \phi^{-1}(\alpha))) \) and \( \{ U \mid \overline{C}(U) = 1 - \alpha \} \overset{d}{=} (\phi(S_1 \phi^{-1}(1 - \alpha)), \ldots, \phi(S_d \phi^{-1}(1 - \alpha))). \)

If \( C \) is a \( d \)-dimensional Archimedean copula as in (1.7), we also prove that:

\[ \text{CoVaR}_{\omega}(X) \text{ and } \text{CTE}_{\omega}(X) \text{ are increasing functions of risk level } \alpha \text{ and under further regularity conditions, decreasing functions of the dependence Archimedean parameter } \theta \text{ (see Corollaries 2.4 and 2.5 in [E16]*; Corollaries 2.2 and 2.4 in [E13]; Corollaries 4.3 and 4.4 in [E9]). Similar results are also proved for our upper-orthant measures.} \]
Example 1 (Bivariate Clayton case) As a matter of example, let us now consider the Clayton family of bivariate copulas. In Table 1.1 we illustrate our $\text{VaR}^{i,\theta}(X,Y)$, $\text{CoVaR}^{i,\theta}(X,Y)$ and $\text{CTE}^{i,\theta}(X,Y)$, for $i = 1,2$. Moreover, it is also possible to derive analogue expressions for the upper-orthant ones.

In Figure 1.1, $\text{VaR}^{i,\theta}(X,Y)$ (left) and $\text{CTE}^{i,\theta}(X,Y)$ (right) are plotted as functions of the risk level $\alpha$ for different values of the parameter $\theta$. As stated above, one can observe that an increase of the dependence parameter $\theta$ tends to lower the $\text{VaR}$ and $\text{CTE}$ up to the perfect dependence case.

<table>
<thead>
<tr>
<th>Copula</th>
<th>$\theta$</th>
<th>$\text{VaR}^{i,\theta}(X,Y)$</th>
<th>$\text{CoVaR}^{i,\theta}(X,Y)$</th>
<th>$\text{CTE}^{i,\theta}(X,Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton $C_\theta$</td>
<td>$(-1, \infty)$</td>
<td>$\frac{\theta}{\theta - 1} \cdot \frac{\alpha^{\theta} - \alpha^{\theta - 1}}{\alpha^{\theta} - 1}$</td>
<td>$(1 + \frac{1}{\theta} - 1)(1 - \omega_1)^{-1/\theta}$</td>
<td>$\frac{1}{2} - \frac{\theta - 1 - \alpha^2(1 + \theta) + 2 \alpha^{1 + \theta}}{\theta - \alpha(1 + \theta) + \alpha^{1 + \theta}}$</td>
</tr>
<tr>
<td>Counter-monotonic $W$</td>
<td>$-1$</td>
<td>$\frac{1 + \alpha}{\ln \alpha}$</td>
<td>$1 - (1 - \omega_1)(1 - \alpha)$</td>
<td>$\frac{1}{2} - \frac{(1 - \alpha)^2}{\alpha - \alpha + \alpha \ln \alpha}$</td>
</tr>
<tr>
<td>Independent $\Pi$</td>
<td>$0$</td>
<td>$\frac{\alpha - 1}{\ln \alpha}$</td>
<td>$\alpha^{1 - \omega_1}$</td>
<td>$\frac{1}{2} - \frac{(1 - \alpha)(1 - \omega_1)}{\alpha - \alpha + \alpha \ln \alpha}$</td>
</tr>
<tr>
<td>$\Pi \times \Pi$</td>
<td>$1$</td>
<td>$\frac{\alpha \ln \alpha}{\ln \alpha}$</td>
<td>$\frac{\alpha}{(1 - \omega_1)(1 - \omega_1) + \alpha}$</td>
<td>$\frac{1}{2} - \frac{1 + \alpha^2(2 \ln \alpha - 1)}{(1 - \alpha)^2}$</td>
</tr>
<tr>
<td>Comonotonic $M$</td>
<td>$\infty$</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\frac{1}{2} - \frac{\alpha}{\ln \alpha}$</td>
</tr>
</tbody>
</table>

Table 1.1: Proposed lower-orthant measures for different bivariate dependence structures.

Figure 1.1: Behavior of $\text{VaR}^{i,\theta}(X,Y)$ (left) and $\text{CTE}^{i,\theta}(X,Y)$ (right) with respect to risk level $\alpha$ for different values of dependence parameter $\theta$ in the bivariate Clayton case (see Table 1.1). The horizontal line in the right panel corresponds to $E[X]$.

1.2.3 Comparing measures using stochastic orders

Using the theory of stochastic ordering, we also analyze the effect of some risk perturbations on these measures. In the same vein as for the univariate measures, we prove that an increase of marginal risks in the sense of the stochastic dominance order yields an increase of the multivariate measure. In this sense, results presented below provide natural multivariate extensions of some classic results in the univariate setting (see, e.g., Denuit and Charpentier [64], Section 3.3.1 in Denuit et al. [65]).
\subsection{Estimation procedures for critical layers and level sets}

Since risk measures introduced in this chapter are based on critical layers and level sets (see Definition 1.2.1), in this section we focus on some estimation procedures of the \(\alpha\)-upper level set \(L(\alpha)\) and for \(\alpha\)-critical layers \(\partial L(\alpha)\) (similar results can be easily stated also for the \(\overline{L}(\alpha)\) and \(\partial \overline{L}(\alpha)\)). Some inferential results for our risk measures in Definitions 1.2.2, 1.2.3 and 1.2.4 are postponed to Section 1.4.

\subsubsection{Plug-in method \cite{E10}; \cite{E18} \textsuperscript{*}}

Most of the existing literature has focused on problem of estimating level sets of the density or, more recently, on regression function (Baíllo et al. \cite{E16}, Cavalier \cite{E39}, Cuevas and Rodríguez-Casal \cite{E54}, Rigollet and Vert \cite{E160}). Mason and Polonik \cite{E138} obtained the asymptotic normality of plug-in level set estimates in the density case. The problem of estimating general level sets under compactness assumptions has been also discussed by Cuevas et al. \cite{E53}. Given that if we consider the level set \(L(\alpha)\) the commonly assumed property of compactness for these sets
(required both in the density and in the regression cases literature) is no more reasonable, we deal with this non-compact setting in [E10] and [E18]*. Considering a consistent estimator $F_n$ of the distribution function $F$, we propose a plug-in approach to estimate $L(\alpha)$ by

$$L_n(\alpha) = \{ x \in \mathbb{R}^d : F_n(x) \geq \alpha \}, \text{ for } \alpha \in (0, 1).$$

As remarked above, to deal with this non-compact setting, given $T > 0$, we set

$$L(\alpha)^T = \{ x \in [0, T]^d : F(x) \geq \alpha \}, \text{ and } L_n(\alpha)^T = \{ x \in [0, T]^d : F_n(x) \geq \alpha \}. \quad (1.8)$$

Our consistency result for critical layers is stated with respect to a criterion of “physical proximity” between sets. More precisely, we consider the consistency in terms of $\lambda(L(\alpha)^T \triangle L_n(\alpha)^T)$, where $\lambda$ stands for the Lebesgue measure on $\mathbb{R}^d$ and $\triangle$ for the symmetric difference.

**Major contributions**

Using these notations, we now establish our consistency result with convergence rate. We can interpret the following theorem as a generalization of Theorem 3 in Cuevas et al. [53] (Section 3) in the case of non-compact level sets.

**Theorem 1.3.1** Let $\alpha \in (0, 1)$. Let $F$ be a twice differentiable distribution function on $\mathbb{R}^d$ satisfying some further regularity conditions on its gradient vector and its Hessian matrix (see Section 2 in [E10] and Section 1 in [E18]*). Let $(T_n)_{n \in \mathbb{N}}$ be a positive increasing truncation sequence. Assume that for each $n$, $F_n$ is measurable. Assume that there exists a positive increasing sequence $(w_n)_{n \in \mathbb{N}}$ such that $w_n \| F - F_n \|_{\infty} \overset{P}{\longrightarrow} 0$. Then, it holds that

$$p_n \lambda(L(\alpha)^T \triangle L_n(\alpha)^T) \overset{P}{\longrightarrow} 0,$$

where the convergence rate $p_n$ depends on $w_n$, $T_n$ and the dimension $d$.

Proof of Theorem 1.3.1 is given in the bivariate case in [E18]*, Theorem 3.1, and in the general $d$-dimensional setting in [E10], Theorem 1 and Corollary 1. Demonstration is essentially based on conditions of the gradient vector and Hessian matrix of $F$ in a reasonably neighborhood of the considered multivariate quantile curve $\partial L(\alpha)$, for a fixed $\alpha \in (0, 1)$.

Moreover, Theorem 1.3.1 provides a convergence rate, which obviously suffers from the well-known curse of dimensionality and is closely related to the choice of the truncation sequence $T_n$.

**1.3.2 Transformation copula based method [E8; E15]**

By using a transformation distributional model (based on a transformed copula and on transformed margins) we construct a parametric estimation procedure for $\partial L(\alpha)$ in Definition 1.2.1 and the associate Kendall distribution $K(\alpha)$.

We just mention here this alternative inferential method in order to build an exhaustive table of contents for this manuscript. However, for sake of clarity and readability, this parametric estimation is postponed to Section 2.2.2, where all necessary notion and preliminaries are also introduced. For further details the interested reader is also referred to [E8] and [E15].

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4In sections 1.3.1 and 1.4.1 we deal with positive random vectors. However the obtained results can be extended in whole $\mathbb{R}^d$ space.
1.3.3 Extreme extrapolation method [E19]

This section is based on [E19]. We provide here a out-sample {method} to estimate the directional multivariate quantiles recently introduced in Torres et al. [191] and Torres et al. [190]. These multivariate quantiles are rotated version in a given direction $u$ of the critical layers in Definition 1.2.1. The inclusion of this direction parameter $u$ is allowed for analyzing the observations (data-cloud) from several interesting perspectives.

Let us define the directional multivariate critical layers, based on the notion of oriented orthant (see also Laniado et al. [130] and Torres et al. [191]).

Definition 1.3.1

1. An oriented orthant in $\mathbb{R}^d$ with vertex $x$ in direction $u$ is defined by, $C^u_x = \{ z \in \mathbb{R}^d : R_u(z - x) \geq 0 \}$, where $u \in \{ z \in \mathbb{R}^d : ||z|| = 1 \}$ and $R_u$ is an orthogonal matrix such that $R_u u = e$, with $e = \frac{\sqrt{d}}{d}[1, ..., 1]'$.

2. The QR oriented orthant with vertex $x$ in direction $u$, denoted as $C^u_{x|u}$, is the oriented orthant satisfying $R_u = Q_e Q'_u$, with $Q_u, Q_e$ the correspondent orthogonal matrices of the QR decomposition.

3. Let $X$ be a $d$–dimensional random vector and $p \in (0, 1)$. Then, the directional multivariate quantile at certain level $1 - p$ in direction $u$ is given by

$$ Q_X(p, u) := \partial \{ x \in \mathbb{R}^d : P(C^u_x) \geq 1 - p \}. \tag{1.9} $$

Remark that the multivariate quantiles based on distributions and survival distributions, are provided through the directions $-e, e$ respectively. Then, Definition 1.3.1 can be seen as a directional extension of multivariate critical layers in Definition 1.2.1.

In the univariate setting, extremes are analyzed considering the two possibilities of exceeding from either distribution or survival distribution and most of the extensions of these analyses to the multivariate setting have also been concentrated on these two types of exceeding (see, for instance, in the bivariate case Embrechts and Puccetti [84]; Shiau [177]; Salvadori [163], and generalized multivariate versions are presented in Fraiman and Pateiro-López [95] and in [E16; E9]).

However, we point out that there are other interesting directions to be taken into consideration. For instance in portfolio optimization, the direction given by the portfolio weights of investments is of particular interest because it takes into account the losses due to the composition of the investment in the portfolio (see Laniado et al. [130]; Torres et al. [191])). In environmental phenomena, the directional approach has also been applied to detect extreme events by considering the direction of maximum variability of the data (see, e.g., Torres et al. [190]). Nelsen and Úbeda Flores [152] develop some directional dependence coefficients which can be used to detect dependence in multivariate distributions not detected by several known measures of multivariate association.

Major contributions

First, in [E19] we prove that a rotation of a regularly varying random vector $X$ is still regularly varying, and the second order regular variation can also be preserved by such a rotation.

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5 Despite their importance, definitions and properties from univariate and multivariate regularly varying theory are omitted in this manuscript for the sake of brevity. The interested reader is referred for instance to Bingham et al. [26], Balkema and Embrechts [17] and Soulier [183].
Proposition 1.3.1 If $X$ has second order multivariate regular variation, then the random vector $QX$ has second order regular variation for any orthogonal transformation $Q$.

Furthermore, by using Proposition 1.3.1 and the bivariate quantile parametrization given in de Haan and Huang [59] extended in a multivariate framework, we propose a characterization of the directional multivariate quantiles at high levels (i.e., $p = p_n \rightarrow 0$ for the sample size $n \rightarrow \infty$).

From Property 3.8 in Torres et al. [191], it holds that $Q_X(p, u) = R_u^p Q_{R_u}X(p, e)$. Then, we aim to estimate $Q_{R_u}X(p, e)$, which according to Definition 1.3.1 is the set of points such that $1 - p = F_u(x)$. To this aim, we generalize in a multivariate framework the polar coordinates parametrization proposed by de Haan and Huang [59]. From now on, any point of $Q_{R_u}X(p, e)$ under the new polar parametrization will be denoted by $x_u(p, \theta)$, $\theta \in [0, 1]^d$.

Remark that under the extreme value assumptions in [E19], if $Q_{R_u}X_1, \ldots, Q_{R_u}X_n$ are independent random vectors distributed as $R_uX$, there exist two sequences $a(n)$, $b(n)$ such that,

$$\lim_{t \rightarrow \infty} t (1 - F_u(a_{u,j}(t) x_{u,j} + b_{u,j}(t); \ j \in I)) = -\ln (G_u(x_u)).$$

Then using (1.10), we provide the following characterization of the elements of $Q_{R_u}X(p, e)$,

$$x_{u,j}(p, \theta) = a_{u,j}(t) \frac{\rho_u(\theta) \theta_j / t p}{\gamma_{u,j}} - 1 + b_{u,j}(t), \quad \text{for all } j \in I,$n

where the scalar function $\rho_u(\theta)$ is given by

$$\rho_u(\theta) := -\ln \left( G_u \left( \frac{\theta_j^{\gamma_{u,j}} - 1}{\gamma_{u,j}}; \ j \in I \right) \right),$$

with $G$, $a$ and $b$ as in (1.10) and $\gamma_{u,j}$ the tail index associated to the $j$--marginal distribution of $F_u$.

Thus, $Q_X(p, u)$ in Definition 1.3.1 is approximated at high levels by the parametrization,

$$Q_X(p, u, \theta) = R_u^p Q_{R_u}X(p, e, \theta), \quad \text{where}$$

$$Q_{R_u}X(p, e, \theta) = \{ x_u(p, \theta), \ \theta \in [0, 1]^d, \ \text{with coordinates } x_{u,j}(p, \theta) \text{ as in (1.11)} \}.$$

In [E19], we propose an estimated version $\hat{Q}_X(p, u, \theta)$ for parametrization in (1.13) by following de Haan and Huang [59]. Then,

$\rightarrow$ if $X$ is a second order multivariate regularly varying random vector and using Proposition 1.3.1, we prove asymptotic normality for the estimated multivariate directional quantile $\hat{Q}_X(p_n, u, \theta)$ for $p_n \rightarrow 0$, when $n \rightarrow \infty$ (see Theorem 5.1 in [E19]).

To conclude, we present the directional quantile curves $Q_X(p = 1/n, e)$ for a centered bivariate $t$--distribution with degrees of freedom $\nu = 3$ (see Figure 1.2). This family of distributions possesses properties such as heavy tails and closure under rotations, which provide a good example for comparing theoretical and estimated solutions. In Figure 1.2 (left) we represent theoretical and estimated directional quantile curves for sample size $n = 500$. We can appreciate the accuracy of the estimations of $Q_X(p = 1/n, e)$. Finally, we repeat the same procedure but using the first PCA direction (see Figure 1.2, centre and right panels). In this case, we can appreciate the good performance of the estimators and the improvements of the identification of the extremes based on the shape of the data.
1.4 Estimation procedures for considered multivariate risk measures

In the following, we propose several inferential procedures for the multivariate risk measures introduced in Definitions 1.2.2, 1.2.3 and 1.2.4 and in Equations (1.5) and (1.6).

1.4.1 Plug-in method [E10]; [E18]*

In [E10] we establish a component-wise consistency result with a convergence rate for the plug-in estimate of the CCTE risk measure in Equation (1.5). To this aim, using the truncated version of the \( \alpha \)-upper level set \( L(\alpha) \) defined in (1.8), we introduce the following quantities.

**Définition 1** Let \( \alpha \in (0, 1) \). Consider a random vector \( X \) with distribution function \( F \) on \( \mathbb{R}^d_+ \) and a covariate random variable \( Y \). Let \( (T_n)_{n \in \mathbb{N}} \) be a positive increasing truncation sequence. We define the truncated theoretical multivariate \( \alpha \)-Covariate-Conditional-Tail-Expectation: \( \text{CCTE}^{T_n}_{\alpha}(X, Y) = \mathbb{E}[Y | X \in L(\alpha)T_n] \), and the associated plug-in estimate as

\[
\hat{\text{CCTE}}^{T_n}_{\alpha,n}(X, Y) = \mathbb{E}_n[Y | X \in L_n(\alpha)T_n]
\]

where \( \mathbb{E}_n \) denotes the empirical version of the expected value.

Using Definition 1, we now establish our component-wise consistency result with convergence rate (for further details see Theorem 4 in [E10]).

**Theorem 1.4.1** Let \( \alpha \in (0, 1) \). Let \( F \) be a twice differentiable distribution function on \( \mathbb{R}^d_+ \), satisfying some further regularity conditions on its gradient vector and its Hessian matrix (see Section 2 in [E10], Section 1 in [E18]*) with an associated density \( f \) such that \( \|f\|_{1+\varepsilon, \lambda} < \infty \), with \( \varepsilon > 0 \). Assume that for each \( n \), \( F_n \) is measurable. Let \( (v_n)_{n \in \mathbb{N}} \) and \( (T_n)_{n \in \mathbb{N}} \) positive increasing sequences such that \( v_n \int_{[0,T_n]^d} |F(x) - F_n(x)|^p \lambda(dx) \xrightarrow{P} 0 \), for some \( 1 \leq p < \infty \). It holds that

\[
\beta_n \left| \hat{\text{CCTE}}^{T_n}_{\alpha,n}(X, Y) - \text{CCTE}^{T_n}_{\alpha}(X, Y) \right| \xrightarrow{n \to \infty} 0,
\]

where the convergence rate \( \beta_n \) depends on \( v_n, T_n, d \) and on conditions for \( f \).
Chapter 1. On some new multivariate risk measures

Proof is based on previous Theorem 1.3.1 about the consistency of \( L_n(\alpha)^T \) and on conditions for the density function \( f \) of \( X \).

By using Theorem 1.4.1, a tractable convergence rate in the case of the empirical distribution \( F_n \) can be derived (see Example 2 in [E10]).

\[ \Rightarrow \text{Furthermore, we state } L_p \text{-consistency with a convergence rate for the estimation of the regression function } r(x) = \mathbb{E}[Y \mid X = x], \text{ for } x \in L(\alpha), \text{ (see Theorem 3 in [E10]).} \]

In a similar vein to [E10], in [E18]*, we consider a plug-in estimate of the lower-orthant bivariate CTE in Definition 1.2.3. Let us introduce the following quantities.

**Definition 2** Let \( \alpha \in (0, 1) \). Consider a random vector \( X = (X_1, X_2) \) with distribution function \( F \) on \( \mathbb{R}^2_+ \). Let \( (T_n^\alpha)^n \in \mathbb{N} \) be a positive increasing truncation sequence. We define the truncated theoretical bivariate \( \alpha \)-Conditional-Tail-Expectation: \( \widehat{\text{CTE}}_{\alpha}^{T_n}(X) = \mathbb{E}[X \mid X \in L(\alpha)^T] \), and the associated plug-in estimate as \( \widehat{\text{CTE}}_{\alpha,n}^{T_n}(X) = \mathbb{E}_n[X \mid X \in L_n(\alpha)^T] \), where \( \mathbb{E}_n \) denotes the empirical version of the expected value.

Under Assumptions of Theorem 1.3.1, suitable regularity conditions on the density \( f \) of \( X \) and using Definition 2,

\[ \Rightarrow \text{we prove the following component-wise consistency result with convergence rate:} \]

\[ \beta_n \left| \text{CTE}^{T_n}_{\alpha}(X) - \widehat{\text{CTE}}_{\alpha,n}^{T_n}(X) \right| \xrightarrow{P} 0, \]

where the convergence rate \( \beta_n \) depends on conditions of the density \( f \) of \( X \) and on the convergence rate \( p_n \) between \( L_n(\alpha)^T \) and \( L(\alpha)^T \) in Theorem 1.3.1 (see Theorem 4.1 in [E18]*).

\[ \Rightarrow \text{Furthermore, a tractable convergence rate in the case of the empirical distribution } F_n \text{ can be easily derived (see Corollary 4.1 in [E18]*).} \]

**Remark 2** The proposed plug-in estimator of \( \text{CTE} \) proposed in [E18]* is based on the consistent estimation of the whole level sets \( L(\alpha) \). Notice that the non-optimal rate of convergence depends on the rate of convergence of the truncation sequence \( (T_n^\alpha)^n \geq 1 \) to infinity. Making the “best choice” for \( (T_n^\alpha)^n \geq 1 \) is not trivial and requires the knowledge of the tail behavior of \( X \), at least in its generic form. In the next section, we try to overcome this drawback.

### 1.4.2 Kendall-process based method [E11]

In [E11], contrarily to the [E18]*’s approach above, we propose a new non-parametric estimator for \( \text{CTE}^\alpha_j(X) = \mathbb{E}[X_j \mid F(X) \geq \alpha] \), for \( j \in I \), in Definition 1.2.3, based on the estimation of the Kendall’s distribution, i.e. the distribution of the multivariate probability integral transformation \( F(X) \) (see Genest and Rivest [100]). Using the central limit theorem for the Kendall’s process (see Equation (1.15) below), proved by Barbe et al. [18], we provide a functional central limit theorem for our estimator. The main advantage of the proposed non-parametric estimate is that it does not require the calibration of extra parameters or sequences.

Our result is based on the Kendall empirical distribution, which is a non-parametric estimator of the Kendall distribution \( K(\alpha) = \mathbb{P}[F(U) \leq \alpha] = \mathbb{P}[C(U) \leq \alpha] \), where \( U := (U_1, \ldots, U_d) \) with
$U_1 = F_1(X_1), \ldots, U_d = F_d(X_d),$ and $C$ is the copula function associated with $X$ by Sklar [180]'s theorem (see also Genest and Rivest [100], Barbe et al. [18]). One can write

$$K_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{C_n(U_i) \leq \alpha + \frac{1-\alpha}{n}\}},$$

(1.14)

where $C_n$ denotes the empirical copula (see Ghoudi and Remillard [102]).

We now state the main result in Barbe et al. [18], which proved that, under regularity conditions on $K$ and on its derivate $k$ (see Assumptions I and II in Barbe et al. [18]), the empirical process

$$\gamma_n(\alpha) = \sqrt{n} \left( K_n(\alpha) - K(\alpha) \right)$$

(1.15)

converges in distribution to a continuous Gaussian process $\gamma$ with zero mean and covariance function $\Gamma$, in the space $D$ of càdlàg functions from $(0,1)$ to $\mathbb{R}$ endowed with the Skorohod topology.

We propose in [E11], a new non-parametric estimator for the lower-orthant CTE based on the Kendall empirical distribution in (1.14).

**Definition 1.4.1** The Kendall-based estimator for the Multivariate $\alpha$-CTE is defined by

$$\widehat{\text{CTE}}^j_n(X) = \frac{1}{n} \sum_{i=1}^{n} X_{ij} \mathbf{1}_{\{C_n(U_i) \geq \alpha \}}$$

for $j \in I$, where $K_n$ is the empirical Kendall estimator of $K$ in (1.14) and $n$ is the sample size.

Using convergence in (1.15), we can prove the weak convergence of the Kendall-based process:

$$\gamma_n(\alpha) = \sqrt{n} \left( K_n(\alpha) - K(\alpha) \right)$$

(1.15)

Using convergence in (1.15), we can prove the weak convergence of the Kendall-based process:

$$\gamma_n(\alpha) = \sqrt{n} (\text{CTE}_n(\alpha) - \text{CTE}_n^j(\alpha))$$

for $j \in I$.

(1.17)

**Major contributions**

Our main result is presented below with a sketch of the proof\(^6\):

$\rightarrow$ **We first prove the coordinatewise convergence.** Under further regularity conditions, the process $\gamma_{n,j}^\text{CTE}$ in (1.17), for $j \in I$, converges weakly in the Skorohod space $D([0,1-\varepsilon], \mathbb{R})$ to a continuous Gaussian process $\gamma_{j}^\text{CTE}$ with zero mean and covariance function $\Gamma_{j}^\text{CTE}$ (see Theorem 3.2 in [E11]).

Sketch of proof. To prove this first step we decompose the process $\gamma_{n,j}^\text{CTE}$ in a sum of subsidiary processes and we study the asymptotic behavior of each of the subsidiary processes (see Section 4 in [E11]). By applying the continuous mapping theorem, we get the result (see Section 5 in [E11]).

$\rightarrow$ **Second, we prove the functional central limit theorem.** Under regularity conditions, the Kendall based process $\gamma_n^\text{CTE}$ in (1.16) converges weakly to a continuous Gaussian process $\gamma_{j}^\text{CTE}$ with zero mean and (cross-)covariance function defined by $\Gamma_{i,j}^\text{CTE}(s,t) = \text{Cov}(\gamma_i(s), \gamma_j(t)), (s,t) \in [0,1]^2, i,j \in I$ (see Theorem 3.3 in [E11]).

\(^6\)Here, weak convergence for processes will always be considered in the space $D$ of càdlàg functions from $[0,1-\varepsilon]$ to $\mathbb{R}^k$ endowed with the Skorohod topology, for some $k \in \mathbb{N}^*$ and $\varepsilon > 0$. 
Let \( j_1 < \ldots < j_i \in I \), if one wants to prove the convergence of the finite-dimensional distributions of \( \gamma_C^{\text{CTR}} \) we consider 
\[
\sum_{k=1}^{n_i} b_k \gamma_i^{\text{CTR}} \] 
for some fixed \( k \in I \). Furthermore, we prove the tightness in \( D ([0, 1 - \varepsilon], \mathbb{R}^d) \) endowed with the Skorohod topology. As each component converges weakly in the Skorohod space \( D ((0, 1 - \varepsilon], \mathbb{R}) \) to a continuous limit, we get the result.

For a detailed discussion on the regularity conditions of the two previous results we refer the interested reader to conditions i), ii) and Assumption I in [E11] (see also Barbe et al. [18]).

**Remark 3** In this section, we presented the [E11]’s estimator which is easy to implement (no parameters to tune) while having good asymptotic properties for moderate to high (but not extreme) risk levels \( \alpha \). For an adapted extreme estimator of the same risk measure the reader is referred to Section 1.4.3.

### 1.4.3 Extreme extrapolation method [E2; E5; E20]

**Extreme extrapolation method for CoVaR in Definition 1.2.4**

We now focus on \( \text{CoVaR}_{\alpha, \omega_i}(X) = \text{VaR}_{\omega_i}(X | X \in \partial L(\alpha)) \) introduced in Definition 1.2.4, for \( i \in I \) (see Equation (1.4)) and we propose a non-parametric extreme estimation procedure for a fixed level \( \alpha \) and extreme level \( \omega_i \). For further details, we refer to [E5].

Let \( T_i := [X_i | X \in \partial L(\alpha)] \), for \( i \in I \), with \( \partial L(\alpha) \) as in Definition 1.2.1 and associated distribution function \( F_i(x | \alpha) = \mathbb{P}[X_i \leq x | X \in \partial L(\alpha)] \). Let \( \omega_i := 1 - p \), with \( p \in (0, 1) \).

Then, in [E5], we consider the \((1 - p)\)-quantile of the random variable \( T_i \), i.e.,
\[
x_i^p := U_{T_i} \left( \frac{1}{p} \right), \quad \text{for } p \in (0, 1),
\] 
(1.18)

where \( U_{T_i}(t) := F_i^{-1}(1 - \frac{1}{t} | \alpha) \) for \( t > 1 \), and \( F_i^{-1}(\cdot | \alpha) \) denotes the left-continuous inverse of \( F_i(\cdot | \alpha) \). We focus on the case with fixed \( \alpha \in (0, 1) \) and \( p_n \to 0 \), as the sample size \( n \to +\infty \).

To model the dependence structure of the multivariate risk vector \( X \), we consider an Archimedean copula as in (1.7). As we will see below, this choice is motivated by the fact that under this assumption the distribution of \( T_i \) and its tail index can be easily obtained (see Proposition 1.4.1 below).

**Major contributions**

Using the notation introduced above, we give below our main results.

**Proposition 1.4.1 (The von Mises condition for \( T_i \))** Let \((X_1, \ldots, X_d)\) be a random vector with Archimedean copula as in (1.7) with twice differentiable generator \( \phi \). Assume that\(^7\) \( \psi \in \mathcal{R}_\rho(1) \), with \( \rho \in [1, +\infty) \). Let \( F_i \) be the twice differentiable distribution function of \( X_i \). Assume that \( F_i \) verifies the von Mises condition with index \( \gamma_i \in \mathbb{R} \).

If \( \rho \in [1, +\infty) \), then \( F_i(\cdot | \alpha) \) verifies the von Mises condition with tail index \( \gamma_i T_i = \frac{2}{p} \).

If \( \rho = +\infty \), then \( F_i(\cdot | \alpha) \) verifies the von Mises condition with tail index \( \gamma_i T_i = 0 \).

\(^7\)Several Archimedean generators satisfying this assumption can be found in Table 1 in Charpentier and Segers [44], with associated \( \rho \) index.
1.4. Estimation procedures for considered multivariate risk measures

Proof of Proposition 1.4.1 is based on the von Mises condition and on the fact that if $\psi \in \mathcal{RV}_\rho(1)$ then $\psi' \in \mathcal{RV}_{\rho^{-1}}(1)$. The interested reader is also referred to Brechmann [31]. Notice that $\gamma_i T_i$ depends on the domain of attraction of the respective margin (i.e., $\gamma_i$) and on the regularly varying index of the generator of the considered Archimedean copula (i.e., $\rho$).

To prove our central limit theorem we introduce the following assumption setting.

**Assumption 1.4.1**

i. Let $(X_1, \ldots, X_d)$ be a random vector with Archimedean copula as in (1.7) with twice differentiable generator $\phi$. Assume that $\psi \in \mathcal{RV}_\rho(1)$, with $\rho \in (1, +\infty)$, i.e., upper tail dependence case.

ii. The upper tail copula of $(X_i, X_j)$, with $i \neq j$, exists, has continuous partial derivatives, and satisfies a second-order condition (see Corollary 2 in Schmidt and Stadtmüller [171]).

iii. For the marginal distribution $X_i$ with assume a $2\mathcal{RV}$ condition on $U X_i$ (see Condition (3.2.4) in de Haan and Ferreira [58]) with tail index $\gamma_i > 0$.

Under Assumption 1.4.1, Proposition 1.4.1 and Proposition 2.2 in [E5] yield, as $n \to \infty$,

$$x_{i,n}^i \sim U X_i \left( \frac{n}{k_U} \right) \left( \frac{k}{n p_n} \right)^{\gamma_i / \rho}, \quad (1.19)$$

where $k = k(n) \to \infty$, $k(n)/n \to 0$, as $n \to \infty$ and $k_U(n)$ the intermediate sequence defined by

$$k_U(n) := n \left\{ 1 - \phi \left[ \left( 1 - \left( 1 - \frac{k(n)}{n} \right)^{1/(d-1)} \right) \phi^{-1}(\alpha) \right]\right\}. \quad (1.20)$$

**Estimation procedure**

We summarize here the estimation procedure provided in [E5]:

a. Let $X_{n-[k_U]}^i$ be the $(n - \lfloor k_U \rfloor)$-th order statistic of $(X_1^i, \ldots, X_d^i)$. Therefore, the natural estimator of $U X_i \left( \frac{n}{k_U} \right)$ in (1.19) is its empirical counterpart, that is, $X_{n-[k_U]}^i$ (e.g., see de Haan and Ferreira [58]), with $k_U$ as in Equation (1.20).

b. We estimate $\gamma_i$ with the Hill estimator (see Hill [108]) with integer auxiliary sequence $k_1$ such that $k_1(n) \to \infty$, $k_1/n \to 0$, $n \to \infty$.

c. We estimate $\rho$ by using the non-parametric rank-based estimator of upper tail dependence coefficient proposed by Schmidt and Stadtmüller [171], with integer auxiliary sequence $k_2$ such that $k_2(n) \to \infty$, $k_2/n \to 0$, $n \to \infty$.

Using steps a, b and c, we can therefore estimate $x_{i,n}^i$ in (1.19) by

$$\hat{x}_{i,n}^i = X_{n-[k_U]}^i \left( \frac{k}{n p_n} \right)^{\gamma_i / \rho}. \quad (1.21)$$

**Consistency results**

In our theoretical results we investigate two different situations: $\rho > 1$ (see Theorem 1.4.2 below) and $\rho = 1$ (see Proposition 1.4.2 below).

**Theorem 1.4.2 (Asymptotic normality of $\hat{x}_{i,n}^i$; upper tail dependence case)** Let $x_{i,n}^i$ be as in (1.18) and $\hat{x}_{i,n}^i$ as in (1.21). Under Assumption 1.4.1, it holds that, for $n \to \infty$,

$$v_n \left( \frac{\hat{x}_{i,n}^i}{x_{i,n}^i} - 1 \right) \xrightarrow{d, n \to \infty} \Xi,$$

where $\Xi$ is a Normal distribution. The convergence rate $v_n$ depends on $d_n := k/(n p_n)$ and on the three intermediate sequences $k_U(n)$, $k_1(n)$ and $k_2(n)$, from steps a, b and c respectively.
Figure 1.3: Boxplots for the ratio $\hat{\gamma}_i^{\hat{p}_n}/x^{\hat{p}_n}_i$, as in Theorem 1.4.2, with $p_n = 1/n$, $1/2n$ for $n = 2000$ for two different 3-dimensional models. Boxplots for some empirical competitor estimators with $p_n = 1/n$ are also displayed. We consider $\alpha = 0.9$ and 500 Monte Carlo simulations.

Proof of Theorem 1.4.2 is based on three central limit theorems. (1) The asymptotic normality of the Hill estimator $\hat{\gamma}_i$ (see Theorem 3.2.5 in de Haan and Ferreira [58]). (2) The asymptotic normality of the rank-based estimator $\hat{\rho}$. The proof of this result follows from Corollary 2 in Schmidt and Stadtmüller [171] and the Delta Method technique. (3) The asymptotic normality of the order statistic $X_i^{\hat{p}_n} - \lfloor k_{U(n)} \rfloor$. To obtain this last result we adapt to our setting the Central Limit Theorem for the intermediate order statistics (see Theorems 2.4.1 and 2.4.2 in de Haan and Ferreira [58] and Theorem 2.1 in Drees [69]).

Proposition 1.4.2 (Asymptotic consistency of $\hat{\gamma}_i^{\hat{p}_n}$; upper tail independence case)

If $\psi \in RV_1(1)$, then one can prove that

$$\frac{\hat{\gamma}_i^{\hat{p}_n}}{x^{\hat{p}_n}_i} \xrightarrow{P} 1, \quad \text{for } n \to \infty.$$ 

We conclude this discussion by pointing out the fact that $k_{U(n)}$ in Theorem 1.4.2 is an unknown sequence. This can be a drawback from a practical point of view, for instance, for the construction of confidence intervals.

Remark 4 (Adaptive version of Theorem 1.4.2) Following the considerations above, in Section 2.4 in [E5], we provide an adaptive version of Theorem 1.4.2 by estimating $k_{U(n)}$ in Equation (1.20). Indeed, interestingly, the sequence $k_{U(n)}$ can be written in terms of the self-nested diagonals of the Archimedean copula introduced by [E14]. The interested reader is referred to Section 2.2.1 for definitions and technicalities. One can write $k_{U(n)} = n \left(1 - \delta_{r(n)}(\alpha)\right)$, where $r(n) := \log \left(1 - \left(1 - \frac{k(n)}{n}\right)^{1/(d-1)}\right) / \log(d)$ and $\delta_{r(n)}$ as Definition 2.2.1. Therefore, using the non-parametric estimator $\hat{\delta}_{r(n)}$ presented in Section 2.2.1, we get the estimator

$$\hat{k}_{U(n)} = n \left(1 - \hat{\delta}_{r(n)}(\alpha)\right), \quad \text{for } \alpha \in (0, 1). \quad (1.22)$$

Consistency of $\hat{k}_{U(n)}$ in (1.22) is given by Lemma 2.2 in [E5] and an adaptive version of Theorem 1.4.2 with $\hat{k}_{U(n)}$ in (1.22) instead of $k_{U(n)}$ in (1.20) is also obtained.

Finite-sample performance of our estimator for two different distributional models can be observed in Figure 1.3. However, we refer to [E5] for a deep simulation study.

Extreme extrapolation method for $\delta_{\text{CE}}$ in (1.6)

We first solve the constrained optimization problem for CE design realization in (1.6) in the Archimedean copula framework. In this case, a closed-form expression is provided:
Proposition 1.4.3 (CE design realization in the Archimedean copula setting)
Let $X = (X_1, \ldots, X_d)$ be the considered random risk vector. Assume that $X$ follows an Archimedean copula as in (1.7) with a twice differentiable generator $\phi$. The solution $\delta_{CE}(\alpha)$ of the maximization problem in Equation (1.6) is given by

$$
\delta_{CE}(\alpha) = \left\{ F_1^{-1}\left( \phi\left( \frac{\phi^{-1}(\alpha)}{d}\right) \right), \ldots, F_d^{-1}\left( \phi\left( \frac{\phi^{-1}(\alpha)}{d}\right) \right) \right\}.
$$  \hfill (1.23)

Sketch of proof. One can write the constrained optimization problem in (1.6) as

$$
\arg \max_{\phi(s_1), \ldots, \phi(s_d)} C(\phi(s_1), \ldots, \phi(s_d)), \quad \text{s.t.:} \quad \sum_{i=1}^d s_i = \phi^{-1}(\alpha), \quad \text{with} \quad s_i \geq 0, \quad \text{for} \quad i \in I.
$$

Firstly, we prove that $C(\phi(s_1), \ldots, \phi(s_d))$ is a schur-concave function (see, e.g., Marshall et al. [137], Durante [70]). Then, by taking $s_i = \phi^{-1}(v_i)$, for $i \in I$, from Theorem 2.21 in Boche and Jorswieck [29], the global maximum in (1.6) is achieved by $v^* = \left( \phi\left( \frac{\phi^{-1}(\alpha)}{d} \right), \ldots, \phi\left( \frac{\phi^{-1}(\alpha)}{d} \right) \right)$. By using the Probability Integral Transform Theorem for each marg, we obtain the result.

Estimation procedure
Using Proposition 1.4.3, we now propose a non-parametric estimation procedure for each component of $\delta_{CE}(\alpha)$ in Equation (1.23) for extreme value $\alpha := \alpha_n \to 1$, for $n \to \infty$.

Remark that, the $i$-th component of $\delta_{CE}(\alpha)$ in Equation (1.23) can be written as the $(1 - p)$-quantile of the random variable $X_i$ with $p = 1 - \phi\left( \frac{\phi^{-1}(\alpha)}{d} \right)$, for $i \in I$.

Then, in [E2], we aim to estimate

$$
\hat{\delta}_{CE,p_n}^i := U_{X_i}\left( \frac{1}{p_n} \right), \quad \text{for} \quad i \in I,
$$

for $p_n := 1 - \phi\left( \frac{\phi^{-1}(\alpha_n)}{d} \right) \to 0$, as $n \to +\infty$, and $U_X(t) := F_i^{-1}(1 - \frac{1}{t})$, for $t > 1$. Therefore, the final estimator is based on the following level plug-in procedure:

1. **Estimation of the random risk level $p_n$.** The starting point here is the fact that $p_n$ can be written as $p_n = 1 - F_{V_i}^{-1}(\alpha_n)$, with $Y := \max\{V_1, \ldots, V_d\}$ and $V_i = F_i(X_i)$, for $i \in I$. So, one can estimate $\hat{p}_n = 1 - F_{V_i}^{-1}(\alpha_n)$ by using the same approximation technique as in Equation (3.1.6) in de Haan and Ferreira [58].

2. **Estimation of $\hat{\delta}_{CE,p_n}^i$ in (1.24).** Since $U_{X_i} \in \mathcal{RV}_{\gamma_i}$, $\hat{\delta}_{CE,p_n}^i \sim U_{X_i}\left( \frac{n}{k_i} \right) \left( k_i / n p_n \right)^{\gamma_i}$, where $k_i = k_i(n) \to \infty$, $k_i(n)/n \to 0$, as $n \to \infty$. Then, we define

$$
\hat{\delta}_{CE,p_n}^i = X_{n-k_i,n}^i \left( k_i / n p_n \right)^{\gamma_i}
$$

with $\hat{p}_n$ from step (1) and $\hat{\gamma}_i$ the Hill estimator (see Hill [108]).

Consistency results
For the proposed estimator we provide:

$\Rightarrow$ A central limit theorem for $\hat{p}_n$ (see Equation (12) in Theorem 3.1 in [E2]).

This result is given using a $2\mathcal{RV}$ condition for the inverse Archimedean generator $\psi$.

$\Rightarrow$ A central limit theorem for $\hat{\delta}_{CE,p_n}^i$ (see Equation (14) in Theorem 3.1 in [E2]).

This result is given using a $2\mathcal{RV}$ condition for $U_{X_i}$. 


Figure 1.4: Critical iso-surface $\partial L_\alpha$ for $\alpha \approx 0.946537$ (1000 years return period, see Salvadori et al. [165]). The star and the dot markers indicate, respectively, the estimator of $\delta_{CE}$ given by Salvadori et al. [165] and our $\hat{\delta}_{CE,\bar{P}_n}$ for the Ceppo Morelli dam data-set.

We conclude this section with a visual representation of $\hat{\delta}_{i,CE,\bar{P}_n}$, for $i = 1, 2, 3$ for the flood peak, volume and initial water level of the Ceppo Morelli dam data-set (see Salvadori et al. [165]). We display the estimator of $\delta_{CE}$ in (1.6) given by Salvadori et al. [165] and our $\hat{\delta}_{CE,\bar{P}_n}$ constructed above (see Figure 1.4). For further details, see Section 5 in [E2].

It should be noted that the Component-wise Excess design realization represented in Figure 1.4 is a point that has the greatest probability of being component-wise exceeded by an extreme realization with return period larger than 1000 years. Therefore, this point could be interpreted as a “safety lower-bound”. Which means, that the structure under design should support realizations having multivariate size the $\delta_{CE}(\alpha)$ for the 1000-years return period. Hence, the underestimation of this quantity can represent a major risk for dam managers and for environmental practitioners.

**Extreme extrapolation method for $\text{CTE}_\alpha$ in Definition 1.2.3**

As remarked before, the Kendall process based estimator of $\text{CTE}_\alpha$ proposed in [E11] and presented in Section 1.4.2 performs well for moderate-high (but not extreme) risk levels $\alpha$. The same consideration holds true for the plug-in estimator of $\text{CTE}_\alpha$ proposed in [E10] and [E18]* and described in Section 1.4.1.

However, risk analysts are frequently interested in extreme risk levels $\alpha$. In this sense, in [E20], we propose a new semi-parametric estimator for $\text{CTE}_\alpha$, based on statistical extrapolation techniques, well designed for extreme risk levels and we prove a central limit theorem. This theoretical result is obtained by adapting results in Cai et al. [36].

**1.5 Perspectives**

In a future perspective, it would be interesting to discuss extensions of our measures to the case of discrete distribution functions, using “discrete multivariate level sets” (see, for instance, Lee and Prékopa [132]).

Another subject of future research should be to experimentally compare the proposed upper and lower-orthant VaR and CTE with other existing multivariate generalizations of these two classic risk measures. Furthermore, Adrian and Brunnermeier [5] defined a systemic risk measure,
called $\Delta \text{CoVaR}$. The in-depth study of $\Delta \text{CoVaR}$ systemic risk measures using our multivariate CoVaR is an interesting open problem.

Due to the fact that the Kendall distribution is not analytically known for elliptical random vectors, it is still an open question whether components of our proposed measures are increasing with respect to the risk level $\alpha$ for such dependence structures. However, numerical experiments in the case of Gaussian copulas support this hypothesis. More generally, the extension of the McNeil and Nešlehová representation (see McNeil and Nešlehová [141]) and the study of the behavior of distribution $[U_i | C(U) = \alpha]$, for $i \in I$, for a generic copula $C$, with respect to $\alpha$, are potential improvements to this chapter.

An interesting future work could be a focus on the optimal choice for the parameter $T_n$ in Sections 1.3.1 and 1.4.1. The asymptotic normality of plug-in level set estimates in the distribution function case by adapting Mason and Polonik [138] is an open problem.

Furthermore, the proposed methods are based on an $i.i.d.$ sample framework. It would be interesting in a future work to include other more realistic types of serial correlation structures and to analyze how this affects the performance of the proposed procedure.

In the last section of this chapter, we focus on the estimation of the multivariate quantile of the hydrological load acting on the structure, i.e., the spillway of the dam. However, in order to evaluate the safety of the dam, one has to consider also the interaction between the hydrological load and the structure. Volpi and Fiori [195] point out the importance of considering the structure in hydraulic design and/or risk assessment problems in a multivariate environment and advise against the uncritical use of design event-based approaches. Indeed, the relationship between the structure and the hydrological loads acting on it is neglected in present chapter and could represent an interesting future study.

Some relevant related literature

In the following we present some papers, among others, where the multivariate measures introduced in this chapter are the starting point for further research. In particular in the works presented below, our measures are theoretically investigated or used for applications or inferential purposes.

- H"urlimann [117] derives tractable integral formulas for our multivariate VaR and CTE in Definitions 1.2.2 and 1.2.3 in the Archimedean copula case. Furthermore he proves that, in case marginal risks satisfy the subadditivity property, the marginal CTE components in Definition 1.2.3 are also sub-additive and hitherto coherent risk measures in the usual sense.

- Further properties and calculation of our multivariate VaR in Definition 1.2.2 are given in Lee and Prékopa [132] and Prékopa [156].

- A directional version of our multivariate VaR in Definition 1.2.2 is proposed by Torres et al. [191] and some axiomatic properties are also derived. Furthermore, Torres et al. [190] identify multivariate extremes and analyze environmental phenomena in terms of the directional version of our multivariate VaR, which allows to consider the data in interesting directions that can better describe for instance an environmental catastrophe.

- To order our multivariate VaR and CTE measures, Sordo [182] defines a multivariate extension of the increasing convex order. This order provides a natural extension of the univariate properties and it is based on a vector-valued functional that capture certain aspects of preferences among vectors.
Chapter 2

Transformation copula models

This chapter is based on papers [E15], [E14], [E8], [E6], [E4] and [E3] in my list of publications. The aforementioned works are divided in two different and complementary parts. After a short introduction, in Section 2.2 we study some global transformation models. These transformations act on the whole domain of the considered distribution function (see [E15], [E14], [E8] and [E4]). Secondly, in Section 2.3 we analyse some local transformation models. These transformations partially modify the copula function in order to fix a target tail behaviour (see [E6] and [E3]). We conclude with some research perspectives on this topic and with an overview of recent papers (written by other researchers) where our transformation copula models are further investigated.

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2.1 Introduction

In this chapter we deal with the study of multivariate distributions (and in particular copulas) by using probability transformations. In our papers, on which the present chapter is based, we show that using transformations has several advantages compared to using classic parametric multivariate distributions. In particular, we focus on:

- the possibility to get analytical expressions both for the multivariate distribution function and for the associated critical layers \( \partial L(\alpha) \), previously introduced in Chapter 1 (see [E15] and [E8]);
- the huge variety of reachable distributions (multimodal, non-exchangeable vector etc.), with the possibility to improve the fit by adding parameters via transformations composition (see [E15] and [E8]);
- some estimation facilities, with the possibility to get very good initial values for parameters even with a large number of parameters (see [E15] and [E8]);
- the possibility to ensure admissibility condition for the transformed distribution or transformed copula (see [E15], [E14], [E4] and [E3]);
- some possible distortions of Archimedean copulas able to deal with asymmetry (see [E15] and [E4]);
- the definition of equivalence classes to select some standardized transformation forms for practical use, for the comparison and the interpretation of the resulting distribution functions (see [E14]);
- a straightforward non-parametric estimation procedure based on the diagonal copula sections (see [E14] and [E8]);
- some transformations allow to build Archimedean generators exhibiting any chosen couple of lower and upper tail dependence coefficients (see [E6] and [E3]).

In the recent literature, many types of transformations of copulas have been considered (see, e.g., Valdez and Xiao [192] or Michiels and De Schepper [145] for a review of some existing transforms). Transformations of bivariate copula, semicopulas and quasi-copulas are studied in Durante and Sempi [74]. Klement et al. [124, 125] focused on transformations of bivariate Archimax copulas. Power transformations of Archimedean generators are proposed in Nelsen [148] (see Theorem 5.4.4).

Most of these copula transformations (but not all of them) are based on an initial copula $C_0$ and on a bounded transformation function $T$. The idea is to exploit characteristics of both functions, $C_0$ and $T$, to generate new copulas from initial ones. Following this literature, in [E15], [E14], [E8] and [E6] we investigate properties of this type of transformation copula model.

More precisely, we take the transformation $T : [0, 1] \to [0, 1]$, where $T$ is an increasing continuous bijection, with $T(0) = 0$ and $T(1) = 1$. We consider here a transformed copula $\tilde{C}_{T,C_0}$, which is transformed from an initial copula $C_0$ using $T$, i.e.,

$$\tilde{C}_{T,C_0}(u_1, \ldots, u_d) = T \circ C_0 \left( T^{-1}(u_1), \ldots, T^{-1}(u_d) \right).$$ (2.1)

The initial copula $C_0$ in (2.1) is not estimated but it is chosen. So it can represent some kind of a priori belief on dependence structure of the data or on the considered problem, that will be transformed in order to improve the fit. Transformations as in (2.1) have been considered for example in Durrleman et al. [75], Valdez and Xiao [192] (Definitions 3.6, in dimension $d = 2$), Hofert [111]. Applications of transformations of copulas to pricing credit derivatives are given in Crane and van der Hoek [52].

A crucial question that we try to investigate in this chapter is the possibility to use the global transformation model in (2.1) to improve the fit on the central part of a given multivariate data-set, starting from an initial copula $C_0$ (see Section 2.2).

Moreover, in this chapter, we will pay particular attention to the Archimedean class of initial copulas as in Equation (1.7) (see [E14], [E6], [E4] and [E3]). In the following, a $d-$ monotone generator, will be called valid generator in dimension $d$ with associated valid copula (see Theorem 2.2 in McNeil and Neslehová [141]).

By using the model in (2.1), an explicit parametric expression of both the generator and its inverse function is available. Indeed, when $\tilde{C}_{T,C_0}$ in (2.1) is a copula, $\tilde{C}_{T,C_0}$ is still Archimedean with transformed generator $\tilde{\phi} = T \circ \phi_0$. However, the Archimedean setting is not always suitable in real-life applications. For example, the dependence between assets from the same industry sector is typically very different from the one that we observe for assets that belong to different sectors. Hence, copulas that accommodate asymmetry are useful. With these considerations in mind, in Section 2.2.3 we investigate extensions of the Archimedean copula family in (1.7) that make it possible to deal with asymmetry (see also [E4]).
2.1. Introduction

Notice that, in general, $\tilde{C}_{T,C_0}$ is not necessarily a copula. The verification of the validity of the transformed generator $\hat{\phi}$ requires the calculation of the signs of its $d$ first derivatives (in the case it is $d$ times differentiable), which can be difficult, especially when $d$ is greater than two, when Faà Di Bruno formula is involved. In Section 2.2.1 we present some results on the admissibility of the transformation $T$ (see also [E14]).

As stated above, in the first part of this chapter (Section 2.2), we will not focus on extreme tail behavior, but on the central part of the $d$–dimensional distribution. Using the necessary few data in the tails would require some specific estimators of tail dependence. This interesting problem is investigated in the second part of the present chapter (Section 2.3). We focus on the possibility to modify the tail dependency by using a transformed distribution without changing the global/central adjustment. More precisely, in Section 2.3.1 we try to answer this question by using the transformation model in (2.1). With the same purpose, in Section 2.3.2, another strategy based on the upper-patched generator is proposed.

Indeed, depending upon targeted applications, understanding the tail behaviour of a copula is of great importance. In many practical problems, like hydrology, finance, insurance, etc. one needs to understand the risk of simultaneous threshold crossing for the considered random variables.

A possible answer strategy by using multivariate crossing models will be presented in Chapter 3. Tail dependence coefficients in the general multivariate case (as defined in De Luca and Rivieccio [61], Lindgren [133]) can also help in this task, as we will see in the present chapter.

**Definition 2.1.1 (Multivariate tail dependence coefficients)** Let the copula $C$ be the distribution of some random vector $U := (U_1, \ldots, U_d)$ with uniform marginals. Recall that $I = \{1, \ldots, d\}$ and consider two non-empty subsets $I_h \subset I$ and $\bar{I}_h = I \setminus I_h$ of respective cardinal $h \geq 1$ and $d - h \geq 1$. Provided that the limits exist, a multivariate version of classic bivariate tail dependence coefficients in Sibuya [178] and Joe [118] is given by

$$
\lambda_{L}^{I_h,\bar{I}_h} = \lim_{u \to 0^+} \mathbb{P} \left[ U_i \leq u, i \in I_h \mid U_i \leq u, i \in \bar{I}_h \right],
$$

$$
\lambda_{U}^{I_h,\bar{I}_h} = \lim_{u \to 1^{-}} \mathbb{P} \left[ U_i \geq u, i \in I_h \mid U_i \geq u, i \in \bar{I}_h \right].
$$

If for all $I_h \subset I$, $\lambda_{L}^{I_h,\bar{I}_h} = 0$, (resp. $\lambda_{U}^{I_h,\bar{I}_h} = 0$) then we say $U$ is lower tail independent (resp. upper tail independent).

Tail dependence measurements in Definition 2.1.1 have been proposed in the literature to explain the asymptotic probability that all random variables in a given set become large, given that random variables of another set are also large. For instance, given $d$ financial asset returns, $\lambda_{U}^{I_h,\bar{I}_h}$ (resp. $\lambda_{L}^{I_h,\bar{I}_h}$) can be interpreted as the probability of very high (resp. low) returns for $h$ assets provided that very high (resp. low) returns have occurred for the remaining $d - h$ assets (for applications in finance and insurance risk management see Durante et al. [71], Embrechts et al. [83] and Lindgren [133]).

The interested reader is referred to Charpentier and Segers [44] for a very precise analysis of the Archimedean copulas tail behaviour in terms of regular variation of the additive generator, including many developments in the (difficult) cases that are close to asymptotic independence. Furthermore some results about tail dependence coefficients of certain transforms of Archimedean copula are given by Hofert [111]. Larsson and Nešlehová [131] proved that the extremal behaviour of a $d$-dimensional Archimedean copulas can be deduced from the associated stochastic representation proposed in McNeil and Nešlehová [141].

In practice, one can find non-parametric estimators of tail dependence coefficients, essentially based on the non-parametric copula estimator and extreme value analysis (see, e.g., de Haan
and Ferreira [58], Schmidt and Stadtmüller [171]). However, even when such tail coefficients are perfectly known, as stated in Larsson and Nešlehová [131], it is rather difficult to construct new Archimedean copula models that exhibit specific tail behaviour. The interested reader is also referred to Embrechts et al. [81]. For instance,

- using usual uni-parametric Archimedean generators does not allow to get both lower and upper tail coefficients in the general case where their values belong to $$(0, 1)$$.

- commonly used families of Archimedean copulas (and also of elliptical copulas) have the property that either $$\lambda_U = \lambda_L$$ or that only one of the coefficients is nonzero (see, e.g., Table 1 of Charpentier and Segers [43]).

Larsson and Nešlehová [131] provided an example in order to overcome this restrictive behaviour of classic Archimedean copulas. The two-parameter BB1, BB4 and BB7 copulas in Joe [118] have both upper and lower tail dependence. Cheung [51] proposed an upper-comonotonic copula with bivariate tail dependence coefficients $$\lambda_U = 1$$. Indeed, when a strong tail dependence corresponds to a dangerous situation, it may be a conservative strategy to change $$C_0$$ into the copula which exhibits perfect upper tail dependence. Nelsen [148] presents a tractable power transformation able to change both lower and upper tail dependence coefficients, i.e., $$\tilde{\phi}(x) = \phi_0(x^{1/\beta})$$, for all $$x \in [0, \infty)$$. By construction, the resulting copula is still Archimedean. However, contrary to proposed transformed generators in Section 2.3, this transformation can affect the global shape of the copula substantially (see further details in [E3]).

Hyperbolic conversion functions for transformation copula models For inferential purposes, it is not always convenient to estimate a transformation $$T$$ in (2.1) with bounding constraints on $$[0, 1]$$. It can be helpful to define a transformation on the whole set $$\mathbb{R}$$ and then rescale it in order to ensure that the resulting transformation maps $$[0, 1]$$ into $$[0, 1]$$. For this reason we focus on transformations that are defined by rescaling a particular real function, called conversion function, onto $$[0, 1]$$.

In this chapter we frequently consider transformations $$T_{f,G} : [0, 1] \rightarrow [0, 1]$$ such that

$$T_{f,G}(u) = \begin{cases} 0 & \text{if } u = 0, \\ G \circ f \circ G^{-1}(u) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1, \end{cases}$$

(2.2)

where $$f$$ is any continuous bijective increasing function, $$f : \mathbb{R} \rightarrow \mathbb{R}$$, called conversion function. The transformation $$T_{f,G}$$ has support $$[0, 1]$$, and the function $$G$$ aims at transferring this support on $$\mathbb{R}$$, in order to allow $$f$$ to be defined on the whole set $$\mathbb{R}$$, without bounding constraints. The function $$G$$ is thus chosen as a continuous and invertible c.d.f with support $$\mathbb{R}$$, i.e., such that $$\forall x \in \mathbb{R}, G(x) \in (0, 1)$$.

Furthermore, some particular shapes of conversion functions $$f$$ in (2.2) help fitting transformed multivariate distributions (see, e.g., [E14], [E8] and [E6]). In this sense, it is important to use parametric forms of $$f$$ that allow, by compositing $$f_1 \circ \ldots \circ f_n$$, to fit any increasing bijection of $$\mathbb{R}$$. As an example, it is easy to fit an increasing target bijection of $$\mathbb{R}$$ by a piecewise linear function, which can be seen as the composition of basic angle functions.

Following these considerations, we focus on hyperbolic conversion functions (i.e., smooth versions of angle functions), where $$f(x) = H_{m,h,p_1,p_2,\eta}(x)$$ and

$$H_{m,h,p_1,p_2,\eta}(x) = m - h + \left(p_1^2 + p_2^2\right) \frac{x - m - h}{2} - \left(p_1 - p_2\right) \sqrt{\left(\frac{x - m - h}{2}\right)^2 + \eta^{p_1 + p_2}},$$

(2.3)
with \( m, h, p_1, p_2 \in \mathbb{R} \) and one smoothing parameter \( \eta \in \mathbb{R} \). For further details see also Bienvenu and Rullière [24] and Bienvenu and Rullière [23]. Remark that \( H \) has been chosen in order to have unbounded real parameters, and to be readily invertible: \( H_{m,-h,-p_1,-p_2,\eta}^{-1}(x) = H_{m,-h,1,p_1,1,p_2,\eta}(x) \). Then one can get easily \((T_{H,G})^{-1} = T_{H^{-1},G}\).

As stated before, when the smoothing parameter \( \eta \) tends to \(-\infty\), the hyperbole in (2.3) tends to the angle function:

\[
A_{m,h,p_1,p_2}(x) = m - h + (x - m - h) \left( e^{p_1} 1_{\{x < m + h\}} + e^{p_2} 1_{\{x > m + h\}} \right).
\] (2.4)

As we will see in Section 2.3.1, these hyperbolic transformations have the important advantage of being flexible in terms of the tail parameters estimation.

We are now ready to go into more detail concerning the results presented in this chapter.

2.2 Global transformations

2.2.1 Non-parametric estimation of transformations and generators [E14]

In this section, we focus on the transformation copula model in (2.1). Our goal is to investigate some admissibility conditions for the transformation \( T \) and to propose a non-parametric estimator for both \( T \) and \( \tilde{\phi} \). These estimators will be provided in any dimensions \( d \geq 2 \). For further details the reader is referred to [E14].

There is a huge literature concerning the estimation of copula structures, see for example Genest and Rivest [100], Joe [119], Scaillet et al. [170], Mendes et al. [144], Autin et al. [12], Hernández-Lobato and Suárez [107]. A comparison of different parametric and non-parametric methods for estimating a copula is given, for example, in Kim et al. [123], essentially in the bivariate case. A particular focus on the dimensionality problem is developed in Embrechts and Hofert [79]. A non-parametric rank-based estimator for the generator of Archimedean copula has been recently proposed by Genest et al. [99]. However this estimator is constructed using successive numerical resolutions of root. Conversely, the proposed estimation procedure in [E14] does not rely on any numerical resolution of root or optimization, in order to simplify both practical use and theoretical analysis.

Our construction is mainly based on the diagonal section of a copula, which is a central tool for Archimedean copulas (see, e.g., Nelsen et al. [150]). The tractable expressions of the obtained estimators play a central role both in the numerical implementation and in the construction of confidence bands (for further details see Section 4.3 in [E14]).

Main contributions

First, we show that

\[
\text{ when the initial copula } C_0 \text{ is the independent one, and when } T \text{ is } d \text{ times differentiable, we can find necessary and sufficient admissibility conditions for } T \text{ in the transformation model (2.1) (see Proposition 2.5 in [E14]).}
\]

Admissibility conditions in the bivariate case are also studied in [E15]. Notice that it can be difficult to obtain tractable admissibility conditions for \( \tilde{C}_{T,C_0} \) in (2.1), also in particular simplified cases. For this reason, in Section 2.3.2 we propose an alternative transformation model such that the transformed generator \( \tilde{\phi} \) is easily guaranteed to be a valid Archimedean generator.
Remark 5 (The problem of uniqueness) Notice that generators and transformations leading to a given copula are not unique (see, e.g., Nelsen [148]). The definition of equivalence classes for both transformations and generators in (2.1) is necessary to select some standardized forms for practical use, for comparison and interpretation of resulting distribution functions. To deal with this problem, we define some equivalence classes for $\tilde{\phi}$ and $T$. For the sake of brevity, these results are omitted here and the interested reader is referred to Section 2.2 in [E14].

The self-nested diagonals introduced below will be the central tool for the construction of our non-parametric estimators.

Definition 2.2.1 (Discrete and extended self-nested diagonals) Consider a $d$-dimensional copula $C$ such that for all $u \in [0,1]$, the diagonal section $\delta_1(u) := C(u, \ldots, u)$ is a strictly increasing function of $u$.

The respective discrete self-nested diagonal of $C$ of order $k$ and $-k$ are the functions $\delta_k$ and $\delta_{-k}$ such that for all $u \in [0,1]$, for all $k \in \mathbb{N}$,

\[
\begin{align*}
\delta_k(u) &= \delta_1 \circ \cdots \circ \delta_1(u), \quad (k \text{ times}) \\
\delta_{-k}(u) &= \delta_{-1} \circ \cdots \circ \delta_{-1}(u), \quad (k \text{ times}) \\
\delta_0(u) &= u,
\end{align*}
\]

(2.5)

where $\delta_{-1}$ is the inverse function of $\delta_1$, so that $\delta_1 \circ \delta_{-1}$ is the identity function.

Functions of the family $\{\delta_r\}_{r \in \mathbb{R}}$ are called (extended) self-nested diagonals of a copula $C$, if $\delta_k(u)$ is as in (2.5), for all $k \in \mathbb{Z}$, and if furthermore

\[
\delta_{r_1+r_2}(u) = \delta_{r_1} \circ \delta_{r_2}(u), \quad \forall \ r_1, r_2 \in \mathbb{R}, \forall \ u \in [0,1].
\]

Discrete self-nested diagonals presented in Definition 2.2.1, correspond to the $k$-fold composition of the diagonal section $\delta_1$ of the copula (see Wysocki [197]). They are defined for $k \in \mathbb{Z}$ (hence justifying the prefix discrete). They can be linked with what is defined as iterates of the diagonal of a t-norm, and with $T$-powers in Alsina et al. [9] (see Lemma 1.3.5. of this book for example, in dimension $d = 2$).

The existence of extended self-nested diagonals of a copula $C$ is automatically guaranteed when $C$ is an Archimedean copula: $\delta_r(x) = \phi(d^r \cdot \phi^{-1}(x))$, for $x \in (0,1)$ and $r \in \mathbb{R}$ (see Lemma 3.4 in [E14]).

We now provide expressions for the transformation $T$ and the transformed generator $\tilde{\phi}$ in terms of the extended self-nested diagonals in Definition 2.2.1.

Proposition 2.2.1 (New expressions of $T$ and $\tilde{\phi}$ using self-nested diagonals)

Consider an Archimedean copula $C_0$ with generator $\phi_0$ and a transformed copula $\tilde{C}$ with generator $\tilde{\phi}$ as in model (2.1). Consider the two associated families of self-nested diagonals $\delta_r^0$ and $\delta_r$, $r \in \mathbb{R}$.

i. If $T(x_0) = y_0$, then the transformation $T$ is such that $T(0) = 0$, $T(1) = 1$ and, for all $x \in (0,1)$,

\[
T(x) = \delta_{r(x)}(y_0), \quad \text{with} \ r(x) \text{ such that} \ \delta_{r(x)}^0(x_0) = x
\]

i.e., $r(x) = \frac{1}{\ln \phi_0(y_0)} \ln \left( \frac{\phi_{-1}(x)}{\phi_0^{-1}(x_0)} \right)$, where $(x_0, y_0) \in (0,1)^2$ can be arbitrarily chosen.
2.2. Global transformations

The generator \( \tilde{\phi} \) is such that, for all \( t \in \mathbb{R}^+ \setminus \{0\} \),

\[
\tilde{\phi}(t) = \tilde{\delta}_{\rho(t)}(y_0), \quad \text{with} \quad \rho(t) = \frac{1}{\ln d} \ln \left( \frac{t}{\phi_0^{-1}(x_0)} \right),
\]

where \((x_0, y_0) \in (0,1)^2\) can be arbitrarily chosen.

Proof of Proposition 2.2.1 is based on the specific expression of the extended self-nested diagonals in the \(d\)-dimensional Archimedean case and on the equivalence classes for both transformations and generators defined in [E14] (see also Remark 5).

Remark 6 The expression of \( \tilde{\phi} \) in Proposition 2.2.1 does only depend on \( \phi_0 \) via the constant \( t_0 = \phi_0^{-1}(x_0) \). In particular, choosing an initial copula \( C_0 \) and constants \((x_0, y_0) \) in \((0,1)^2\) can be simply reduced to the choice of \((t_0, \varphi_0) \in \mathbb{R}^+ \setminus \{0\} \times (0,1)\) such that \( \tilde{\phi}(t_0) = \varphi_0 \), with \( t_0 = \phi_0^{-1}(x_0) \) and \( \varphi_0 = y_0 \). Then, item ii. can be also written as: \( \tilde{\phi}(t) = \tilde{\delta}_{\rho(t)}(\varphi_0) \), with \( \rho(t) = \frac{1}{\ln d} \ln \left( \frac{t}{t_0} \right) \). Notice that the role of \((x_0, y_0)\) and \((t_0, \varphi_0)\) is to select a generator among all equivalent generators (see also Remark 5).

Based on Proposition 2.2.1 for Archimedean families of copulas, we provide in [E14] some straightforward non-parametric estimators for \(T\) and \(\tilde{\phi}\) (see Algorithm 1 below). To this aim, we assume that an estimator of the diagonal of the copula \(\delta\) and of \(\delta_{-1}\) are available. We denote respectively \(\hat{\delta}_{\rho_1}\) and \(\hat{\delta}_{\rho_{-1}}\) these estimators. Remark that some consistent estimators for \(\delta_1\) and \(\delta_{-1}\) are provided in the literature (see, e.g., Deheuvels [62], Deheuvels [63], Fermanian et al. [91]). A comparison between several estimators is presented in Omelka et al. [154]. Relevant papers related to the convergence of empirical copula process are also Rüschendorf [162] and Segers [173]. The interested reader is referred to Segers [174] where an extension of the empirical copula is considered by using a hybrid estimator.

In Algorithm 1 below, we summarize the principal steps of this estimation procedure. However, some technicalities are omitted here and the interested reader is referred to Section 4.1 in [E14].

Remark 7 (Identifiability problem and Archimedean diagonal sections) The diagonal of an Archimedean copula is essential to describe it. Indeed, it is known that if \(C\) is an Archimedean 2-copula with diagonal section \(\delta\), and \(\delta'(1^-) = 2\), then \(C\) is uniquely determined by its diagonal section (see Frank [96]). Moreover, in Erdely et al. [89] the authors prove that for \(d \geq 3\), if \(C\) is an Archimedean \(d\)-copula whose diagonal section \(\delta\) satisfies \(\delta'(1^-) = d\), then \(C\) is uniquely determined by its diagonal. Some constructions of copulas starting from the diagonal section are given for example in Nelsen et al. [150] and Wysocki [197]. A counterexample to a statement asserted in Erdely et al. [89] concerning the generator and the diagonal section of an Archimedean copula is provided in Fernández-Sánchez et al. [93].

Notice that estimators based on the diagonal section proposed in this section only use partial information about the dependence and thus might not be efficient in order to capture tail dependence (as was pointed out by Hofert et al. [113]). Indeed, if the condition \(\delta'(1^-) = 2\) is not fulfilled, there may be more than one Archimedean copula with diagonal \(\delta\). An example is constructed in Alsina et al. [9] (page 155). To show that the situation of many Archimedean copulas having the same diagonal is far from exceptional, a recipe to construct further examples is given in Segers [172].
Algorithm 1 Sketch of non-parametric estimation procedure for $T$ and $\tilde{\phi}$

**Input parameters**

Choose $(x_0, y_0)$, arbitrary values in $(0, 1)^2$
Choose $(t_0, \varphi_0)$, arbitrary values in $\mathbb{R}^+ \setminus \{0\} \times (0, 1)$
Choose $C_0$ and $\varphi_0$

respectively, the initial $d$-dimensional Archimedean copula and the associated generator
Choose $\hat{\delta}_1$, an estimator for $\delta_1$, and its inverse $\hat{\delta}_1^{-1}$

**Estimation**

Define the function $\hat{\delta}_k(x)$ (resp. $\hat{\delta}_{-k}(x)$), for $x \in [0, 1]$
by using the $k-$fold composition of $\hat{\delta}_1$ (resp. $\delta_{-1}$), for $k \in \mathbb{N} \setminus \{0\}$, see Equation (2.5)

Define the function $\hat{\delta}_r(x)$, for any $r \in \mathbb{R}$ and $x \in [0, 1]$
by using an interpolation of $\hat{\delta}_k$ and $\hat{\delta}_{k+1}$, with $k = \lfloor r \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part (see Lemma 3.6 in [E14])

Get $\hat{T}(x)$, for any $x \in [0, 1]$ using Proposition 2.2.1
i.e., $\hat{T}(x) = \hat{\delta}_{r(x)}(y_0)$, with $r(x) = \frac{1}{\ln d} \ln \left( \frac{\varphi_0^{-1}(x)}{\varphi_0^{-1}(x_0)} \right)$

Get $\hat{\phi}(t)$, for all $t \in \mathbb{R}^+ \setminus \{0\}$ using Proposition 2.2.1
i.e., $\hat{\phi}(t) = \hat{\delta}_{\rho(t)}(\varphi_0)$, with $\rho(t) = \frac{1}{\ln d} \ln \left( \frac{t}{t_0} \right)$

---

Then, some non-parametric estimators of the Archimedean generator, like our estimator or the one described in Genest et al. [99], do not appear to be well-adapted to describe the upper tail dependency in the Archimedean multivariate structures (see Embrechts and Hofert [77]). A deep analysis about this drawback is developed in Section “Upper tail dependence” in [E14].

Finally, convergence properties of estimators in Algorithm 1 are studied in Section 4.2 in [E14]. We quantify the estimation error of $\hat{\phi}$ in terms of the error of the estimation of $\hat{\delta}_1$.

$\rightarrow$ First, we derive the estimation error on any $r-$fold composition $\hat{\delta}_r$, for $r \in \mathbb{R}$ (see Proposition 4.8 in [E14]).

$\rightarrow$ Then, we use the previous result to control the estimation error of $\hat{\phi}$ by giving some confidence bands (see Proposition 4.9 in [E14]).

In the aforementioned paper, we compare the finite sample performance of our generator estimate with one proposed in Genest et al. [99] by using $\lambda$ function (originally introduced in Genest and Rivest [100] for inferential purposes).

Estimation of the transformed copula $\tilde{C}$ proposed above can be represent the starting point to fit the multivariate distribution of a risk vector $X$. This is the main purpose of the following section.

2.2.2 Transformed distributions and critical layers [E8; E15]

By using the copula transformation model discussed in Section 2.2.1, this section aims at:

- giving a parametric representation of the multivariate distribution $F$ of a risk vector $X$
  (in [E8] we deal with hydrological risks, in particular $X$ presents rain measurements, in [E15] insurance losses data-set is considered);
- giving direct estimation procedure for this representation;

- giving closed parametric expressions, both for critical layers \( \partial L(\alpha) \) in Definition 1.2.1 and Kendall distribution \( K(\alpha) \) (see Section 1.3.2);

- adapting this methodology to some asymmetric dependencies (as, for instance, non-exchangeable random vectors) with the possibility to improve the fit by adding parameters (via transformations composition);

- showing some estimation facilities, as the possibility to get very good initial values for parameters even with a large number of parameters.

To answer the issues introduced above, we consider the following distributional model:

\[
\tilde{F}(x_1, \ldots, x_d) = \tilde{C}(\tilde{F}_1(x_1), \ldots, \tilde{F}_d(x_d)), \quad \text{with (2.6)}
\]

\[
\tilde{C}(u_1, \ldots, u_d) = T \circ C_0(T^{-1}(u_1), \ldots, T^{-1}(u_d)), \quad \text{(2.7)}
\]

\[
\tilde{F}_i(x) = T \circ T_i^{-1} \circ F_i(x), \quad \text{for } i \in I, \quad \text{(2.8)}
\]

where \( F_1, \ldots, F_d \) are given initial marginal distribution functions, and where \( C_0 \) is a given initial Archimedean copula with generator \( \phi_0 \) as in (1.7). Hence the distribution \( \tilde{F}(x_1, \ldots, x_d) \) is built from transformed marginals \( \tilde{F}_i, i \in I \) and a transformed Archimedean copula \( \tilde{C} \) as in (2.1), under regularity conditions. Internal transformations \( T_i : [0, 1] \rightarrow [0, 1] \) are continuous non-decreasing functions, such that \( T_i(0) = 0, T_i(1) = 1 \), for \( i \in I \). For further details the reader is referred to [E8] and [E15].

Notice that the class of reachable copulas \( \tilde{C} \) does obviously rely on the initial copula \( C_0 \). The impact of the choice of the initial copula \( C_0 \) has been studied in Section 2.3 in [E15]. Furthermore the relationship between the copula transformation \( T \) and the changing of the risk level \( \alpha \) in the associated critical layers \( \partial L(\alpha) \) is investigated in Section 2.1 in [E15].

Main contributions

In the following we detail a semi-parametric estimation procedure in order to easily fit the distributional model in (2.6)-(2.8) (preliminaries and details can be found in [E15]). The proposed estimation is straightforward, it has a tunable number of parameters and it does not rely on any optimization procedure. Furthermore this adjustment is flexible, it can be adapted to different types of data (see Section 5 in [E15] for a multimodal distribution example; see Section 7 in [E8] for a nested model in the case of a non-exchangeable vector).

Non-parametric estimation

We use here a smooth estimator \( \hat{\delta}_1 \) of the diagonal section and of its inverse \( \hat{\delta}_{-1} \), detailed in Omelka et al. [154], and directly inspired by the one of Deheuvels [62]. Remark that, as a consequence of Equation (2.8), the main problem in our procedure is to estimate the external transformation \( T \). By using Algorithm 1 in Section 2.2.1, one can get Algorithm 2 below.

Since we aim here at proposing some parametric estimators for transformations \( T \) and \( T_i \), we prove in Proposition 2.2.2 below that starting from given thresholds \( q_j^{(i)} \in (0, 1) \), it is possible

\[\text{Notice that the considered Return Period in Salvadori et al. [166] and Gräler et al. [105] can be expressed using Kendall’s function: } \text{RP}_\Delta(\alpha) = \frac{\Delta \alpha}{1 - \alpha}, \text{ where } \Delta \alpha > 0 \text{ is the (deterministic) average time elapsing between two consecutive dangerous events and } \alpha \in (0, 1).\]
Proposition 2.2.2 (Set of points for $T$ and $T_i$). Let $\hat{T}$ and $\hat{F}_i$ as in Algorithm 2. Let $J_i \subset \mathbb{N}$ be finite sets of indexes and $\mathcal{Q}_i =\left\{q_j^{(i)} \right\}_{j \in J_i}$ be a finite given set of targeted quantile levels, $q_j^{(i)} \in (0, 1)$, $j \in J_i$, $i \in I$. Then,

$$\hat{T}_i = F_i \circ \hat{F}_i^{-1} \circ \hat{T} \text{ is passing through all points of } \Omega_i(\mathcal{Q}_i, \hat{T}),$$

where $\Omega_i(\mathcal{Q}, \hat{T}) = \left\{(\alpha_j^i, \beta_j^i) \right\}_{j \in J_i}$, with $\alpha_i^j = \hat{T}^{-1}(q_j^{(i)})$, $\beta_i^j = F_i \circ \hat{F}_i^{-1}(q_j^{(i)})$. \hfill (2.9)

Let $J \subset \mathbb{N}$ be a finite set of indexes and $\mathcal{Q} = \left\{q_j^{(T)} \right\}_{j \in J}$ be a given set of targeted quantile levels, $q_j^{(T)} \in (0, 1)$, $j \in J$. Then, obviously,

$$\hat{T} \text{ is passing through all points of } \Omega(\mathcal{Q}),$$

where $\Omega(\mathcal{Q}) = \left\{(\alpha_j, \beta_j) \right\}_{j \in J}$, with $\alpha_j = q_j^{(T)}$, $\beta_j = \hat{T}(\alpha_j)$. \hfill (2.10)

Proof is based on the fact that all considered estimators are supposed to be continuous and invertible functions.

Furthermore in [E8], we prove that

$\Rightarrow$ using $\Omega_i$ in (2.9), one can select a further parametric estimator $T_{\Omega_i}$ passing through points of $\Omega_i$ and such that

$$\hat{F}_{\Omega_i}^{-1}(q) = \hat{F}_i^{-1}(q) \text{ for all } q \in \mathcal{Q}_i,$$

i.e., quantiles of the target $\hat{F}_i$ are identified with quantiles of the transformed margin using $T_{\Omega_i}$, for each chosen quantile level $q \in \mathcal{Q}_i$ and for any $i \in I$ (see Proposition 2.2 in [E8]).

$\Rightarrow$ An analogous (weaker) result is also proved for the external transformation $T$ (see Proposition 2.3 in [E8] for further details).
Given the considered subsets $\Omega$ and $\Omega_i$, $i \in I$, one can get piecewise linear estimators. However, these estimators are not differentiable everywhere on $(0, 1)$. Then, in the following, we provide differentiable parametric estimators passing through the points of $\Omega$ and $\Omega_i$, $i \in I$.

### Parametric estimation

We consider the particular class of $T_{f,G}$ given in (2.2) with $f$ an hyperbolic conversion function as in (2.3). Furthermore, we chose here $G(\cdot) = \logit^{-1}(\cdot)$. As proved in Proposition 3.1 in [E8] (see also Proposition 4.1 in [E15]), in the chosen hyperbolic conversion model one can get very good initial values for the parameters. In the following, we summarize this result (see Proposition 3.1 in [E8] and Proposition 4.1 in [E15]).

**Proposition 2.2.3 (Suited parameters from a given set of points)** Given a set of points $\Omega$, one can build a set of parameters $\theta$, such that

- **a.** a piecewise linear composition of angles in (2.4) with parameters $\theta$, called $A_{\theta}$, links point $(0, 0)$, points of $\Omega$, and point $(1, 1)$.
- **b.** an associated composition of hyperboles in (2.3) with parameters $\theta$, called $H_{\theta, \eta}$, converges pointwise to $A_{\theta}$, as $\eta$ tends to $-\infty$.

Then, the continuous and differentiable transformation $H_{\theta, \eta}$ can fit as precisely as desired the given set of points $\Omega$, when $\eta$ tends to $-\infty$. From Proposition 2.2.3, we define our final smooth parametric estimators $T$ and $T_i$, $i \in I$, respectively for external and internal transformations (see Definition 2.2.2 below). They are passing respectively through the sets of points $\hat{\Omega}(Q)$ and $\hat{\Omega}_i(Q_i, T)$ obtained by the non-parametric preliminary procedure step (see Proposition 2.2.2).

**Definition 2.2.2 (Smooth estimation of $T$ and $T_i$, $i \in I$)** Let $\eta \in \mathbb{R}$ and $\eta_i \in \mathbb{R}$, $i \in I$ be given smoothing parameters. Using Algorithms 1, 2, Propositions 2.2.2 and 2.2.3, one defines

$$
\begin{align*}
\bar{T} &= H_{\bar{\theta}, \eta}, \\
\bar{T}_i &= H_{\bar{\theta}_i, \eta_i}, & \text{for } i \in I,
\end{align*}
$$

where $\bar{\theta}$ (resp. $\bar{\theta}_i$) is the vector of suited parameters associated to the set of points $\hat{\Omega}(Q)$ (resp. $\hat{\Omega}_i(Q_i, T)$) built as in Proposition 2.2.2.

Notice that once given thresholds sets $Q$ and $Q_i$ and smooth parameters $\eta$ and $\eta_i$, all estimated parameters are directly and analytically defined, so that we do not need here any inversion or optimization procedure.

### Final parametric results

The previous parametric model allows to get various analytical results for both the multivariate distribution function in model (2.6)-(2.8), its associated critical layers and Kendall’s function (see [E8]).

- $\rightarrow$ The final parametric transformed Archimedean generator in (2.7) is given by $\tilde{\phi}(t) = \bar{T}(\phi_0(t))$.
- $\rightarrow$ The corresponding estimated transformed multivariate distribution in (2.6) is given by

$$
\tilde{F}(x_1, \ldots, x_d) = H_{\bar{\theta}, \eta} \circ C_0(\bar{H}^{-1}_{\bar{\theta}_1, \eta_1} \circ F_1(x_1), \ldots, \bar{H}^{-1}_{\bar{\theta}_d, \eta_d} \circ F_d(x_d)).
$$

(2.11)
The associated parametric $\alpha$-critical layers as in Definition 1.2.1 are given by

$$\partial \tilde{L}(\alpha) = \{ (F_1^{-1} \circ \mathcal{H}_{\hat{\theta}, \eta_1}^{-1}(u_1), \ldots, F_d^{-1} \circ \mathcal{H}_{\hat{\theta}, \eta_d}^{-1}(u_d)), \text{ for } (u_1, \ldots, u_d) \in (0, 1)^d \text{ and } C_0(u_1, \ldots, u_d) = \mathcal{H}_{\hat{\theta}, \eta}^{-1}(\alpha) \}, \text{ for } \alpha \in (0, 1),$$

where a direct analytic expression $\mathcal{H}_{\hat{\theta}, \eta}^{-1}$ is given by Remark 2 in [E8] (see also Proposition 2.4 in [E15]).

The estimated transformed Kendall distribution $\tilde{K}$ is given by

$$\tilde{K}(\alpha) = \alpha + \sum_{i=1}^{d-1} \frac{1}{i!} \left(-\phi_0^{-1}(T^{-1}(\alpha))\right)^i \left(T \circ \phi_0\right)^{(i)} \left(\phi_0^{-1}(T^{-1}(\alpha))\right), \text{ for } \alpha \in (0, 1),$$

where the notation $f^{(i)}$ corresponds to the $i$-th derivatives of a function $f$ (see also Genest and Rivest [101]).

We briefly illustrate the quality of our final parametric estimation in Table 2.1 (left), where the estimated survival diagonal of the transformed $\tilde{F}$ for a 5-dimensional rainfall data-set is presented in logarithmic scale. Some classic Archimedean copula models are also displayed (Gumbel, Clayton, Frank) in order to quantify the misspecification model error. However, for more accurate numerical illustrations, we refer the reader to Sections 6 and 7 in [E8].

Since the presented method involves different notations and tuning parameters, to improve clarity, a comprehensive scheme of the estimation procedure is presented in Table 2.1 (right) (the reader is also referred to [E8] for a more detailed scheme).

![Diagonal fit](image)

Table 2.1: Left: Estimation of the survival diagonal in logarithmic scale for a 5-dimensional rainfall data-set (see Sections 6 and 7 in [E8]). Empirical diagonal is presented in black thick line; diagonal of the transformed model $\tilde{F}$ in full red line; diagonals of classic parametric Gumbel, Clayton and Frank models are displayed as specified in the legend. Right: Comprehensive scheme of the estimation procedure.
Finally, in [E8], we adapt our flexible methodology in the case of some asymmetric dependencies. We propose a transformed nested Archimedean copula model, with two nested levels (see, e.g., Hofert and Pham [114]). This transformed nested model can be seen as a first investigation in this sense. The importance of taking asymmetry into account in our models and to be able to built transformations for non-exchangeable random vectors is the central topic of the next section.

2.2.3 Asymmetric multivariate Archimedean copulas with quadratic form [E4]

The symmetry of Archimedean copulas class is often considered to be a rather strong restriction, especially in large dimensional applications. It implies that all multivariate projections of the same dimension are equal. To circumvent exchangeability, Archimedean copulas can be nested within each other under certain admissibility conditions. The resulting copulas are called nested Archimedean copulas. In the last decade the nested Archimedean copulas have been studied from different points of view (theoretically, computationally, in view of applications and so on). The interested reader is referred to Hofert [109, 110], Hofert and Mächler [112], McNeil [140]. There are other strategies to generalize Archimedean copulas in order to avoid symmetry such as the Hierarchical Kendall copulas (see Brechmann [31]), or Liouville copulas (see McNeil and Neslehová [143]). The interested reader is also referred to Genest and Neslehová [98] for a survey work on non-exchangeability for bivariate copulas.

An asymmetric generalization of the Archimedean copula class, containing both the Archimedean and the extreme-value copulas as a special case, are the Archimax copulas (see Capéraà et al. [38] for the bivariate case, Charpentier et al. [41] for the multivariate case).

In the following, we propose a possible extension of the Archimedean copula family that makes it possible to deal with asymmetry (for further details, see [E4]). Our extension is based on the observation that when applied to the copula, the inverse function of the generator of an Archimedean copula as in (1.7) can be expressed as a linear form of generator inverses. In [E4], we propose to add a distortion term to this linear part, which leads to asymmetric copulas (see Definition 2.2.3).

**Definition 2.2.3 (Considered model and basic required assumptions)** Let us denote the column vectors of length \(d\) by \(u = (u_1, \ldots, u_d)\), \(\psi(u) = (\psi(u_1), \ldots, \psi(u_d))\) and \(1 = (1, \ldots, 1)\). We define, for all \(u \in [0,1]^d\), a function \(C_{\phi,G,\Sigma}(u)\) as

\[
C_{\phi,G,\Sigma}(u) = \phi \left( 1^t \psi(u) + z \left( g(u)^t \Sigma h(u) \right) \right),
\]

where \(\phi\) is a valid strict Archimedean generator with regular inverse function \(\psi\), \(g : [0, 1]^d \to \mathbb{R}^d\) and \(h : [0, 1]^d \to \mathbb{R}^d\) are two vector-valued continuous functions, \(z : \mathbb{R} \to \mathbb{R}\) is a continuous real function. The index \(G\) in \(C_{\phi,G,\Sigma}\) is a vector function combining \(g, h, z\), i.e., \(G := (g, h, z)\). In the following, we will denote \(\mathcal{F}\) the class of functions \(C_{\phi,G,\Sigma}\) as in (2.12).

Parameters of this new class of copulas are grouped within a matrix \(\Sigma = (\sigma_{ij})_{i,j \in I}\), thus facilitating some usual applications as critical layers determination (see Property 2 in [E4]), upper tail dependence coefficient in Definition 2.1.1 (see Property 4 in [E4]) or estimation (see Property 5 in [E4]).

**Main contributions**

In [E4] we specify constraints on both parameters and \(g, h\) and \(z\) functions. These constraints come from the choices and desired features of the proposed model in (2.12).
Remark 8 Interestingly, for some particular choices of parameters and \( g, h \) and \( z \) functions, \( C_{\phi,G,\Sigma} \) copulas can provide natural multivariate extensions of Farlie-Gumbel-Morgenstern or Gumbel-Barnett copula models (see Nelsen [148] for the bivariate cases). In Figure 2.1 (left)
(resp. right) we provide a scatterplot of data from the distorted 3-dimensional copula $C_{\phi,G,\Sigma}$ in the case of “extended Gumbel-Barnett model” (resp. “extended Farlie-Gumbel-Morgenstern model”).

Let conclude this section by studying the impact of transformations as in (2.1) on the distorted copula $C_{\phi,G,\Sigma}$. To this aim, we consider a copula $C_{\phi,G,\Sigma}$ as in Equation (2.12) and the transformed associated one, i.e., $\tilde{C}_{\phi,G,\Sigma}(u) = T \circ C_{\phi,G,\Sigma}(T^{-1}(u))$, where $T^{-1}(u) = (T^{-1}(u_1), \ldots, T^{-1}(u_d))$. Consider the multivariate functions $\tilde{g}(u) = g(T^{-1}(u))$ and $\tilde{h}(u) = h(T^{-1}(u))$. Then, it holds that

\[
\tilde{C}_{\phi,G,\Sigma}(u) = C_{\tilde{\phi},\tilde{G},\Sigma}(u), \quad \text{where} \quad \tilde{\phi} = T \circ \phi \quad \text{and} \quad \tilde{G} = (\tilde{g}, \tilde{h}, z),
\]

(see Property 1 in [E4]). As remarked before, in the Archimedean case, $T$ in (2.1) is preserving the Archimedean structure, and thus the symmetry. For $C_{\phi,G,\Sigma}$, the asymmetry depends on the matrix $\Sigma$, which is, unsurprisingly, the same in $C_{\phi,G,\Sigma}$ and in $C_{\tilde{\phi},\tilde{G},\Sigma}$.

### 2.3 Local transformations

“It is rather difficult to construct new Archimedean copula models that exhibit specific tail behaviour”. (Larsson and Nešlehová [131])

#### 2.3.1 Multivariate transformed tail dependence coefficients [E6]

Despite their utility, there are however few works on the transformed tails of Archimedean copulas. Such knowledge would help building new classes of Archimedean copulas exhibiting desired tail behaviour. The flexibility of the transformed Archimedean copulas proposed in this section, coupled with a good knowledge of tail dependence coefficients, will be the starting point of the construction of copulas with given/target tail dependence coefficients.

In the following, we show that for some particular transformations $T$ and starting from some particular initial Archimedean copulas $C_0$, it is possible to produce Archimedean copulas $\tilde{C}_{T,C_0}$ as in (2.1) having tunable regular variation properties, and thus to get specific targeted multivariate lower and upper tail coefficients in Definition 2.1.1 (for further details, see [E6]).

**Main contributions**

Results gathered below are based on properties of regularly varying ($RV$) functions. The interested reader is referred for instance to Bingham et al. [26] and Soulier [183].

**Theorem 2.3.1 (Multivariate tail coefficients of transformed Archimedean copula)**

Let $\psi_0$ be an initial generator and $T$ be an admissible transformation as in (2.1). Assume that $\psi_0 \in RV_{-r_0}(0)$, with $r_0 \in [0, +\infty]$ and $T \in RV_\lambda(0)$, with $\lambda \in (0, +\infty)$. Let $\tilde{r} = r_0/\lambda$. Then the transformed multivariate lower tail dependence coefficient associated to $\tilde{\phi}$ is given by:

\[
\tilde{\lambda}_{L}^{(h,d-h)} = \begin{cases} 
\text{see Theorem 2.4 in [E6]}, & \text{if } \tilde{r} = 0, \\
\frac{d}{\tilde{\lambda}} r_0^{-1} (d - h)^{\tilde{\lambda}} r_0^{-1}, & \text{if } \tilde{r} \in (0, +\infty), \\
1, & \text{if } \tilde{r} = +\infty.
\end{cases}
\]

Assume now that $\psi_0 \in RV_{\rho_0}(1)$, with $\rho_0 \in [1, +\infty]$ and $1 - T \in RV_\bar{\lambda}(1)$, with $\bar{\lambda} \in (0, \rho_0]$. Let $\tilde{\rho} = \rho_0/\bar{\lambda}$. Then the transformed multivariate upper tail dependence coefficient associated to $\tilde{\phi}$
is given by:

\[
\lambda_{U}(h,d-h) = \begin{cases} 
\text{see Theorem 2.5 in [E6]}, & \text{if } \tilde{\rho} = 1, \\
\frac{\sum_{i=1}^{d} C_i^d (-1)^{i-1} \tilde{\rho}_0^{i-1}}{\sum_{i=1}^{d-h} C_i^{d-h} (-1)^{i-1} \tilde{\rho}_0^{i-1}}, & \text{if } \tilde{\rho} \in (1, +\infty), \\
1, & \text{if } \tilde{\rho} = +\infty,
\end{cases}
\]

where \(C_i^d\) is the number of \(i\)-combinations from a given set of \(d\) elements, for \(i \leq d\).

Proof of Theorem 2.3.1 can be seen as a multivariate extension using transformed copulas in (2.1) of Theorem 4.4 in Juri and Wüthrich [122] and Theorem 3.9 in Juri and Wüthrich [121] (both in classic Archimedean bivariate setting). It is based essentially on properties of regularly varying functions.

↪ For sake of brevity, the more complicated (and interesting) situations of asymptotic independence (i.e., \(\tilde{\rho} = 1, \tilde{\tau} = 0\)) are omitted in this manuscript. The interested reader is referred to Theorem 2.4 and 2.5 in [E6].

Let us consider the hyperbolic conversion transformation model defined in (2.2)-(2.3). The chosen conversion function has an asymptote easily linked to the associated parameters. Then this model helps parameterizing the transformation, since the asymptotes slopes are given. More precisely,

- \(H_{m,h,p_1,p_2,\eta}(x)\) in (2.3) tends to a continuous piecewise linear function when the smoothing parameter \(\eta\) tends to \(-\infty\) (for an illustration see Figure 2.2).

- \(H_{m,h,p_1,p_2,\eta}(x)\) in (2.3) has an asymptote \(ax + b\) at \(-\infty\) with \(a = e^{p_1}\), and an asymptote \(\alpha x + \beta\) at \(+\infty\) with \(\alpha = e^{p_2}\).

- We can easily obtain a local copula transformation by choosing \(H_{m,h,p_1,p_2,\eta}(x)\) close to \(x\) for some \(x\), in particular if \(p_1 = 0\) or \(p_2 = 0\) (for an illustration see Figure 2.2, where the bisector of the plane is represented using a red line).

Under these considerations, transformations in (2.2)-(2.3) allow to build Archimedean generators exhibiting any chosen couple \((\lambda_L, \lambda_U)\) without modifying the distribution in the central part of its domain, i.e., they are local transformations. This implies the possibility to modify the tail dependency of the transformed distribution without changing the global adjustment.

![Figure 2.2](image_url)  
**Figure 2.2:** \(H_{m,h,p_1,p_2,\eta}(x)\) for different values of parameters \(m, h, p_1, p_2 \in \mathbb{R}\) and smoothing parameter \(\eta \in \mathbb{R}\). **Left panel:** \(m = 3, h = 2, \eta = 3, p_1 = 0, p_2 = 2\). **Right panel:** \(m = 1, h = 1, \eta = -1, p_1 = 3, p_2 = 0\). Bisector of the plane is represented using a red line.

The following result describes the relationship between the asymptotes of the chosen transformations in (2.2)-(2.3) and the regular variation of the transformed tails.
**Proposition 2.3.1** (Multivariate tail coefficients using conversion model in (2.2)-(2.3))

Assume that \( \psi_0 \in \mathcal{RV}_{-r_0}(0) \), with \( r_0 \in (0, +\infty) \) and \( \psi_0 \in \mathcal{RV}_{\rho_0}(1) \), with \( \rho_0 \in [1, +\infty) \). Assume that the distribution \( G \) in (2.2) also satisfies \( G'/G \in \mathcal{RV}_{g}(-\infty) \), with \( g \in (0, +\infty) \) and \( G'/\tilde{G} \in \mathcal{RV}_{\gamma}(-\infty) \), where \( \tilde{G} = 1 - G \), \( G' \) the density of \( G \) and \( \gamma \in (0, +\infty) \).

Then, the transformed multivariate tail coefficients in Theorem 2.3.1 can be written as:

\[
\tilde{\lambda}^{(h,d-h)}_L = d^{-\alpha} r_0^{-1} (d - h)^{\alpha} r_0^{-1} \quad \text{and} \quad \tilde{\lambda}^{(h,d-h)}_U = \frac{\sum_{i=1}^{d-h} C_{d-i}^{i}(-1)^i \cdot i^{\alpha} \rho_0^{-1}}{\sum_{i=1}^{d-h} C_{d-h}^{i}(-1)^i \cdot i^{\alpha} \rho_0^{-1}}.
\]

**Remark 9** By using Proposition 2.3.1, when fitting some data, it is thus possible to propose a fit that respects some estimated tail dependence coefficients, by deducing parameters \( p_1 \) and \( p_2 \) of \( H_{m,h,p_1,p_2,\eta} \) from tail coefficients and by estimating other parameters \( m, h \) and \( \eta \). The methodology detailed in [E8] and [E15] can be adapted for this purpose (see previous Section 2.2.2).

Let us illustrate the bivariate case, for which expressing slopes \( a \) and \( \alpha \) as functions of \( \tilde{\lambda}^{(h,d-h)}_L \) and \( \tilde{\lambda}^{(h,d-h)}_U \) is straightforward and it does not require any numerical inversions. Indeed \( \tilde{\lambda}^{(1,1)}_L = 2^{-a^{\alpha} r_0^{-1}} \) and \( \tilde{\lambda}^{(1,1)}_U = 2 - 2^{a^{\alpha} r_0^{-1}} \). Then we can easily find \( a = e^{p_1} \) and \( \alpha = e^{p_2} \) and we get

\[
p_1 = \frac{1}{g} \ln \left( -r_0 \frac{\ln \tilde{\lambda}^{(1,1)}_L}{\ln 2} \right) \quad \text{and} \quad p_2 = \frac{1}{\gamma} \ln \left( \rho_0 \frac{\ln (2 - \tilde{\lambda}^{(1,1)}_U)}{\ln 2} \right).
\]

An illustration of Proposition 2.3.1 and Remark 9 with an initial Clayton copula \( C_0 \) and \( G(\cdot) = \logit^{-1}(\cdot) \) (i.e., \( g = \gamma = 1 \)) in detailed in Section 3.2 in [E6].

**2.3.2 Upper-patched generators [E3]**

In this section, we aim at proposing a new local transformation of a given initial generator \( \phi_0 \) such that

- the transformed generator \( \tilde{\phi} \) is (easily) guaranteed to be a valid Archimedean generator (see Proposition 2.3.2) with absolutely continuity property\(^2\) for the associated transformed copula (see Proposition 2.3.3);

- the upper tail dependence coefficient in Definition 2.1.1 can be chosen (see Proposition 2.3.4);

- the likelihood of a data-set is not reduced by the proposed local transformation (see Proposition 2.3.5).

The results in this section have a potential interest in risk management, since they allow to construct copulas with different tail dependence behaviour. The interested reader is also referred to Durante et al. [72]. As in Section 2.3.1, (1) we are looking for local transformations that would be able to exhibit any chosen tail dependence coefficient, and that would also preserve the shape of a copula on its central part (for further details see [E3]). (2) The approach here does not aim at leaving the Archimedean class of copulas: the transformed copula will stay within this class. Differently from Section 2.3.1, we provide here simple admissibility conditions ensuring the validity of the transformed generator.

\(^2\)Remark that this latter property is crucial for statistical inference.
Definition 2.3.1 (Proposed upper-patched generator) Let $t_0 \in (0, +\infty)$ be a given real value, and $d \geq 2$. The initial generator $\phi_0 : \mathbb{R}^+ \to [0, 1]$ and transformation generator $\phi_D : \mathbb{R}^+ \to [0, 1]$ are such that

- $\phi_0$ is a valid $d$-dimensional Archimedean generator,
- $\phi_D$ is a valid $d$-dimensional Archimedean generator, such that $k$-th derivatives $\phi_D^{(k)}(d_0) = 0$, for all $k = 0, \ldots, d-2$ and for some $d_0 \in (0, t_0]$. This implies, in particular, that $\phi_D$ is a non-strict generator with end-point $d_0 = \inf \{t \in \mathbb{R}^+ : \phi_D(t) = 0\} \leq t_0$.

An upper-patched generator $\tilde{\phi}$ is given by

$$\tilde{\phi}(t) = \begin{cases} p_{d-1}(t) + (1 - p_d(0))\phi_D(t), & \text{if } t < t_0, \\ \phi_0(t), & \text{if } t \geq t_0, \end{cases}$$

for all $t \in \mathbb{R}^+$, where $p_{d-1}(t) = \sum_{i=0}^{d-1} \phi_0^{(i)}(t_0) \frac{(t - t_0)^i}{i!}$ is the Taylor expansion of $\phi_0(t)$ at order $d-1$.

In Figure 2.3 we give an illustration of the upper-patched generator $\tilde{\phi}$ proposed in Definition 2.3.1 when $d = 2$ (left panel) and $d = 3$ (right panel).

![Upper-patched generator in dimension d=2](image1)

![Upper-patched generator in dimension d=3](image2)

Figure 2.3: Upper-patched generators $\tilde{\phi}$ (red line). **Left panel:** We consider the case $d = 2$ and $\phi_D$ is generator 4.1.2 in Nelsen [148] with parameter $\theta_D = 2$ and end-point $d_0 = 1$. **Right panel:** We consider the case $d = 3$ and $\phi_D$ is generator 4.1.15 in Nelsen [148] with parameter $\theta_D = 2$ and end-point $d_0 = 1$. In both plots, the function $p_{d-1}(t)$, for $t \leq t_0$, is the blue dashed tangent line of $\phi_0(t)$ at abscissa $t_0 = 2$. The considered initial Clayton copula $C_0$ with parameter $\theta = 3$ is represented by the black curve in both panels.

Main contributions

**Proposition 2.3.2 (Validity of the upper-patched generator)** Let $\tilde{\phi}$ be an upper-patched transformed generator as in Definition 2.3.1. Then $\tilde{\phi}$ is a valid $d$-dimensional Archimedean generator.

Sketch of proof. The initial generator remains unchanged on $[t_0, +\infty]$, so that the change acts only on the upper tail dependence behavior the copula. On $[0, t_0]$, due to constraints on derivatives at $t = t_0$ and convexity conditions, the upper-patched generator must be above the Taylor expansion $p_{d-1}(t)$, so that we just add a new transformation generator $\phi_D$, with suitable normalisation constant $1 - p_d(0)$ to ensure that $\tilde{\phi}(0) = 1$. Using a non-strict generator
2.3. Local transformations

with \( \phi_D(t_0) = 0 \) ensures the continuity of \( \tilde{\phi} \), but the supplementary constraints on derivatives of \( \phi_D \), in Definition 2.3.1, are needed in order to ensure the \( d \)-monotonicity (and then the admissibility) of \( \tilde{\phi} \).

Proposition 2.3.3 (Absolute Continuity) Let \( \tilde{\phi} \) be an upper-patched transformed generator as in Definition 2.3.1. Consider one supplementary order differentiability condition: assume that \( \phi_0 \) and \( \phi_D \) are \( (d + 1) \)-monotone on \( \mathbb{R}^+ \). Furthermore, assume that \( \phi_D^{(k)}(t_0) = 0 \) for all \( k = 1, \ldots, d - 1 \), then \( \tilde{\phi} \) is a valid \( d \)-dimensional Archimedean generator which induces an absolutely continuous copula.

Proof of Proposition 2.3.3 is based on Proposition 4.1 (ii) in McNeil and Nešlehová [141].

Remark that, modifying a generator \( \phi(x) \) for small values of \( x \) as in Definition 2.3.1 allows to change the upper tail dependence coefficient of the copula (see Charpentier and Segers [44] and [E6]). More precisely, in the following we show that the regular variation properties of the proposed upper-patched generator \( \tilde{\phi} \) are directly linked to those of the transformation generator \( \phi_D \).

Proposition 2.3.4 (Regular index for the upper-patched generator) Consider two generators \( \phi_0 \) and \( \phi_D \) as in Definition 2.3.1. If \( 1 - \phi_D \in \mathcal{R}V_{1/\rho}(0) \) with \( \rho \in [1, +\infty) \), then \( 1 - \tilde{\phi} \in \mathcal{R}V_{1/\rho}(0) \).

Furthermore, the patched upper tail dependence coefficient is given by

\[
\lambda_U^{(h,d-h)} = \begin{cases} 
0, & \text{if } \rho = 1, \\
\frac{\sum_{i=1}^{d}(-1)^i C_{d-i}^{-1/\rho}}{\sum_{i=1}^{d}(-1)^i C_{d-i}^{1/\rho}}, & \text{if } \rho \in (1, +\infty), \\
1, & \text{if } \rho = +\infty.
\end{cases} \tag{2.13}
\]

From Proposition B.1.9 in de Haan and Ferreira [58], we obtain that \( 1 - \tilde{\phi} \circ I \in \mathcal{R}V_{\max(-1,-1/\rho)}(+\infty) \). Hence the first result. The second one comes down from Charpentier and Segers [44] and [E6] (see Section 2.3.1).

Finally we show that, on a given multivariate data-set, it is possible to choose a target upper tail dependence behaviour without reducing the likelihood of the copula on pseudo-observations.

Proposition 2.3.5 (Likelihood improvement) Let \( C_0 \) be the Archimedean copula associated to an initial generator \( \phi_0 \). Consider an upper-patched generator \( \tilde{\phi} \) as in Definition 2.3.1. Denote \( \tilde{\phi}^{-1} \) the inverse function of \( \tilde{\phi} \) and let \( \tilde{C}(u_1, \ldots, u_d) = \tilde{\phi}(\tilde{\phi}^{-1}(u_1) + \ldots + \tilde{\phi}^{-1}(u_d)) \), then

\[
\tilde{C}(u_1, \ldots, u_d) = C(u_1, \ldots, u_d), \quad \text{for all } (u_1, \ldots, u_d) \in [0, \phi(t_0)]^d. \tag{2.14}
\]

Let \( \mathcal{L}_0 \) the likelihood function under \( C_0 \) on given (pseudo-) observations \( U_1, \ldots, U_n \in (0,1)^d \). Let \( \lambda_U \) the upper tail coefficient of the patched generator \( \tilde{\phi} \) and \( \tilde{\mathcal{L}} \) be the corresponding likelihood function. Let \( \lambda \in [0,1] \) be a chosen upper tail dependent coefficient, one can find \( t_0 \in \mathbb{R}^+ \) and an associated non-strict generator \( \phi_D \) such that

\[\lambda_U^{(h,d-h)} = \lambda \quad \text{and} \quad \tilde{\mathcal{L}} \geq \mathcal{L}_0.\]

Sketch of proof. One can find a point \( t_0 \) such that \( \tilde{\mathcal{L}} = \mathcal{L}_0 \) on a domain \([0, \phi(t_0)]^d \). Since Equation (2.13) gives a bijection between \( \rho \in [0, +\infty] \) and \( \lambda_U^{(h,d-h)} \in [0,1] \), we can find \( \phi_D \) such that \( \lambda_U^{(h,d-h)} = \lambda \), whatever the choice of \( t_0 \). Furthermore, from Equation (2.14), we can choose \( t_0 \) such that \( \tilde{\mathcal{L}} = \mathcal{L}_0 \): the likelihood is ensured to be not reduced, but other choices of \( t_0 \) can possibly improve the likelihood.
2.4 Perspectives

An inferential problem is that it can be difficult to fit both the global shape of the copula (or of the considered distribution) and its tail dependence. It thus seems that there is a need to propose a flexible parametric estimation of the generator of an Archimedean copula with given tail dependence coefficients. Results presented in this chapter could help adjusting parametrically both the tail and the central part of the considered copula. One can imagine for example starting from a copula exhibiting a good fit on the central part of the multivariate data-set, and applying transformations in Section 2.3 to improve the fit of the tails. Or starting from a copula exhibiting a good fit on the tails and transforming it, as proposed in Section 2.2, in order to improve its central part. Or more generally finding the best transformation to fit both the tails and the central part of a given multivariate data-set, starting from a given copula \( C_0 \). Then, using results in Sections 2.2.1, 2.2.2, 2.3.1 and 2.3.2, the derivation of a complete estimation procedure both for the center of the distribution and for the tails could be an interesting topic for future researches.

An open interesting point is also a whole benchmark study to compare different available estimators of the generator of an Archimedean copula to those proposed in Section 2.2. In this sense a development of \( \lambda \) function study, started in Section 2.2.1, could be an important future work. Furthermore, the measure of the goodness of fit and the construction of specific tests, based on the proposed non-parametric estimated generator, are interesting perspectives.

As remarked above, the presented estimation does not require any optimization procedure. However an optimization can improve the quality of the estimation. Firstly the choice of thresholds \( Q_i \) and the smoothing parameters \( \eta, \eta_i \) can be optimized, instead of being arbitrarily chosen. Also the parameters linked to the estimation of the Archimedean copula can be optimized, as the choice of a generator among its equivalence class via the point \( (x_0, y_0) \) (see Section 2.2.1), or the kernel for smoothing empirical diagonal of the copula (see Section 2.2.2).

One limitation of transformations in Sections 2.2.1, 2.2.2, 2.3.1 and 2.3.2 is that they transform Archimedean copulas into other Archimedean copulas. The resulting copula is thus symmetric in the sense that it does not vary if margins are permuted. A way to cope with this problem is proposed in Section 2.2.3. Natural extensions of the results presented in this section would be the determination of a stochastic representation of random variables corresponding to the distorted copula \( C_{\phi,G,\Sigma} \) and developments on parameters estimation under validity constraints. Another interesting future study could be the investigation of the link between the proposed distorted copulas \( C_{\phi,G,\Sigma} \) and the Archimax ones (see, e.g., Capéraà et al. [38], Charpentier et al. [41], Saminger-Platz et al. [169]).

Some relevant related literature

In the following we present some papers, among others, where the transformation copula models introduced in this chapter are considered for further researches.

- Binois et al. [27] establish a connection between Pareto fronts and critical layers in Definition 1.2.1. Furthermore, they estimate the considered Pareto front in the Archimedean copula case, by using our non-parametric estimator of the generator based on the empirical diagonal section (see [E14] and Section 2.2.1 in this manuscript).

- In Durante et al. [73], a general way to transform a given copula by means of a univariate function is presented. The obtained copula can be interpreted as the result of a global shock
affecting all the components of a system modeled with the original copula. Transformation copula models presented in this chapter can be included in this analysis (see Remark 3.1 in Durante et al. [73]).

In order to study tail dependence, Grazian and Liseo [106] estimate the multivariate tail dependence coefficients in Definition 2.1.1 using a Bayesian inference for a semi-parametric copula model.
Chapter 3

On some multivariate crossing problems

This chapter is based on papers [E12] and [E1] in my list of publications. First, we introduce some problems linked to the multivariate crossing models and some related literature references. Then, we devote a section to each article before concluding with some research perspectives on this topic.

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3.1 Introduction

Various phenomena in biological, physical or economic sciences are amenable to the description by level-crossings of random processes (see, e.g., Azaïs and Wschebor [14], Blake and Lindsey [28]). Examples of these phenomena are spike coordination of neurons in the brain (see, e.g., Tchumatchenko et al. [187]), insurance risk assessment (see, e.g., McNeil et al. [142]) or stress levels generated by ocean waves (see, e.g., Ehrenfeld et al. [76] Azaïs et al. [13]). Therefore a number of mathematical studies in recent decades have focused on the statistical properties of level-crossings arising from stationary processes. However, this literature largely addresses the properties of one level-crossing process and rarely deals with the coordinated level-crossings of multivariate processes.

A prominent application where correlated level-crossings are of particular importance is neuroscience. Recent empirical work, as some in vitro experiments on pyramidal cells, has shown that the spikes can be approximated by a Gaussian level-crossing process (see Tchumatchenko et al. [187]; Burak et al. [33]). The spikes of two neurons are then modeled as upward level-crossing times of two cross-correlated fluctuating Gaussian potentials (see, e.g., Destexhe et al. [66]). Following this model, we consider the random, temporally correlated Gaussian zero mean process $V_j(t)$ which represents the voltage of a neuron $j$, for $j = 1, 2$ (see [E12] for further details). In Section 3.2, we address the following two features of the level-crossings of these multiple correlated Gaussian processes.

- First, we aim to understand how many more coincident level-crossings at a given time are expected if the underlying Gaussian random processes are correlated. Remark that, the spike count dependencies across neurons is a topic of current research efforts (see, e.g., Chapter 8 in Dayan and Abbott [57]).
- Second, we would clarify whether level-crossing counts derived from multiple correlated processes are jointly Gaussian. Indeed, in this case a simple statistical measure of dependence is the Pearson correlation coefficient. Remark that correlation coefficient measurements are often used to measure the strength of interdependencies in a pair of neurons (see, e.g., de la Rocha et al. [60]; Shea-Brown et al. [176]; Lampl et al. [128]; Vilela and Lindner [194]).

As mentioned above, recent empirical neurological studies allow to accept to model neuronal spikes by using Gaussian level-crossing processes. However, a statistical multivariate Gaussianity test based on the level up-crossings could be an appropriate mathematical tool to test and validate this model assumption. In fact, the neurological signals that are collected when studying brain activity are commonly observed through their level sets, as for instance through their spikes (see Worsley [196] and Lindner and Schimansky-Geier [134]). A natural question concerns the type of the signal distribution: Gaussian or Poisson? Gaussian or Oscillator?

The statistical test that we build in Section 3.3, together with the alternatives that we have considered, could be an appropriate tool for studying this problem. Indeed, in Section 3.3, we use the information given by some level functionals of a single realisation of a real valued stationary isotropic random field in order to infer some information on the whole distribution of the field. This can be a very realistic situation in many real-life applications when the whole trajectory of the considered phenomenon can not be observed (see, for instance, the neurological application described above, see also Section 3.2).

Previously, Lindgren [133] provides estimators for the second spectral moment of an univariate Gaussian process that are based on the number of up-crossings at various levels. Cabaña [34] builds a test of isotropy, for a two dimensional random field, that is based on the perimeter and the area of excursion sets. Let us also mention Adler et al. [4] where the authors start from the observation of a neurological space-time signal at some moderate levels and deduce some parameters that help in estimating the probability of exceeding very high values. Not far from this theme, one can find the question of exceedances or the study of extreme values, when considering high levels (see, e.g., Rice [159], Worsley [196], Taylor et al. [185]).

The level functional considered in $[E1]$ is the Euler characteristic (EC) of the excursion sets above some levels. To be more precise, in Section 3.3 we aim at proving that the function that associates $u$ to the mean excursions EC at level $u$ provides a kind of signature of the distribution of the random field under study. To this aim,

- Firstly, we extend the results in Estrade and León [90] by showing a Central Limit Theorem for the joint excursions EC concerned with different levels and disjoint domains. A numerically tractable formula for the associated asymptotic variance in the univariate Gaussian case is also provided.

- Secondly, using theoretical results mentioned above, we propose a statistical methodology to test the Gaussianity distribution versus some specific alternatives and we implement that on simulated data-sets to evaluate the finite-sample performance of our strategy.

### 3.2 A joint level-crossing model for neuronal spikes $[E12]$

In the following, we address the statistics of coincident level-crossings arising from two Gaussian zero mean processes that share a common latent source. This situation is illustrated in Figure 3.1 (A).
3.2. A joint level-crossing model for neuronal spikes [E12]

Figure 3.1: (A) Spike correlations can arise from common input in a neuronal network. (B) We consider coincident level-crossings arising from two Gaussian processes \( V_j \), for \( j = 1, 2 \), that share a common latent source. Whenever the voltage crosses a threshold \( \psi \) from below a spike \( s_j(t) \), for \( j = 1, 2 \), is emitted. Spikes are indicated by vertical solid lines. Vertical dotted lines indicate the width of a time bin \( T \) used to compute spike counts \( U_{[0,T]}^j(\psi) \), \( j = 1, 2 \).

As motivated in Section 3.1, we begin by defining the random, temporally correlated Gaussian zero mean process \( V_j(t) \) which represents the voltage of a neuron \( j \)

\[
V_j(t) = \int_{-\infty}^{\infty} e^{i\lambda} f_V^1(\lambda) (\sqrt{1-r} dW_j(\lambda) + \sqrt{r} dW_c(\lambda)),
\]

with \( f_V \) is a combination of filters \( f_V(\lambda) = \gamma(\lambda) g(\lambda) \), where \( \gamma \) represents the membrane filter and \( g \) the synaptic filter. By \( W_c \) we denote the common noise component. Moreover if \( A \cap B = \emptyset \) then \( W(A) \) and \( W(B) \) are independent Gaussian random variables. The correlation strength \( r \), \( r = [0,1) \), denotes the presynaptic overlap of neurons 1 and 2 generated in a neuronal network.

The auto- and crosscorrelation functions between \( V_i \) and \( V_j \) are, respectively

\[
C_{V_i}(\tau) = \langle V_i(0)V_i(\tau) \rangle = \sigma_i^2 \delta(\tau),
\]

\[
C_{V_iV_j}(\tau) = \langle V_i(0)V_j(\tau) \rangle = r \sigma_i \sigma_j \delta(\tau), \quad \text{for } j, k \in \{1,2\},
\]

where \( \delta(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} f_V(\lambda) d\lambda \) and \( \tau \) is the considered delay.

As illustrated in Figure 3.1 (B), neurons communicate using brief pulses, the so called spikes (see, e.g., Dayan and Abbott [57]). We are interested in how the level-crossings of one Gaussian process can be used to predict the level-crossing probability of the partner process in a specific time interval relative to the observed level-crossing in one process.

If \( Q(\varepsilon) = [t,t+\varepsilon] \times [\tau,\tau+\varepsilon] \), we can write the upward crossings as

\[
\mathbb{E}[U_{Q(\varepsilon)}(\psi)] = \mathbb{E}[\#\{(t_1,t_2) \in Q(\varepsilon) : V_1(t_1) = \psi, V_2(t_1 + t_2) = \psi, 1_{V_1(t_1) \geq 0, V_2(t_1+t_2) \geq 0}\}],
\]

with the voltages \( V_j \) and their derivatives \( V'_j \), for \( j = 1, 2 \). Then, following Tchumatchenko et al. [187], we can define the correlation of two spike trains as:

\[
\langle s_1(t)s_2(t+\tau) \rangle := \lim_{\varepsilon \to 0} \frac{\mathbb{E}[U_{Q(\varepsilon)}(\psi)]}{\varepsilon^2} = \mathbb{E}[V_1'(0)1_{V_1(0) \geq 0} V_2'(\tau)1_{V_2'(\tau) \geq 0}] = \psi, V_2(\tau) = \psi|p_\tau(\psi,\psi),
\]

where \( p_\tau(\psi,\psi) \) is the joint Gaussian density of the vector \( (V_1(0), V_2(\tau)) \). We now introduce our quantity of interest, i.e., the conditional firing rate

\[
\nu_{\text{cond}}(\tau) = \langle s_1(t)s_2(t+\tau) \rangle / \sqrt{\nu_1\nu_2},
\]

with \( \nu_j = \frac{\sigma_j V'_j}{2\pi \sigma_j} \exp\left(-\frac{\psi^2}{2\sigma_j^2}\right) \) the firing rate of neuron \( j \), for \( j = 1, 2 \) (see, e.g., Tchumatchenko et al. [187]).
Main contributions

In Proposition 3.2.1 below we provide closed-form expressions for the conditional level-crossing correlation function in (3.2). Remark that in the literature available mathematical results for conditional upward crossings in Gaussian processes currently comprise mostly variance and moments for one level-crossing process (see Chapters 3-5 in Azaïs and Wschebor [14]) as well as the low and high correlation limit in pairs of processes (see Tchumatchenko et al. [186, 187]). As yet, a comprehensive closed-form solution covering the complete level-crossing cross-correlation function is currently lacking. This is exactly the aim of Proposition 3.2.1 below.

**Proposition 3.2.1** For all correlation strengths \( r \in [0, 1] \), it holds that

\[
\nu_{\text{cond}}(\tau) = C_Z(\tau) \frac{p_r(\psi, \psi)}{\sqrt{\sigma_1^2 \sigma_2^2}}
\]

where \( p_r(\cdot, \cdot) \) is a joint Gaussian distribution for voltages \( V_j \) and \( C_Z(\tau) \) is a tractable series obtained by using Hermite polynomials and a regression Gaussian model (for the derivation and the complete explicit expression of \( C_Z \) the interested reader is referred to Sections 3 and 6.1 in [E12]).

The index \( Z \) in \( C_Z \) is a vector function combining the variances \( \sigma_{V_j} \) and \( \sigma_{V_j'} \), the threshold \( \psi \), the correlation function \( c(\tau) \) and the correlation strength \( r \), i.e., \( Z := (\sigma_{V_1}, \sigma_{V_2}, \sigma_{V_1'}, \sigma_{V_2'}, \psi, c(\tau), r) \).

Proof of Proposition 3.2.1 is based on the application of a regression Gaussian model and the Mehler’s Formula (see Corollary 10.7 in Azaïs and Wschebor [14]). We provide here the MATHEMATICA 8 (Wolfram Research) code to iteratively calculate \( \nu_{\text{cond}}(\tau) \) in Proposition 3.2.1, for all \( r \in [0, 1] \).

In Figure 3.2 (A,B) below, we illustrate \( \nu_{\text{cond}}(\tau) \) obtained in Proposition 3.2.1 for progressively large truncation orders \( n \) for the tractable series \( C_Z(\tau) \). We consider a first truncation order \( n = 1 \) and a large \( n \) limit (\( n = 10 \)). In Figure 3.2 (C,D) below, we show \( \nu_{\text{cond}}(\tau) \) versus \( \tau \) as in Proposition 3.2.1 for varying correlation strength \( r \in [0, 1] \). We chose here \( c(\tau) = \cosh(\tau/\tau_s)^{-1} \)

Notice that \( \nu_{\text{cond}}(\tau) \) for two identical neurons \( (\sigma_{V_1} = \sigma_{V_2}) \) is a symmetric function while for a pair of heterogonous neurons with different rates \( (\sigma_{V_1} \neq \sigma_{V_2}) \), \( \nu_{\text{cond}}(\tau) \) is asymmetric.

Our second main result is the joint Gaussianity of upcrossing spike counts (see Theorem 3.2.1).

**Theorem 3.2.1** Let \( V_j(t), j \in \{1, 2\} \) be two processes satisfying Equation (3.1), with covariance \( C_{V_{ij}}(\tau) = E[V_i(0)V_j(\tau)] \) where \( i, j = 1, 2 \) and a common spiking threshold \( \psi \). We rescale the voltages \( V_j(t) \) and the threshold \( \psi \) to obtain processes \( X_j(t) \) with unit variance and unit variance of the derivatives. The number of \( \psi \)-level up-crossings in a time interval \( T \) for process \( X_j \) is given by \( U_{X_j}^{0,T}(\psi) \). We assume regularity conditions on moments of \( U_{X_j}^{0,T}(\psi) \) and the Geman’s Condition for \( c(\tau) \) (see, e.g., Theorem 3 in Kratz and León [126]). Then,

\[
\frac{1}{\sqrt{T}} \left( \begin{array}{c} U_{X_1}^{0,T}(\psi_1) - E[U_{X_1}^{0,T}(\psi_1)] \\ U_{X_2}^{0,T}(\psi_2) - E[U_{X_2}^{0,T}(\psi_2)] \end{array} \right) \xrightarrow{T \to \infty} \mathcal{N} \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array} \right),
\]

where the count covariances \( a_{ij} \) with \( i, j \in \{1, 2\} \) are three convergent series with \( 0 < a_{ii} = \sum_{q=1}^{\infty} \sigma_{X_i,q}^2(q) \) and \( 0 < a_{12} = \sum_{q=1}^{\infty} \sigma_{X_1,q} \sigma_{X_2,q} \). The asymptotic Pearson correlation coefficient \( \rho \) defined by

\[
\lim_{T \to \infty} \frac{\text{Cov}(U_{X_1}^{0,T}(\psi_1), U_{X_2}^{0,T}(\psi_2))}{\sqrt{\text{Var}(U_{X_1}^{0,T}(\psi_1)) \text{Var}(U_{X_2}^{0,T}(\psi_2))}} = \frac{a_{12}}{\sqrt{a_{11}a_{22}}},
\]

will also converge to the respective ratio of the asymptotic covariances and variances.
Sketch of proof. First, to take advantage of the available Gaussianity proofs that are typically derived for unit variance processes, we rescale the voltages \( V_j(t) \) and the threshold \( \psi \) to obtain processes \( X_j(t) \) with unit variance and unit variance of the derivatives. Then, we use a modified Breuer-Major-Theorem to prove joint Gaussianity and show that any linear combination of level-crossing counts of the two processes is also Gaussian.

3.3 Euler characteristic of excursion sets and a related Gaussianity test [E1]

“Suppose that we are given a real valued random field \( f \) on \( \mathbb{R}^3 \), which we observe on the unit cube \([0,1]^3\). Our problem is that we are not certain what the distribution of the field is. For simplicity, we shall assume that the choices are Gaussian and \( \chi^2 \), where both are assumed to be stationary and isotropic. Then one way to choose between the two is to calculate, from the data, an empirical Euler characteristic curve, …” Adler [3]

Adler’s words describe precisely the main goal of this section. Indeed, we deal here with a standard stationary isotropic random field \( X: \mathbb{R}^d \to \mathbb{R} \) and we assume it is partially observed through the Euler characteristic (EC) of excursion sets. We aim at providing a methodology for a test of Gaussianity based on this information. Further regularity conditions on the considered random field \( X \) can be found in Assumption (A) in [E1].

For any \( u \in \mathbb{R} \) and compact cube \( T \subset \mathbb{R}^d \), we call “excursion set of \( X \) above the level \( u \) within the domain \( T \)” the following set \( \{ t \in T : X(t) \geq u \} \), and \( | \cdot | \) denotes without any ambiguity, either the absolute value, or the \( d \)-dimensional Lebesgue measure. Let \( \lambda > 0 \) (resp. \( r \)) the second spectral moment (resp. the covariance function) of \( X \).
The considered EC of excursion sets is equal to a sum of two terms. The first one only depends on the restriction of \( X \) to the interior of \( T \), it is given by the quantity \( \varphi(X,T,u) \) defined in Equation (3.3) below. The second one exclusively depends on the behaviour of \( X \) on the \( l \)-dimensional faces of \( T \), with \( 0 \leq l < d \). From now on, we focus on the term \( \varphi(X,T,u) \), named as “modified Euler characteristic” in Estrade and León [90], and we still call it Euler characteristic. It is defined by the following,

\[
\varphi(X,T,u) = \sum_{k=0}^{d} (-1)^k \mu_k(T,u), \quad \text{where} \quad \mu_k(T,u) = \# \{ t \in \hat{T} : X(t) \geq u, X'(t) = 0, \text{index}(X''(t)) = d - k \},
\]

with \( \hat{T} \) the interior of \( T \) and the “index” stands for the number of negative eigenvalues.

**Euler characteristic, extrema and up-crossings**

We develop below two special cases (in dimension \( d = 1 \) and \( d = 2 \)) to highlight the link between the extrema of \( X \) above \( u \), the number of up-crossings at level \( u \) and the considered EC.

**Dimension one.** When \( d = 1 \), Equation (3.3) becomes

\[
\varphi(X, [0,T], u) = \# \{ \text{local maxima of } X \text{ above } u \text{ in } (0,T) \} - \# \{ \text{local minima of } X \text{ above } u \text{ in } (0,T) \}.
\]

Morse’s theorem says that this quantity is linked with the number of up-crossings \( U(X, [0,T], u) := \# \{ t \in [0,T] : X(t) = u, X'(t) \geq 0 \} \), by the relation

\[
\varphi(X, [0,T], u) = 1_{\{X(0) > u, X'(0) < 0\}} + 1_{\{X(T) > u, X'(T) > 0\}} = U(X, [0,T], u) + 1_{\{X(0) > u\}}.
\]

Taking expectation and using stationarity, one get \( \mathbb{E}[\varphi(X, [0,T], u)] = \mathbb{E}[U(X, [0,T], u)] \). So, in this univariate case, we focus here, on average, exactly on same quantity of Section 3.2, i.e., the number of up-crossings.

**Dimension two.** When \( d = 2 \), Equation (3.3) can be rewritten in the following way. With the notations introduced within this equation, \( \mu_0(T,u) \) denotes the number of local maxima above \( u \), \( \mu_2(T,u) \) denotes the number of local minima above \( u \) and \( \mu_1(T,u) \) the number of local saddle points above \( u \). Hence,

\[
\varphi(X,T,u) = \# \{ \text{local extrema of } X \text{ above } u \text{ in } \hat{T} \} - \# \{ \text{local saddle points of } X \text{ above } u \text{ in } \hat{T} \}.
\]

We are now ready to go into more detail concerning the results presented in this section.

**Main contributions**

In particular, in [E1] we extend Theorem 2.6 in Estrade and León [90] by showing that the random variables

\[
Z_k = |T_i|^{-1/2} (\varphi(X,T_i,u_k) - \mathbb{E}[\varphi(X,T_i,u_k)]), \quad \text{for } i = 1, \ldots, m; \ k = 1, \ldots, p;
\]

with \( \varphi(X,T_i,u_k) \) the EC of a standard Gaussian field \( X \) associated to the level \( u_k \) in the domain \( T_i \) and \( \mathbb{E}[\varphi(X,T_i,u_k)] \) the theoretical Gaussian mean of excursions EC at level \( u_k \), are asymptotically jointly distributed as Gaussian when the disjoint domains \( T_i \) grow up and satisfy asymptotic independence. More precisely,
Theorem 3.3.1 Assume that $X$ is Gaussian and satisfies regularity conditions (see Assumption (A) in [E1]).

(a) Let $T_1$ and $T_2$ be two cubes in $\mathbb{R}^d$ such that $|T_1| = |T_2|$ and $\text{dist}(T_1, T_2) > 0$ and let $u_1$ and $u_2$ belong to $\mathbb{R}$. Let $T_i^{(N)} = \{Nt : t \in T_i\}$ the image of the fixed cube $T_i$ by the dilation $t \mapsto Nt$, for $i = 1, 2$. For any integer $N > 0$, we introduce

$$Z_i^{(N)} = |T_i^{(N)}|^{-1/2} \left( \varphi(X, T_i^{(N)}, u_i) - \mathbb{E}[\varphi(X, T_i^{(N)}, u_i)] \right), \quad \text{for } i = 1, 2.$$ 

As $N \to +\infty$, $(Z_1^{(N)}, Z_2^{(N)})$ converges in distribution to a centered Gaussian vector with diagonal covariance matrix

$$\begin{pmatrix} V(u_1) & 0 \\ 0 & V(u_2) \end{pmatrix}$$

where $V(u_i)$ is obtained by Equation (3.5) below.

(b) Let $T$ be a cube in $\mathbb{R}^d$ and let $u_1$ and $u_2$ belong to $\mathbb{R}$. Let $T^{(N)} = \{Nt : t \in T\}$ the image of a fixed cube $T$ by the dilation $t \mapsto Nt$. For any integer $N > 0$, we introduce

$$\zeta_i^{(N)} = |T^{(N)}|^{-1/2} \left( \varphi(X, T^{(N)}, u_i) - \mathbb{E}[\varphi(X, T^{(N)}, u_i)] \right), \quad \text{for } i = 1, 2.$$ 

As $N \to +\infty$, $(\zeta_1^{(N)}, \zeta_2^{(N)})$ converges in distribution to a centered Gaussian vector with covariance matrix

$$\begin{pmatrix} V(u_1) & V(u_1, u_2) \\ V(u_1, u_2) & V(u_2) \end{pmatrix},$$

where $V(u_1, u_2)$ is prescribed by Equation (3.5) below and $V(u_i) = V(u_i, u_i)$, for $i = 1, 2$.

Sketch of proof. Concerning item (a), we first have to establish that the fields $Z_1^{(N)}$ and $Z_2^{(N)}$ are asymptotically independent. Intuitively, it comes from the fact that the distance between the cubes $T_1^{(N)}$ and $T_2^{(N)}$ goes to infinity and the covariance function of $X$ has a sufficient rate of decay at infinity due to the regularity conditions (see Assumption (A) in [E1]). Finally, one can use the same arguments as those in Estrade and León [90] (which were inspired by the Breuer-Major Theorem of Nourdin et al. [153]) to establish that any linear combination $xZ_1^{(N)} + yZ_2^{(N)}$ has a Gaussian limit in distribution. Item (b) is proved in Estrade and León [90], Theorem 2.5. The derivation of the integral form of the covariances $V(u_1, u_2)$ is obtained in Proposition 3.3.1 below. Obviously $V(u_1) = V(u_1, u_1)$ and $V(u_2) = V(u_2, u_2)$.

We give below an integral expression of the asymptotic covariances $V(u_1, u_2)$ and variances $V(u_i) = V(u_i, u_i)$, for $i = 1, 2$, in Theorem 3.3.1.

Proposition 3.3.1 Assume that $X$ is Gaussian and satisfies regularity conditions (see Assumption (A) in [E1]) and let $T$ be a cube in $\mathbb{R}^d$. Then for any $u$ in $\mathbb{R}$,

$$V(u_1, u_2) = \lim_{N \to \infty} \text{Cov} \left( \frac{\varphi(X, T^{(N)}, u_1)}{|T^{(N)}|^{1/2}}, \frac{\varphi(X, T^{(N)}, u_2)}{|T^{(N)}|^{1/2}} \right) < +\infty,$$

is given by

$$V(u_1, u_2) = \int_{\mathbb{R}^d} G(u_1, u_2, t) D(t)^{-1/2} - C(u_1)C(u_2) \ dt + (2\pi)^{-d/2} g(\max(u_1, u_2)), \quad (3.5)$$

where

$$G(u_1, u_2, t) = \mathbb{E}[1_{[u_1, \infty)}(X(0)) 1_{[u_2, \infty)}(X(t)) \det(X''(0)) \det(X''(t)) | X'(0) = X'(t) = 0],$$

$$D(t) = (2\pi)^{2d} \det(\lambda^2 I_d - r''(t)^2),$$

$$C(u_j) = (2\pi)^{-(d+1)/2} \lambda^{d/2} H_{d-1}(u_j) e^{-u_j^2/2},$$

$$g(u_j) = \mathbb{E}[1_{[u_j, \infty)}(X(0)) | \det(X''(0))],$$

for $j = 1, 2$ and $H_k$ stands for the Hermite polynomial of order $k$. 

Furthermore, in the univariate Gaussian case, we derive a new numerically tractable formula for the asymptotic variance $V(u_i, u_i)$ in (3.5) (see Proposition 8 in [E1]).

Theorem 3.3.1 and Proposition 3.3.1 are used in [E1] to build a Gaussianity test for a standard random field. The strategy of the proposed test is sketched in the following.

I. If the H0 hypothesis: “the random field $X$ is Gaussian” holds true, then the test statistic based on the sum of the scaled $Z_i^k$’s in (3.4) follows an appropriate chi-squared distribution. So, we can get high $p$-values of goodness-of-fit tests for the chi-squared distribution associated to the considered test statistics based on $Z_i^k$’s.

Conversely,

II. if the underlying standard random field $X$ is not Gaussian with a given second spectral moment (in particular if $X$ is a $\chi^2$ or a Kramer oscillator process), we deliberately center again, as in the I. case, the $Z_i^k$’s variables in (3.4) by using the (wrong) theoretical Gaussian mean of excursions $EC$ at level $u_k$ with the same second spectral moment.

In this second case, we obtain very small goodness-of-fit $p$–values for the chi-squared distribution associated to the considered test statistics based on $Z_i^k$’s. Then we are able to reject the H0 hypothesis under considered alternatives.

![Figure 3.3: Theoretical $u \mapsto E[\varphi(X,T,u)]$ for $|T| = 200$ (full line). We display, for different levels of $u$, the empirical counterpart (red dots) with associated empirical intervals, based on 300 Monte Carlo simulations. First row, $d = 1$ Left: Standard Gaussian univariate process with covariance function $r(t) = e^{-t^2}$ and $\lambda = 2$. Right: Standard Kramer oscillator process with $\lambda = 2$. Second row, $d = 2$ Left: Standard Gaussian field with covariance function $r(t) = e^{-||t||^2}$. Right: Standard $\chi^2$ field with $\lambda = 2$.](image)

For sake of brevity we omitted here the necessary study of the theoretical mean of excursions $EC$ under H0 hypothesis and under the considered alternatives. The interested reader is referred to Sections 2 and 3 in [E1].

We only include Figure 3.3 above to provide a visual evidence of different shapes of the mean excursions $EC$ for several random fields ($d = 2$) or random processes ($d = 1$). Large numerical
illustrations of Theorem 3.3.1 and Proposition 3.3.1 are provided in Sections 4 and 5 in [E1] for both univariate and bivariate cases. We also show the finite-sample performance of the proposed test of Gaussianity by evaluating $p$-values of goodness-of-fit test on simulated data-sets.

3.4 Perspectives

We consider methodological works described in this chapter as a first step; there are many open questions and possible future researches extending and going further these studies.

A first interesting point is the investigation of some neuronal spikes models in the case of correlated non-Gaussian processes. However, in this case, it could be hard to provide tractable and closed-form expressions as in Section 3.2.

It is not difficult to imagine that the same question of Gaussianity is actually asked in other real-life situations, and that the potential alternatives take various shapes, like a chi-square when the process under study is always positive, or an oscillator when the process presents an almost periodic structure (sea waves) or a shot noise process when sudden peaks are observed. Our methodology could also serve as a goodness-of-fit test in the opposite way. For instance, in the area of geostatistics, Gabriel et al. [97] observe the Euler characteristic of excursion sets to detect abrupt changes on soil data sets. They use as an a priori model for the data a chi-square random field. Then, a natural extension to the study presented in Section 3.3 could be a test of chi-square distribution versus not chi-square. In order to go further in this direction, it would be necessary to establish Central Limit Theorems for the Euler characteristic of excursion sets for new classes of processes or random fields.

Another extension of Section 3.3 could concern non-continuous random fields, like some types of shot noise processes. A first study in this sense is developed in [E1], Section 6. For instance, Boolean model can be seen as an excursion above level one of a specific shot noise model with jumps (see, e.g., Hug et al. [116]). Some recent references deal with two dimensional shot noise, Bierné and Desolneux [25] and Lachièze-Rey [127] by studying the perimeter of excursion sets, the mean total curvature function and the mean excursion EC of bivariate shot noise random fields. A related statistical analysis based on these Minkowski functionals is the object of a working paper in collaboration with Anne Estade and Céline Duval, Université Paris 5 - Descartes (see [E22]).

From an applications point of view, in the analysis of the Cosmic Microwave Background (CMB) radiation, the random field under study is defined on the two dimensional celestial sphere. Hence, in order to be applied, our Gaussianity test should be extended beyond the Euclidean case. Let us mention that recent studies, like Cammarota and Marinucci [37], Cheng and Xiao [50], Marinucci and Vadalàmani [136] for instance, contain theoretical results on high energy behavior of spherical random fields that could provide the required background for a test of Gaussianity in the same spirit as the one presented here.

We conclude this chapter with the citation of the following paper based on our theoretical results presented in Section 3.2.

By using the mathematical tractability of the joint level-crossing model proposed in [E12] (see closed-form solutions for both auto- and cross-correlation functions in Section 3.2), Dettner et al. [67] derive a minimal set of spike features containing the complete information of a neuron.

For the sake of completeness, we summarize below the papers written during my three years of Ph.D. thesis.

**On Multivariate Extensions of Value-at-Risk \([E16]^*\)**

This paper represents the starting point of several works in my list of publications about multivariate extensions of risk measures. For its crucial rule, \([E16]^*\) is also presented in Chapter 1 of this manuscript.

In this paper, we introduce two alternative extensions of the classic univariate Value-at-Risk (VaR) in a multivariate setting (see Definition 1.2.2 in Chapter 1). The two proposed multivariate VaR are vector-valued measures with the same dimension as the underlying risk portfolio. The lower-orthant VaR is constructed from level sets of multivariate distribution functions whereas the upper-orthant VaR is constructed from level sets of multivariate survival functions (see Definition 1.2.1 in Chapter 1). Several properties have been derived. In particular, we show that these risk measures both satisfy the positive homogeneity and the translation invariance property. Comparison between univariate risk measures and components of multivariate VaR are provided. We also analyze how these measures are impacted by a change in marginal distributions, by a change in dependence structure and by a change in risk level. Interestingly, these results turn to be consistent with existing properties on univariate risk measures. Illustrations are given in the class of Archimedean copulas.

**Estimating Bivariate Tail: a copula based approach \([E17]^*\)**

This paper deals with the problem of estimating the tail of a bivariate distribution function. The general idea of our procedure is to decompose the estimation of \(P(X \leq x, Y \leq y)\), for \(x, y\) above some marginal thresholds \(u_X, u_Y\), in the estimation of three different regions. For the joint upper tail region \([u_X, x] \times [u_Y, y]\) we use a new non-parametric estimator coming from a general extension of the POT (Peaks-Over-Threshold) method. This is mainly based on a two-dimensional version of the Pickands-Balkema-de Haan Theorem. This new estimator adopts a distributional point of view for the tail, following Juri and Wüthrich [122] (see also Charpentier and Juri [42]). We introduce a new parameter that describes the nature of the tail dependence, and we provide a way to estimate it. We construct a two-dimensional tail estimator and study its asymptotic properties.

**Plug-in estimation of level sets in a non-compact setting with applications in multivariate risk theory \([E18]^*\)**

This paper is the starting point of different works in my list of publications. For this reason, \([E18]^*\) is also briefly discussed in Chapter 1 of this manuscript.

This paper deals with the problem of estimating the level sets \(L(\alpha) = \{F(x) \geq \alpha\}\), with given \(\alpha \in (0, 1)\), of an unknown distribution function \(F\) on \(\mathbb{R}^2\). A plug-in approach is followed. That is, given a consistent estimator \(F_n\) of \(F\), we estimate \(L(\alpha)\) by \(L_n(\alpha) = \{F_n(x) \geq \alpha\}\). In our setting no compactness property is a priori required for the level sets to estimate. We state consistency results with respect to the Hausdorff distance and the volume of the symmetric difference. Our results are motivated by applications in multivariate risk theory. Indeed, we propose and estimate a new bivariate version of the Conditional-Tail-Expectation by conditioning the two-dimensional random vector to be in the level set \(L(\alpha)\) (see Definitions 1.2.1 and 1.2.3, in Chapter 1, in the bivariate case).
Other works \[E7\]; \[E24\]

Since this manuscript is completely devoted to the multivariate framework, we give here a short description of results obtained in \[E7\]. Indeed this article deals with an interesting univariate risk measure: expectiles. Multivariate extensions of expectiles risk measures have been recently proposed by Maume-Deschamps et al. \[139\].

Furthermore, we propose a summary of the Technical Report \[E24\] for SIAVB. This report, produced in September 2013, was the object of the Master 2 degree in Probability and Statistics of University \textit{Paris Sud} of Fan JIA, under my co-supervision.

Risk Management with Expectiles \[E7\]

Expectiles are a one-parameter family of coherent univariate risk measures that have been recently suggested as an alternative to VaR and to CTE. Expectiles benefit from the property of elicitability that corresponds to the existence of a natural backtesting methodology (see, e.g., Embrechts and Hofert \[80\] and Ziegel \[199\]). In \[E7\], we provide a transparent financial meaning of expectiles in terms of their acceptance sets as being the amount of money that should be added to a position in order to have a prespecified, sufficiently high gain-loss ratio.

In \[E7\], we also compare them with univariate VaR and CTE and we study their asymptotic behaviour when the risk level $\alpha \to 1$. In Bellini et al. \[22\] it was shown that for a Paretian tail with tail index $\beta > 2$, expectiles are ultimately (i.e., for $\alpha$ large enough) less conservative than the corresponding quantiles. On the contrary, when $\beta < 2$ the opposite inequality holds. In \[E7\] we provide the extension of this asymptotic comparison between quantiles and expectiles to the Gumbel and Weibull maximum domain of attraction, in which expectiles are ultimately less conservative than quantiles. Furthermore, in the domain of attraction of Fréchet and in the more complicated cases of the Weibull and Gumbel domain of attractions, we provide an explicit expression for the asymptotic behaviour of expectiles.

The explicit asymptotic approximations of expectiles obtained in \[E7\] are the starting point of the extrapolation estimation procedure proposed by Daouia et al. \[56\].

Extreme value modelling of rainfalls and flows in Bièvre region \[E24\]

The main goal of this work is to study the control system in Bièvre region (in the south of Paris) installed by SIAVB (Syndicat Intercommunal pour l’Assainissement de la Vallée de la Bièvre) in order to evaluate its efficiency. To this aim, we find out the best fitting distributions of rainfalls and flows measured in Bièvre region during ten years (2003-2013). Both Block Maxima and Peak Over Threshold (POT) methods are applied and compared (see also Engeland et al. \[88\]).

We deal with important trends and seasonal factors. First, we remove at the beginning the trends and seasonal factors, then we apply GEV and GP distributions to the remaining stationary series. Second, we use the point process, which incorporates trends and seasonal variations in considered models, instead of previously deseasonalizing the data (see also Smith \[181\]). With these models, we obtain estimates of $T$-years return levels for different return periods.


