Complexity results for identifying codes in planar graphs

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Abstract

Let G be a simple, undirected, connected graph with vertex set $V(G)$ and $\mathcal{C} \subseteq V(G)$ be a set of vertices whose elements are called *codewords*. For $v \in V(G)$ and $r \geq 1$, let us denote by $I_r^{\mathcal{C}}(v)$ the set of codewords $c \in \mathcal{C}$ such that $d(v, c) \leq r$, where the distance $d(v, c)$ is defined as the length of a shortest path between v and c. More generally, for $A \subseteq V(G)$, we define $I_r^{\mathcal{C}}(A) = \cup_{v \in A} I_r^{\mathcal{C}}(v)$, which is the set of codewords whose minimum distance to an element of A is at most r. If r and l are positive integers, $\mathcal{C}$ is said to be an $(r, \leq l)$-identifying code if one has $I_r^{\mathcal{C}}(A) \neq I_r^{\mathcal{C}}(A')$ whenever A and A' are distinct subsets of $V(G)$ with at most l elements. We consider the problem of finding the minimum size of an $(r, \leq l)$-identifying code in a given graph. It is already known that this problem is NP-hard in the class of all graphs when $l = 1$ and $r \geq 1$. We show that it is also NP-hard in the class of planar graphs with maximum degree at most three for all (r, l) with $r \geq 1$ and $l \in \{1, 2\}$. This shows, in particular, that the problem of computing the minimum size of an $(r, \leq 2)$-identifying code in a given graph is NP-hard.

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1 Notation and definitions

By *graph* we mean an undirected graph without loops nor multiple edges. If G is a graph, we denote respectively by $V(G)$ and $E(G)$ the sets of vertices

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and edges of G. An edge $\{x, y\} \in E(G)$ with $x, y \in V(G)$ will be simply denoted by xy. We refer to [3] for basic notions such as adjacent vertices, paths, cycles or the neighbourhood of a vertex. Let us recall the distinction between a *subgraph* and an *induced subgraph*: a subgraph of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, whereas if $W \subseteq V(G)$, the subgraph *induced by G on W* is the graph $G[W]$ whose vertex set is W, and whose edges are all the edges $xy \in E(G)$ with x and y in W.

From now on, we consider only connected graphs; for $v \in V$ and $r \in \mathbb{N}$, the *ball* of radius r centered at v is the set

$$B(v, r) = \{w \in V(G) : d(v, w) \leq r\}$$

where $d(v, w)$ denotes the number of edges in any shortest path between v and w, i.e. the *length* of any shortest path between v and w. Whenever $d(v, w) \leq r$, we say that v and w *r-cover* each other (or simply *cover* if there is no ambiguity). A vertex $c \in V(G)$ is said to *r-separate* (or simply *separate*) vertices x and y if c r-covers one of them and does not r-cover the other.

In this paper, what we call a *code* is simply a set of vertices $\mathcal{C} \subseteq V(G)$, and we refer to its elements as *codewords*. A code $\mathcal{C}$ is said to be *r-covering* (or *r-distance-dominating*) if every vertex $v \in V(G)$ is r-covered by at least one codeword $c \in \mathcal{C}$. A code $\mathcal{C}$ is said to be *r-separating* if for every pair of distinct vertices $x \neq y$ of G there exists a codeword $c \in \mathcal{C}$ which r-separates x and y.

An *r-identifying code* is a code which is both r-covering and r-separating. Equivalently, $\mathcal{C} \subseteq V(G)$ is an r-identifying code if all the sets

$$I_r^{\mathcal{C}}(v) = B(v, r) \cap \mathcal{C}$$

for $v \in V(G)$ are non-empty and different.

More generally, if $\mathcal{C}$ is a code and A is a subset of $V(G)$ we denote by $I_r^{\mathcal{C}}(A)$ the set of codewords which r-cover at least one element of A, i.e.

$$I_r^{\mathcal{C}}(A) = \bigcup_{v \in A} B(v, r) \cap \mathcal{C}.$$

If r and l are positive integers, an *$(r, \leq l)$-identifying code* is a code $\mathcal{C}$ such that the sets $I_r^{\mathcal{C}}(A)$ for all $A \subseteq V(G)$ with $|A| \leq l$ are different. Note that, in this case, as $I_r^{\mathcal{C}}(\emptyset) = \emptyset$, then $I_r^{\mathcal{C}}(A) \neq \emptyset$ for $1 \leq |A| \leq l$; thus an $(r, \leq 1)$-identifying code is simply an r-identifying code.

We recall that the *degree* of a vertex $v \in V(G)$ is the number $\delta(v)$ of vertices $w \in V(G)$ such that $vw \in E(G)$. The maximum degree of G is defined as

$$\Delta(G) = \max_{v \in V(G)} \delta(v).$$

Finally, a graph is *planar* if it can be drawn in the plane in such a way that its edges do not cross. For precise definitions and additional background about graphs, we refer once again to [3]. The reader will also need basic knowledge in algorithmic complexity such as polynomial reduction and NP-completeness; for these notions we refer to [5].

2 Introduction and main results

The problem of finding an identifying code of minimum size in a graph has been introduced and studied in [10]; the original motivation was fault detection in processor systems. It was shown in [2] that the computation of the minimum size of an r-identifying code in a given graph is NP-hard for any $r \geq 1$; furthermore it was proved in [6] that this problem is APX-hard for $r = 1$. In particular, this implies if $P \neq NP$ that there exists a constant $c > 1$ such that no polynomial algorithm gives an efficiency ratio better than c. Indeed there is, for all $r \geq 1$, a polynomial approximation algorithm which computes an r-identifying code with efficiency ratio $O(\log |V(G)|)$, but sublogarithmic ratios are intractable (see [15]). For a nearly comprehensive bibliography about identifying codes, see [13].
In this paper we prove NP-hardness results for the restriction of this same problem to the class of planar graphs with maximum degree at most three; this class is quite restrictive, as connected graphs with maximum degree at most two are paths and cycles where the size of a minimum r-identifying code is known exactly in most cases (see [1], [8], [14] and [16]). We also study the problem of finding the minimum size of an $(r, \leq 2)$-identifying code and prove its NP-hardness in the class of planar graphs with maximum degree at most three for all $r \geq 1$, which of course implies its NP-hardness in the class of all graphs. These codes have been investigated in [12], [7] and [11], but to our knowledge at this day no complexity result is known about them: our results show, in particular, that the problem of computing the minimum size of an $(r, \leq 2)$-identifying code in a given graph is NP-hard.

Let us denote by Π_3 the class of planar graphs with maximum degree at most three, and let r and l be positive integers. The problem that we study is precisely the following one:

<u>MIN $(r, \leq l)$-ID-CODE IN Π_3</u>
- INSTANCE: a graph $G \in \Pi_3$ and an integer k;
- QUESTION: is there an $(r, \leq l)$-identifying code $\mathcal{C}$ of G with $|\mathcal{C}| \leq k$?

Our results can be summarized in the following theorem:

Theorem 1. *The problem* MIN $(r, \leq l)$-ID-CODE IN Π_3 *is NP-complete for* $l \in \{1, 2\}$ *and all* $r \geq 1$.

3 Proofs of the complexity results

3.1 The vertex cover problem

Let G be a graph. An edge $e = xy \in E(G)$ is said to be *covered* by a vertex $v \in V(G)$ if v and e are incident, i.e. if $v = x$ or $v = y$. A *vertex cover* in G is a code $\mathcal{C} \subseteq V(G)$ such that every edge of G is covered by a at least one codeword $c \in \mathcal{C}$. Equivalently, $\mathcal{C}$ is a vertex cover if

$$\forall e = xy \in E(G), \quad x \in \mathcal{C} \text{ or } y \in \mathcal{C}.$$

It is well known that the problem of finding the minimum cardinality of a vertex cover in a given graph is NP-hard ([9]); furthermore, it was proved in [4] that this problem remains NP-hard in the class of planar graphs whose maximum degree is at most three. More precisely, the following problem is NP-complete:

<u>Min Vertex Cover in Π_3</u>
- Instance: a graph $G \in \Pi_3$ and an integer k;
- Question: is there a vertex cover $\mathcal{C}$ of G with $|\mathcal{C}| \leq k$?

If r and l are fixed and a code $\mathcal{C}$ is given in a graph G, as all distances between vertices of G can be computed in polynomial time, we can also compute all the sets $I_r^{\mathcal{C}}(A)$ for $A \subseteq V(G)$ with $|A| \leq l$ and compare them in polynomial time, and thus check that $\mathcal{C}$ is an $(r, \leq l)$-identifying code: therefore the problem Min $(r, \leq l)$-ID-code in Π_3 belongs to NP. We will complete the proof of Theorem 1 by showing that Min Vertex Cover in Π_3 polynomialy reduces to the problem Min $(r, \leq l)$-ID-code in Π_3 for all $r \geq 1$ and $l \in \{1, 2\}$; we will consider four cases depending on the values of r and l: $r = 1$, $l = 1$ (Section 3.2), $r = 2$, $l = 1$ (Section 3.3), $r \geq 3$, $l = 1$ (Section 3.4), and $r \geq 1$, $l = 2$ (Section 3.5).

3.2 Reduction for $r = 1$ and $l = 1$

In this section, we set $r = 1$ and $l = 1$. Let us consider an instance of Min Vertex Cover in Π_3, i.e. a planar graph G with maximum degree at most three and an integer k. We construct a graph G' by replacing every edge $e = xy$ of G by the structure C_e counting 9 vertices and 11 edges specified by Fig. 1. Note that we do not consider x and y as elements of the structure C_e; thus C_e is not a graph (since it contains edges without containing their ends), but we will denote by $V(C_e)$ the set of its nine vertices as we would have done if it were one. See also Fig. 2 for an example of transformation.
If G has n vertices and m edges, then G' has $n+9m$ vertices and $11m$ edges; thus G' can be constructed from G in polynomial time. It is easy to check that G' is also planar and has maximum degree three. The key-result for the reduction is the following:

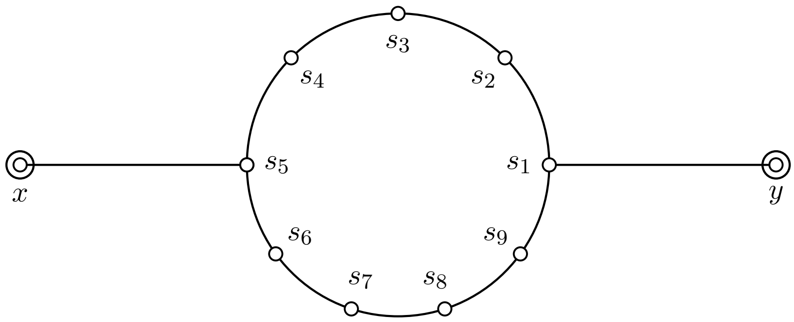

Figure 1: The structure C_e which replaces an edge $e = xy$ of G in G' .

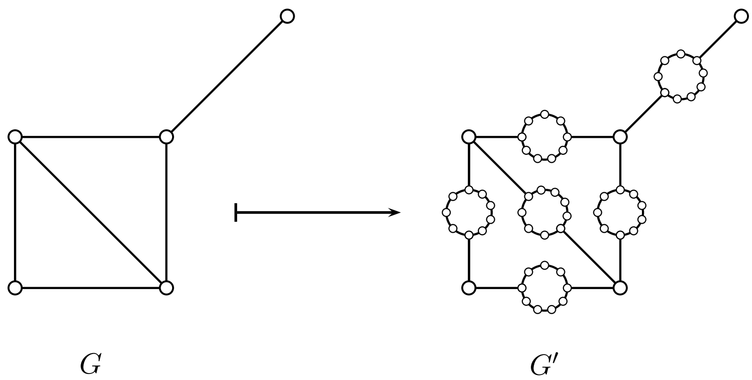

Figure 2: A graph G and the graph G' obtained through our transformation.

Proposition 2. *With notation above, G admits a vertex cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$ if and only if G' admits a 1-identifying code $\mathcal{C}'$ with $|\mathcal{C}'| \leq k + 5m$.*

Proof. Let us fix our notation first: if G is the original graph and G' is the transformed one, we consider the vertex set $V(G)$ of G as a subset of $V(G')$. Therefore $V(G')$ can be partitioned in $|E(G)| + 1$ sets:

$$V(G') = V(G) \cup \bigcup_{e \in E(G)} V(C_e).$$

Let $\mathcal{C}$ be a vertex cover for G with $|\mathcal{C}| \leq k$; we construct an identifying code $\mathcal{C}'$ for G' by adding to $\mathcal{C}$ five vertices in each set $V(C_e)$ for all $e \in E(G)$; thus the code $\mathcal{C}'$ will have cardinality $|\mathcal{C}'| = |\mathcal{C}| + 5m \leq k + 5m$ as stated in the proposition. For each edge $e = xy \in E(G)$, the choice of the set $\mathcal{C}'_e$ of five vertices in $V(C_e)$ added to $\mathcal{C}$ will depend on the case $x \in \mathcal{C}$ or $y \in \mathcal{C}$; recall that as we have requested $\mathcal{C}$ to be a vertex cover of G, we have at least one of x and y in $\mathcal{C}$. As the structure is symmetric, without loss of generality let us assume that $x \in \mathcal{C}$. Then we define (see Fig. 3)

$$\mathcal{C}'_e = \{s_1, s_2, s_5, s_6, s_9\}.$$

When this is done for all $e \in E(G)$, the code $\mathcal{C}'$ is defined as

$$\mathcal{C}' = \mathcal{C} \cup \bigcup_{e \in E(G)} \mathcal{C}'_e.$$

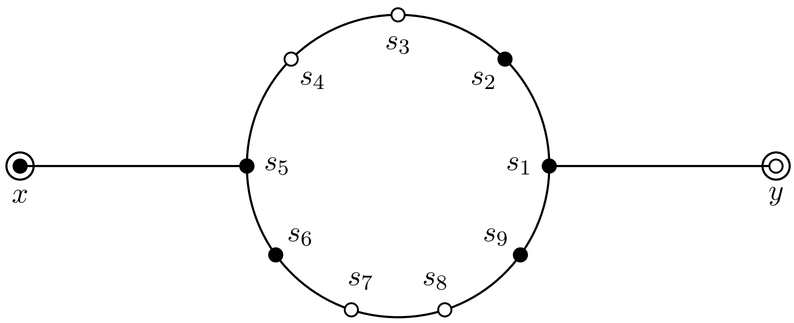

Figure 3: The code $\mathcal{C}'$ in C_e if $x \in \mathcal{C}$. Codewords are in black, non-codewords in white, with the exception of y which may be a codeword or not.

It remains to prove that $\mathcal{C}'$ is a 1-identifying code; since for each edge $e = xy \in E(G)$ the set of vertices covered by $\{s_1, s_2, s_5, s_6, s_9\}$ is precisely $V(C_e) \cup \{x, y\}$, it is easily seen that it suffices to check that for every edge $e = xy$ of G, all vertices in

$$V(C_e) \cup \{x, y\}$$

are covered and pairwise separated by $\mathcal{C}'$. We summarize in the following table which codewords cover the different vertices. Note that in C_e both x and y may belong to $\mathcal{C}$; this has no consequence since *containing* an identifying code is a sufficient condition for a set of vertices to *be* an identifying code.

vertices /codewords	x	s_1	s_2	s_5	s_6	s_9
x	$\bullet$			$\bullet$		
y		$\bullet$				
s_1		$\bullet$	$\bullet$			$\bullet$
s_2		$\bullet$	$\bullet$			
s_3			$\bullet$			
s_4				$\bullet$		
s_5	$\bullet$			$\bullet$	$\bullet$	
s_6				$\bullet$	$\bullet$	
s_7					$\bullet$	
s_8						$\bullet$
s_9		$\bullet$				$\bullet$

Conversely, suppose that there exists a 1-identifying code $\mathcal{C}'$ for G' with $|\mathcal{C}'| \leq k + 5m$. First we prove:
If neither x nor y belongs to $\mathcal{C}'$, then $|V(C_e) \cap \mathcal{C}'| \geq 6$. (1)
We just have to recall why a 1-identifying code on a cycle on 9 vertices must have at least 6 codewords. To simplify notation, let us consider that our vertices are integers modulo 9. If a is a vertex of the cycle, then $a + 1$ and $a + 2$ must be separated by a codeword, therefore a or $a + 3$ must be codewords. The same argument proves that there must be a codeword in $\{a + 3, a + 6\}$, and another in $\{a + 6, a\}$. So there must be at least two codewords in $\{a, a+3, a+6\}$ for each vertex a. Applied to $a \in \{1, 2, 3\}$, this argument shows that there are at least two codewords in each of the sets $\{1, 4, 7\}$, $\{2, 5, 8\}$ and $\{3, 6, 9\}$, therefore $|V(C_e) \cap \mathcal{C}'| \geq 6$.

Regardless of the fact that x or y belong to $\mathcal{C}'$, we have $|V(C_e) \cap \mathcal{C}'| \geq 5$. (2)
As in the previous case, there must be a codeword in $\{a, a + 3\}$ for every a except in the cases when x or y can be used to separate $a+1$ from $a+2$, i.e. except for $a \in \{3, 4, 8, 9\}$. So there must be at least one codeword in each of the pairs:

$$\{s_1, s_4\} \;\; \{s_2, s_5\} \;\; \{s_5, s_8\} \;\; \{s_6, s_9\} \text{ and } \{s_7, s_1\}.$$

We consider three cases:

- if neither s_1 nor s_5 are codewords, then s_4, s_2, s_8 and s_7 must be, as well as one of the pair $\{s_6, s_9\}$, so we need five codewords at least in $V(C_e)$;
- if s_1 and s_5 are both codewords: as at least one vertex in the pair $\{s_6, s_9\}$ is a codeword we can suppose by symmetry that $s_6 \in \mathcal{C}'$; but s_3 and s_8 are not covered by s_1, s_5, s_6, x nor y; therefore we need at least two other codewords in $V(C_e)$;

– if s_1 is a codeword but s_5 is not (the other case being the same by symmetry), then s_2 and s_8 are codewords as well as one of the pair $\{s_6, s_9\}$. In both cases we still need to cover s_4 with a codeword in $V(C_e)$.

Let us define $\mathcal{C}$ as the trace of $\mathcal{C}'$ on $V(G)$, i.e.

$$\mathcal{C} = \mathcal{C}' \cap V(G).$$

Recall that $V(G)$ is a subset of $V(G')$ and so $\mathcal{C}$ is a code in G. We would like to use $\mathcal{C}$ in order to build a vertex cover of G. It may happen for an edge $e = xy \in E(G)$ that in G' neither x nor y belongs to $\mathcal{C}'$, and in this case the edge xy of G is not covered by the code $\mathcal{C}$. Let p be the number of edges $e \in E(G)$ which are not covered by $\mathcal{C}$. From (1) and (2) we have

$$|\mathcal{C}| \leq |\mathcal{C}'| - 6p - 5(m - p) = |\mathcal{C}'| - 5m - p \leq k - p.$$

As there are p uncovered edges, if we add to $\mathcal{C}$ one codeword on each edge uncovered by $\mathcal{C}$, we get a vertex cover of G with at most k codewords. $\square$

3.3 Reduction for $r = 2$ and $l = 1$

For $r = 2$, we use a different strategy for the reduction. The basic idea is to replace every edge $e = xy \in E(G)$ by a chain on 4 vertices and 5 edges: see Fig. 4. Consider the central vertices a and b on the chain; as a 2-identifying code has to separate a from b, we deduce that x or y must be a codeword. Hence the trace on G of a 2-identifying code of G' will be a vertex cover of G.

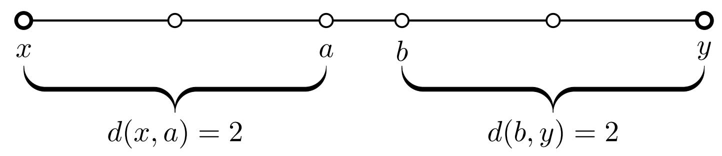

Figure 4: A 2-identifying code will have to contain x or y to separate a from b.

Clearly, this reduction is not entirely satisfactory because we cannot control the total number of codewords in the identifying code. In order to do so, we will add other devices around the chain.
Let $G \in \Pi_3$ and k be an integer. We construct from G a graph G' by replacing every edge $e = xy$ in $E(G)$ by a structure C_e counting 11 vertices and 14 edges: see Fig. 5, where the chain $xv_5v_7v_8v_6y$ plays the role of the

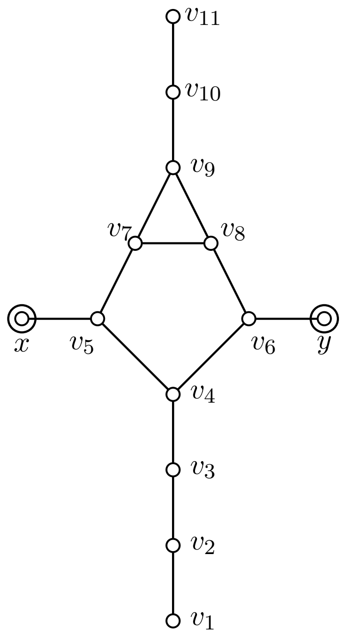

Figure 5: The structure C_e which replaces an edge $e = xy$ of G for $r = 2$.

chain in Fig. 4. As in the previous case we do not consider x and y as elements of $V(C_e)$.
Clearly, G' can be constructed in polynomial time from G, and $G' \in \Pi_3$. Let $m = |E(G)|$. We will prove:

Proposition 3. *G admits a vertex cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$ if and only if G' admits a 2-identifying code $\mathcal{C}'$ with $|\mathcal{C}'| \leq k + 5m$.*

Proof. Once again, $V(G)$ is a subset of $V(G')$; so if we consider a vertex cover $\mathcal{C}$ of G with $|\mathcal{C}| \leq k$, we can add codewords to $\mathcal{C}$ in order to construct a 2-identifying code for G'. We do this in the following way: if $e = xy \in E(G)$, then x or y is in $\mathcal{C}$. Suppose without loss of generality that $x \in \mathcal{C}$. We add to $\mathcal{C}$ the 5 codewords v_3, v_4, v_6, v_8 and v_9 on the structure C_e (see Fig. 6). When this is done for all $e \in E(G)$, we get a code $\mathcal{C}'$ on G' with $|\mathcal{C}'| = |\mathcal{C}| + 5m$, hence $|\mathcal{C}'| \leq k + 5m$. To check that $\mathcal{C}'$ is a 2-identifying code on G', we just have to see that for every $e = xy \in E(G)$, the vertices x, y and the 11 vertices of C_e are covered by different subsets of $\{x, v_3, v_4, v_6, v_8, v_9\}$. This is clearly sufficient because for a given $e = xy \in E(G)$, a vertex belongs to $V(C_e) \cup \{x, y\}$ if and only if it is 2-covered by v_3, v_4 or v_9. It remains to check the following table:

vertices /codewords	x	v_3	v_4	v_6	v_8	v_9
x	•		•			
y			•	•	•	
v_1		•				
v_2		•	•			
v_3		•	•	•		
v_4	•	•	•	•	•	
v_5	•	•	•	•	•	•
v_6		•	•	•	•	•
v_7	•		•	•	•	•
v_8			•	•	•	•
v_9				•	•	•
v_{10}					•	•
v_{11}						•

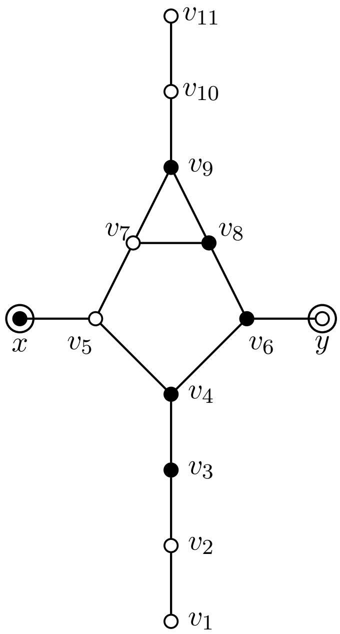

Figure 6: The codewords in C_e for $r = 2$.

Conversely, suppose that $\mathcal{C}'$ is a 2-identifying code of G' with $|\mathcal{C}'| \leq k + 5m$. Then if $e = xy \in E(G)$, consider the codewords on C_e:

- v_1 and v_2 must be separated by $\mathcal{C}'$ so $v_4 \in \mathcal{C}'$;
- v_1 must be covered by $\mathcal{C}'$ so $\mathcal{C}' \cap \{v_1, v_2, v_3\} \neq \emptyset$;
- v_2 and v_3 must be separated by $\mathcal{C}'$ so $\mathcal{C}' \cap \{v_5, v_6\} \neq \emptyset$;

- v_{10} and v_{11} must be separated by $\mathcal{C}'$ so $\mathcal{C}' \cap \{v_7, v_8\} \neq \emptyset$;
- v_{11} must be covered by $\mathcal{C}'$ so $\mathcal{C}' \cap \{v_9, v_{10}, v_{11}\} \neq \emptyset$.

These five facts show that in each structure C_e, we have $|\mathcal{C}' \cap V(C_e)| \geq 5$, therefore the trace of $\mathcal{C}'$ on $V(G)$

$$\mathcal{C} = \mathcal{C}' \cap V(G)$$

has cardinality at most

$$|\mathcal{C}| \leq |\mathcal{C}'| - 5m \leq k.$$

Moreover, in each struture C_e, v_7 and v_8, which play the same role as a and b in Fig. 4, must be separated by $\mathcal{C}'$, so x or y must belong to $\mathcal{C}'$, hence to $\mathcal{C}$. Thus $\mathcal{C}$ is a vertex cover of G'. □

3.4 Reduction for $r \geq 3$ and $l = 1$

For $r \geq 3$ our reduction uses the same idea as for $r = 2$, but the structure is slightly different. It is easier to present the proof for $r = 3$, but the general case is essentially the same.
Fix $r = 3$ and let $G \in \Pi_3$. We replace every edge $e = xy$ of $E(G)$ by the structure specified by Fig. 7. In particular, there are 5 cycles with attached paths on this structure that we call *suns*. A single sun is displayed on Fig. 8. There are, in Fig. 7, 189 vertices, not counting x and y, and our transformation is polynomial; moreover, the new graph G' is clearly in Π_3. Now we will prove the following proposition, where $m = |E(G)|$:

Proposition 4. *G admits a vertex cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$ if and only if G' admits a 3-identifying code $\mathcal{C}'$ with $|\mathcal{C}'| \leq k + 70m$.*

Proof. First suppose that we have a vertex cover $\mathcal{C}$ of G with $|\mathcal{C}| \leq k$. Then we construct a 3-identifying code $\mathcal{C}'$ on G' by adding 70 codewords to $\mathcal{C}$ on each structure C_e; these codewords correspond to 14 codewords on each sun, as in Fig. 8, thus $|\mathcal{C}'| \leq k + 70m$. We leave the readers to convince themselves that this defines a 3-identifying code on G'; just note that as in the case $r = 2$, the central vertices a and b are separated because x or y belongs to $\mathcal{C}$ and therefore is a codeword of $\mathcal{C}'$. Once again, it will be sufficient to check that $\mathcal{C}'$ 3-separates and 3-covers vertices inside each structure C_e if we first observe that $\mathcal{C}'$ enables us, for any vertex $v \in V(G')$, just by looking at the set $I_r^{\mathcal{C}'}(v)$, to know if v belongs to $V(G)$ or to identify the structure C_e to which v belongs.
Conversely, if $\mathcal{C}'$ is a 3-identifying code on G' with $|\mathcal{C}'| \leq k + 70m$, then consider a single sun of a structure C_e (see Fig. 9). In order to separate vertices v_4 and v_5, v_1 must be a codeword. In order to cover v_5, we must

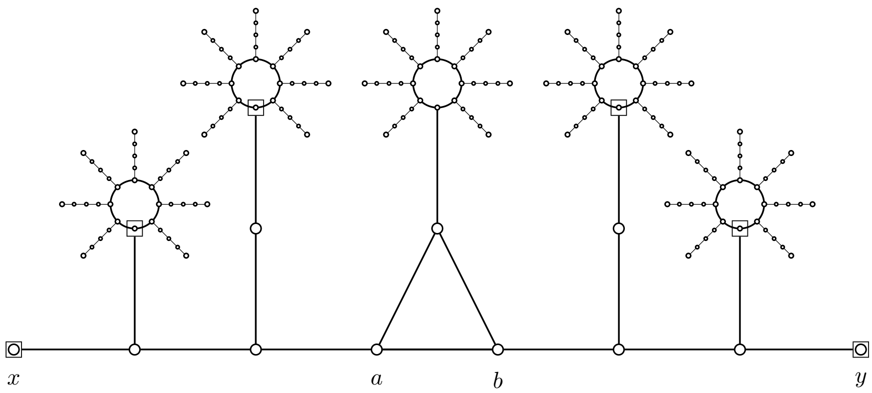

Figure 7: The structure C_e which replaces an edge $e = xy$ of G for $r = 3$. Squared vertices belong to $B(a, 3)\Delta B(b, 3)$.

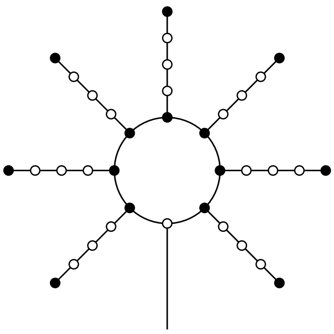

Figure 8: A sun and its codewords for $r = 3$.

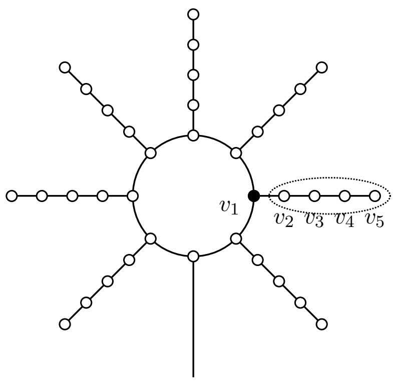

Figure 9: Necessary codewords in a 3-identifying code in a sun.

have $\mathcal{C}' \cap \{v_2, v_3, v_4, v_5\} \neq \emptyset$. The same argument holds for every ray of the sun, therefore we need at least $2 \times 7 = 14$ codewords in it; as there are five suns in C_e we need at least 70 codewords on C_e.
The code $\mathcal{C}'$ being 3-identifying on G', it necessarily separates the vertices a and b, i.e. there is at least one codeword in the symmetric difference $B(a,3)\Delta B(b,3)$. This set is composed of six vertices (see Fig. 7): x, y and four vertices belonging to C_e. Note that none of these six vertices was counted above in the 70 codewords. Hence we have (recall that x and y do *not* belong to C_e):

- if x or y belong to $\mathcal{C}'$ then $|\mathcal{C}' \cap V(C_e)| \geq 70$;
- if neither x nor y belongs to $\mathcal{C}'$ then $|\mathcal{C}' \cap V(C_e)| \geq 71$.

The conclusion goes exactly like in the case $r = 1$: let us define

$$\mathcal{C} = \mathcal{C}' \cap V(G)$$

which is a code on G and let p be the number of edges $e \in E(G)$ uncovered by $\mathcal{C}$. Then

$$|\mathcal{C}| = \left|\mathcal{C}' \cap V(G)\right| \leq \left|\mathcal{C}'\right| - 71p - 70(m-p) \leq k - p$$

because we supposed $|\mathcal{C}'| \leq k + 70m$. Thus if we add to $\mathcal{C}$ one codeword on each of the p edges of G uncovered by $\mathcal{C}$, we get a vertex cover of G, whose cardinal is at most k. □

For $r \geq 4$ our proof is exactly the same as in the case $r = 3$ but the structure has to be adapted. If we consider an edge $e = xy$ of G, we replace it by a path on $2r$ extra vertices and attach five suns to the path as on Fig. 10. Each sun is now a cycle on $2r + 2$ vertices with $2r + 1$ rays of length $r + 1$. With this structure replacing every edge of G, we obtain a graph G', satisfying the same properties as in the case $r = 3$. One can prove the following proposition exactly as it was done for $r = 3$.

Proposition 5. *G admits a vertex cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$ if and only if G' admits an r-identifying code $\mathcal{C}'$ with $|\mathcal{C}'| \leq k + 10(2r+1)m$.*

3.5 Reduction for $r \geq 1$ and $l = 2$

For our transformation we need to define a restriction of the Min Vertex Cover in Π_3 problem. Let Π'_3 be the class of graphs G such that:

- G belongs to Π_3, i.e. G is planar and every vertex of G has degree at most three;
- no vertex of G has degree one;

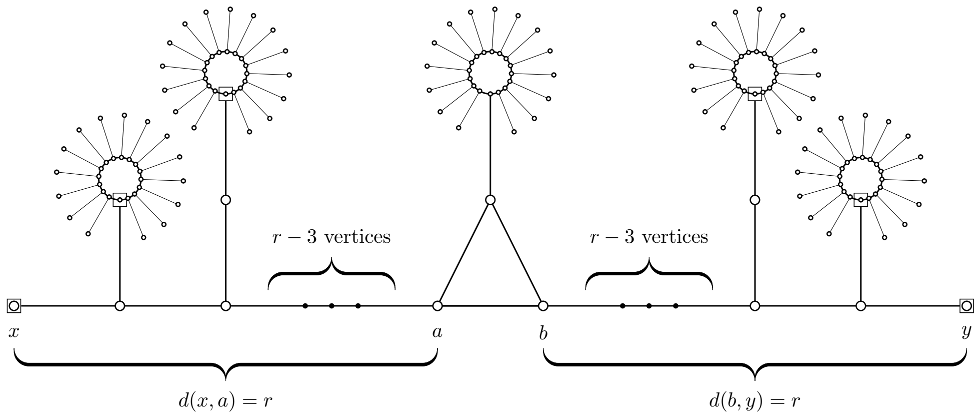

Figure 10: The structure C_e in the general case $r \geq 3$.

– if v_1 and v_2 are distinct vertices of G with degree 2, there exist vertices v_1' and v_2', distinct from v_1 and v_2, such that v_1' is adjacent to v_1 but not to v_2, and v_2' is adjacent to v_2 but not to v_1.

We define a problem Min Vertex Cover in Π_3' by analogy with the problem Min Vertex Cover in Π_3. We will prove the following result:

Lemma 6. *The problem* Min Vertex Cover in Π_3' *is NP-complete.*

Before proving this lemma let us begin with three preliminary results.

Lemma 7. *Let G be a graph and $xy \in E(G)$ be an edge such that the degree of the vertex x is 1, and let $G' = G[V(G) \setminus \{x, y\}]$ be the graph obtained when we remove x, y and all their incident edges from G. Then the minimum cardinality of a vertex cover in G equals the minimum cardinality of a vertex cover in G' plus* 1 *(see Fig. 11).*

Proof. Suppose that $\mathcal{C}$ is a minimum vertex cover in G; then exactly one of x and y belongs to $\mathcal{C}$, because if we had x and y in $\mathcal{C}$, then $\mathcal{C} \setminus \{x\}$ would also be a vertex cover of G. Hence $\mathcal{C}' = \mathcal{C} \setminus \{x, y\}$ is a vertex cover of G' and $|\mathcal{C}'| = |\mathcal{C}| - 1$. Conversely, if $\mathcal{C}'$ is a (minimum) vertex cover of G', then $\mathcal{C}' \cup \{y\}$ is a (minimum) vertex cover of G. □

Lemma 8. *Let G be a graph and x be a vertex of degree 2 whose neighbours y and z are adjacent, and let $G' = G[V(G) \setminus \{x, y, z\}]$ be the graph obtained when we remove x, y, z and all their incident edges from G. Then the minimum cardinality of a vertex cover in G equals the minimum cardinality of a vertex cover in G' plus* 2 *(see Fig. 12).*

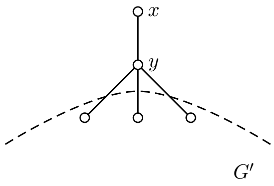

Figure 11: Illustration of Lemma 7.

Proof. Suppose that $\mathcal{C}$ is a minimum vertex cover in G; then exactly two of the three vertices x, y and z belong to $\mathcal{C}$, because we need at least two of them to cover the edges xy, yz and zx, and if we had x, y and z in $\mathcal{C}$ then $\mathcal{C} \setminus \{x\}$ would still be a vertex cover of G. So $\mathcal{C}' = \mathcal{C} \setminus \{x, y, z\}$ is a vertex cover of G' and $|\mathcal{C}'| = |\mathcal{C}| - 2$. Conversely, if $\mathcal{C}'$ is a (minimum) vertex cover of G', then $\mathcal{C}' \cup \{y, z\}$ is a (minimum) vertex cover of G. $\square$

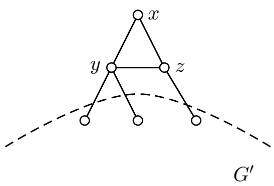

Figure 12: Illustration of Lemma 8.

Note that the previous two lemmas could be generalized in the following way: if x is a simplicial vertex in G, i.e. a vertex x with degree $\delta \geq 1$ whose δ neighbours induce a clique in G, then the minimum cardinality of a vertex cover in G is equal to the mimimum cardinality of a vertex cover in $G[V \setminus B(x, 1)]$ plus δ. [1]

Lemma 9. *Let G be a graph and x, y be two distinct vertices of G with degree 2, sharing the same neighbours z and z' (in particular x and y are non adjacent). Let $G' = G[V(G) \setminus \{x, y, z, z'\}]$ be the graph obtained when we remove x, y, z, z' and all their incident edges from G. Then the minimum cardinality of a vertex cover in G equals the minimum cardinality of a vertex cover in G' plus 2 (see Fig. 13).*

Proof. If $\mathcal{C}$ is a minimum vertex cover in G, then exactly two of the four vertices x, y, z and z' belong to $\mathcal{C}$; the fact that $zz' \in E(G)$ or not does not matter. The conclusion follows as in the previous two lemmas. $\square$

Proof of Lemma 6. Suppose that G belongs to Π_3, but not to Π'_3. Then either:

- there exists a vertex x of degree one in G, and we will apply Lemma 7;

[1] We thank the referee for this remark.

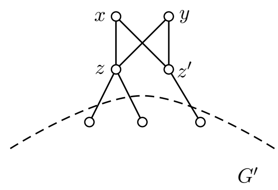

Figure 13: Illustration of Lemma 9.

– there exist vertices $x \neq y$ of degree two such that every neighbour of x is either y or adjacent to y. Then:

  – if x and y are adjacent, let z be their common neighbour. Then we can apply Lemma 8;

  – if x and y are non adjacent, let z and z' be their (common) neighbours; we can apply Lemma 9.

In the three cases, we obtain a 'reduced' graph G' and we can trivially compute the size of a minimum vertex cover in G if we know the size of a minimum vertex cover in G'. Clearly, the reduced graph G' is planar and has maximum degree at most three. If this graph does not belong to Π'_3, we can reduce it once again, and so on until we obtain an 'irreducible' graph G'', i.e. a graph which belongs to Π'_3. Thus we obtain by an algorithm which is obviously polynomial a graph $G'' \in \Pi'_3$ and an integer k such that the size of a minimum vertex cover in G is equal to the size of a minimum vertex cover in G'' plus k. This proves that the problem Min Vertex Cover in Π'_3 is algorithmically harder than Min Vertex Cover in Π_3. As the converse is obviously true and since Min Vertex Cover in Π_3 is NP-complete, we have proved Lemma 6. □

Let us now come back to $(r, \leq 2)$-identifying codes. Let $G \in \Pi'_3$; we replace every edge $e \in E(G)$ of G by a structure C_e which can be seen on Fig. 14. The structure C_e is made of a cycle on $4r + 3$ vertices, attached to x and y by a path on $2r + 1$ vertices. As before, we do not consider that x and y belong to $V(C_e)$.
Clearly, if $G \in \Pi'_3$ then $G' \in \Pi_3$, and G' can be computed in polynomial time from G. The reduction of the problem Min Vertex Cover in Π'_3 to the problem Min $(r, \leq 2)$-ID-code in Π_3 will be obtained if we prove:

Proposition 10. *G admits a vertex cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$ if and only if G' admits an $(r, \leq 2)$-identifying code $\mathcal{C}'$ with $|\mathcal{C}'| \leq k + (6r + 4)m$.*

In order to shorten the proof of Proposition 10, we need two other technical lemmas:

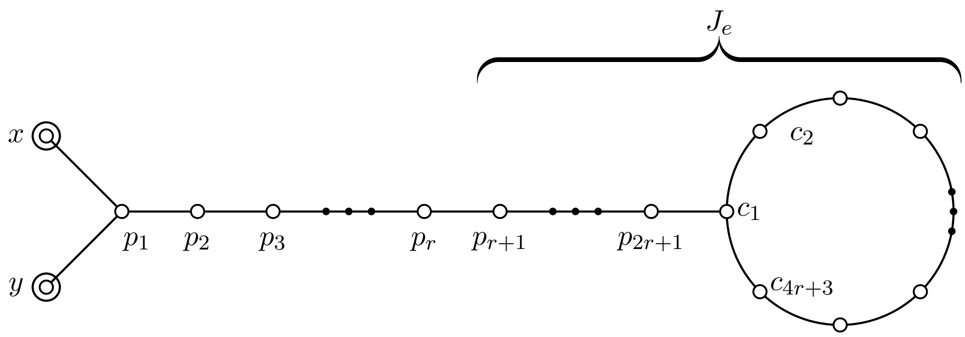

Figure 14: The structure C_e which replaces an edge $e = xy$ of G in the case $l = 2$.

Lemma 11. *Let $r \geq 1$ and H be a graph consisting of a path on $3r + 1$ vertices*

$$p_{-r+1}\ p_{-r+2} \cdots p_{-1}\ p_0\ p_1\ p_2 \cdots p_{2r+1},$$

a cycle on $4r + 3$ vertices

$$c_1\ c_2 \cdots c_{4r+3}\ c_1$$

and the edge $p_{2r+1}c_1$ which connects the path to the cycle (see Fig. 15). Let W be the set of vertices

$$W = \{p_1, p_2, \cdots, p_{2r+1}\} \cup \{c_1, c_2, \cdots, c_{4r+3}\}$$

and let $\mathcal{C} = V(H)$ be the code consisting of all vertices in H. Then all the identifying sets $I_r^{\mathcal{C}}(A)$, for all $A \subset W$ with $|A| \leq 2$, are distinct.

Proof. The lemma is proved if we show that $I_r^{\mathcal{C}}(A_1) \neq I_r^{\mathcal{C}}(A_2)$ in the 15 cases given by the Table below.

$A_1 =$	$\{p_{i_1}\}$	$\{c_{i_1}\}$	$\{p_{i_1}, p_{i_2}\}$ $(i_1 < i_2)$	$\{c_{i_1}, c_{i_2}\}$	$\{p_{i_1}, c_{i_2}\}$
$A_2 = \{p_{j_1}\}$	1	2	3	4	5
$A_2 = \{c_{j_1}\}$		6	7	8	9
$A_2 = \{p_{j_1}, p_{j_2}\}$ $(j_1 < j_2)$			10	11	12
$A_2 = \{c_{j_1}, c_{j_2}\}$				13	14
$A_2 = \{p_{j_1}, c_{j_2}\}$					15

The cases $2, 4, 7, 9, 11, 14$ are easy to check: in cases $2, 4, 11$, the codeword p_{j_1-r} r-covers p_{j_1} and neither c_{i_1} nor c_{i_2}, so $I_r^{\mathcal{C}}(A_1) \neq I_r^{\mathcal{C}}(A_2)$; similarly, in cases $7, 9, 14$, p_{i_1-r} covers p_{i_1}, and neither c_{j_1} nor c_{j_2}.
In case 1, assuming without loss of generality that $j_1 < i_1$, p_{j_1-r} covers p_{j_1}, not p_{i_1}.

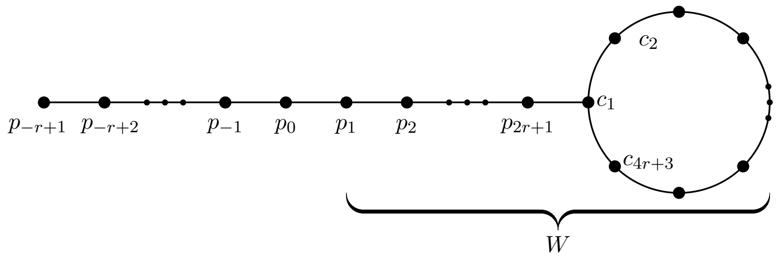

Figure 15: The graph H in Lemma 11.

In case 3, if $j_1 = i_1$, then either p_{i_2+r}, if it exists, or a codeword of type c covers p_{i_2}, not p_{j_1}; otherwise, set $\mu = \min\{i_1, j_1\}$ and consider $p_{\mu-r}$.
In cases $5, 12$, one of $c_{i_2 \pm r \mod 4r+3}$ covers c_{i_2}, and neither p_{j_1} nor p_{j_2}.
Cases $6, 8$ are also easy to handle.
In case 10, consider $p_{\min\{i_1,j_1\}-r}$ if $i_1 \neq j_1$; if $i_1 = j_1$, then $p_{\max\{i_2,j_2\}+r}$, if it exists, or a codeword of type c will do.
Case 15 is easy, both if $i_1 = j_1$ and if $i_1 \neq j_1$.
We are left with case 13, where, instead of showing that $I_r^{\mathcal{C}}(\{c_{i_1}, c_{i_2}\}) \neq I_r^{\mathcal{C}}(\{c_{j_1}, c_{j_2}\})$, we equivalently show that, given $I_r^{\mathcal{C}}(\{c_1, c_2\})$, we can uniquely recover c_1 and c_2.
It is quite straightforward to observe that, with computations carried modulo $4r+3$, the set $I_r^{\mathcal{C}}(\{c_1, c_2\})$ is of the form $\{c_\alpha, c_{\alpha+1}, \cdots, c_{\alpha+\beta}\}$, plus maybe vertices of type p, where $2r+1 \leq \beta \leq 4r+1$. Then $\{c_{\alpha+r}, c_{\alpha+\beta-r}\} = \{c_1, c_2\} \mod 4r+3$. □

Lemma 12. *Let $r \geq 1$ and H be a graph consisting of a path on $2r+1$ vertices*

$$p_1\ p_2 \cdots p_{2r+1},$$

a cycle on $4r+3$ vertices

$$c_1\ c_2 \cdots c_{4r+3}\ c_1$$

and the edge $p_{2r+1}c_1$ which connects the path to the cycle (see Fig. 16). Let $\mathcal{C}$ be the code on H defined by

$$\mathcal{C} = \{p_{r+1}, p_{r+2}, \cdots, p_{2r+1}\} \cup \{c_1, c_2, \cdots, c_{4r+3}\}.$$

Then $\mathcal{C}$ is an r-identifying code on H.

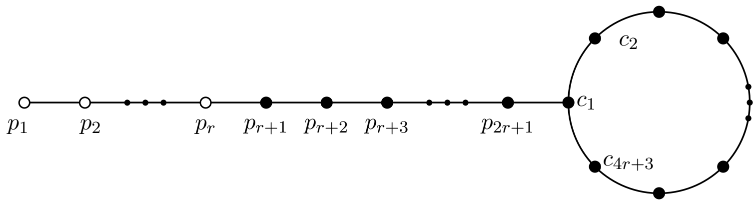

Figure 16: The graph H in Lemma 12. Codewords are in black.

Proof. The proof being much simpler than the proof of Lemma 11, we leave it to the readers who will easily convince themselves that the sets $I_r^{\mathcal{C}}(v)$ are all non-empty and distinct for all $v \in V(H)$. $\square$

Proof of Proposition 10. Consider a vertex cover $\mathcal{C}$ of G with $|\mathcal{C}| \leq k$. As in the previous sections, $V(G)$ is a subset of $V(G')$, and so $\mathcal{C}$ is a subset of $V(G')$. We construct an $(r, \leq 2)$-identifying code $\mathcal{C}'$ of G' by adding to $\mathcal{C}$ all the vertices in $V(C_e)$ for each $e \in E(G)$ (cf. Fig. 14). As each structure C_e counts $6r + 4$ vertices, the code $\mathcal{C}'$ will have cardinality

$$\left|\mathcal{C}'\right| = |\mathcal{C}| + (6r+4)m \leq k + (6r+4)m.$$

Before checking that $\mathcal{C}'$ is an $(r, \leq 2)$-identifying code of G', let us recall that $V(G')$ is partitioned in the following way:

$$V(G') = V(G) \cup \bigcup_{e \in E(G)} V(C_e).$$

Consider a set of vertices $A \subseteq V(G')$ with $|A| \in \{1, 2\}$; if we are given $I_r^{\mathcal{C}'}(A)$, we want to identify A. If $e \in E(G)$ let us define J_e as the following subset of $V(C_e)$ (see Fig. 14):

$$J_e = V(C_e) \setminus \{p_1, \cdots, p_r\}.$$

We will repeatedly use the following obvious fact:

$$I_r^{\mathcal{C}'}(A) \cap J_e \neq \emptyset \text{ if and only if } A \cap V(C_e) \neq \emptyset. \tag{3}$$

Consider three cases:

- *First case: there exist two distinct edges e_1, e_2 in $E(G)$ such that*

$$I_r^{\mathcal{C}'}(A) \cap J_{e_i} \neq \emptyset \textit{ for } i \in \{1, 2\}.$$

By (3) we must have $|A| = 2$ and A consists of two vertices v_1 and v_2 respectively belonging to $V(C_{e_1})$ and $V(C_{e_2})$. Since $e_1 \neq e_2$, we have $I_r^{\mathcal{C}'}(v_2) \cap J_{e_1} = \emptyset$; if we denote by H be the graph induced by G' on $V(C_{e_1})$ (cf. Fig. 16), Lemma 12 implies that J_{e_1} is an r-identifying code on H and thus v_1 can be identified. By the same argument v_2, and thus A can be identified.

• *Second case: there is a single edge $e_1 \in E(G)$ such that $I_r^{\mathcal{C}'}(A) \cap J_{e_1} \neq \emptyset$.*
In this case there are three possibilities:

– either $|A| = 2$ and A consists of two vertices x and v_1, where $x \in V(G)$ and $v_1 \in V(C_{e_1})$;

– or $|A| = 2$ and A consists of two vertices $v_1, v_2 \in V(C_{e_1})$;

– or $|A| = 1$ and A consists of a single vertex $v_1 \in V(C_{e_1})$.

We can simply detect if a vertex $x \in V(G)$ belongs to A: such a vertex x is r-covered by the vertices p_r in each structure C_e such that e is incident with x in G. Suppose that $x \in A$. Since $G \in \Pi_3'$, x has a degree at least 2 in G, and so there must exist two vertices p_r and p_r' in two distinct structures C_e and $C_{e'}$ such that p_r and p_r' belong to $I_r^{\mathcal{C}'}(A)$. If this happens, we easily find x as the common endpoint of e and e'. Conversely, if $A \cap V(G) = \emptyset$, we know that $A \subset V(C_{e_1})$ and the vertices p_r in structures C_e with $e \neq e_1$ cannot belong to $I_r^{\mathcal{C}'}(A)$.
Thus we have proved that in the above three possibilities, we know if we are in the first one and then we can identify x. It will remain to identify the vertex $v_1 \in C_{e_1}$; as x does not r-cover the vertices of J_{e_1}, we know by Lemma 12 that this can be done.

In the case $A \cap V(G) = \emptyset$, we must determine if $|A| = 1$ or $|A| = 2$, and then find A. Let $e_1 = xy$, with $x, y \in V(G)$; since $\mathcal{C}$ is a vertex cover of G we know that x or y belong to $\mathcal{C}$, and thus to $\mathcal{C}'$. Let us suppose without loss of generality that $x \in \mathcal{C}'$. The degree of x in G is at least two, therefore there must exist an edge $e_2 = xz$ incident with e_1 in x. Let us denote by $p_1', p_2', \cdots, p_r'$ the vertices on the path of the structure C_{e_2}. If we consider the union of the path $p_{r-1}' \cdots p_1'$ in C_{e_2}, with the vertex x and the set $V(C_{e_1})$ (see Fig. 17), we find an (isometric) induced subgraph of G' where every vertex is a codeword. Since we know that $A \subset V(C_{e_1})$, Lemma 11 can be applied, with x playing the role of p_0, and thus A can be identified.
• *Third case: for all $e \in E(G)$ we have $I_r^{\mathcal{C}'}(A) \cap J_e = \emptyset$.*
Then by (3), A consists of one or two vertices which belong to $V(G)$. Let us denote by F the set of edges $e = xy \in E(G)$ such that the vertex p_r of C_e belongs to $I_r^{\mathcal{C}'}(A)$. If $e = xy \in F$, with $x, y \in V(G)$, we know that $x \in A$ or $y \in A$, and the converse is also true: thus F is precisely the set of edges of

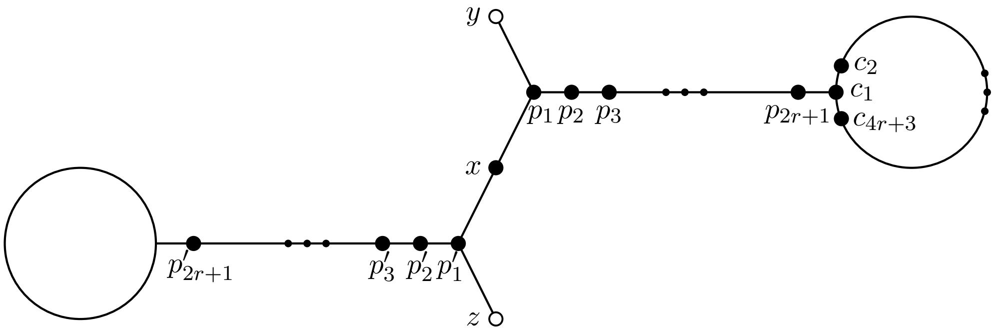

Figure 17: An induced subgraph where Lemma 11 can be applied.

G which are incident in G with the vertex, or the two vertices, in A. Can we find A if F is given ?
Let us consider the graph $G[F]$ whose edge set is F and whose vertex set consists of the elements of $V(G)$ incident with at least one edge in F (this is not always an induced subgraph of G). We claim that we can identify A by looking at the degrees of its vertices in G and in $G[F]$.
Note that if $z \in A$, then z has the same degree in $G[F]$ as in G, and this degree is two or three because $G \in \Pi'_3$; and if $z \in V(G[F]) \setminus A$, then z, in $G[F]$, can have neighbours only in A, so its degree is one or two.
Thus if $|A| = 1$, every vertex of $G[F]$ has degree one in $G[F]$ with the exception of the vertex x such that $A = \{x\}$, whereas if $|A| = 2$ there must exist at least two vertices in $G[F]$ with degree at least two. Therefore by counting the number of vertices with degree at least two in $G[F]$, we can tell if $|A|$ is equal to 1 or 2, and identify A in the former case.
If $|A| = 2$, let $A = \{x, y\}$. Because $G \in \Pi'_3$, we have three cases:
(i) If there is no vertex of degree three in $G[F]$, we have $\delta(x) = \delta(y) = 2$; then there must exist vertices z and z', distinct from x and y, such that z is adjacent to x and not to y and z' is adjacent to y and not to x. Thus z and z' are vertices of $G[F]$ and have degree one in $G[F]$: this enables us to identify x and y as the only vertices adjacent to vertices of degree one in $G[F]$.
(ii) If there are two vertices with degree three in $G[F]$, we have $\delta(x) = \delta(y) = 3$: these are precisely the only two vertices with degree three in $G[F]$, and they can be identified as such.
(iii) We are left with the case when, without loss of generality, $\delta(x) = 3$ and $\delta(y) = 2$; then x is identified as the only vertex with degree three in $G[F]$.
If at least one of the neighbours of y has degree one in $G[F]$, then, as in case (i), y can be identified by the fact that it is adjacent to a vertex of degree one in $G[F]$. So now we assume that the two neighbours of y, z_1 and z_2, have degree two or three in $G[F]$; note that if z_i has degree two in $G[F]$ then it is adjacent to x and y in $G[F]$, and if it has degree three in $G[F]$ then $z_i = x$.

If x and y are non adjacent, then z_1, z_2 are adjacent to x, y, and y can be identified as the only vertex in $G[F]$ not adjacent to x (which has already been identified). And if x and y are adjacent, then we have the following edges in $G[F]$: $yx = yz_1$, xz_2, yz_2, xz_3. Now how to know whether it is y or z_2 which belongs to A? They cannot both have degree two in G, because the characterization of Π'_3 states that they should have distinct neighbours. So z_2 has degree three in G, and y is identified by the fact that it has degree two in G and in $G[F]$.

Given G and $G[F]$, let us recapitulate how we can determine the unknown set A, where $|A| \leq 2$. If there is no edge in $G[F]$, then $A = \emptyset$. If there is only one vertex with degree at least two in $G[F]$, then this vertex is the only element in A; otherwise, $|A| = 2$. If there are two vertices with degree three in $G[F]$, then these vertices are the two elements in A. If there is no vertex with degree three in $G[F]$, then the elements in A are the two vertices adjacent to vertices of degree one in $G[F]$. Finally, if there is exactly one vertex with degree three in $G[F]$, this vertex belongs to A, and we name it x. Then if there is one vertex which is not adjacent to x in $G[F]$, this vertex is the second element in A; otherwise, the second element in A is the vertex with degree two in G and in $G[F]$.

Conversely, consider an $(r, \leq 2)$-identifying code $\mathcal{C}'$ of G' with

$$\left|\mathcal{C}'\right| \leq k + (6r+4)m.$$

First we will show that for every $e \in E(G)$ we must have $V(C_e) \subset \mathcal{C}'$. To do this, note that since $\mathcal{C}'$ is an $(r, \leq 2)$-identifying code, if we find in G' two distinct vertices a and b such that

$$(B(a,r) \cup B(b,r))\, \Delta B(b,r) = \{c\}, \tag{4}$$

where $c \in V(G')$ and Δ stands for the symmetric difference of sets, then, since we must have

$$I_r^{\mathcal{C}'}(\{a,b\}) \neq I_r^{\mathcal{C}'}(b),$$

we conclude that necessarily $c \in \mathcal{C}'$. Let us write $(a,b) \longrightarrow c$ if (4) is true for three vertices a, b and c. The following facts are easily checked on Fig. 14.

$$(p_{r+1}, p_{r+2}) \longrightarrow p_1$$

$$(p_{r+2}, p_{r+3}) \longrightarrow p_2$$

$$\cdots$$

$$(p_{2r}, p_{2r+1}) \longrightarrow p_r$$

$$(p_{2r+1}, c_1) \longrightarrow p_{r+1}$$

so we must have $p_i \in \mathcal{C}'$ for $i \in \{1, 2, \cdots, r+1\}$. Furthermore

$$(p_2, p_1) \longrightarrow p_{r+2}$$

$$(p_3, p_2) \longrightarrow p_{r+3}$$

$$\cdots$$

$$(p_{r+1}, p_r) \longrightarrow p_{2r+1}$$

and so we also have $p_i \in \mathcal{C}'$ for $i \in \{r+2, r+3, \cdots, 2r+1\}$. The same argument can be applied on the cycle:

$$(c_{r+1}, c_{r+2}) \longrightarrow c_1$$

$$(c_{r+2}, c_{r+3}) \longrightarrow c_2$$

$$\cdots$$

$$(c_{4r+2}, c_{4r+3}) \longrightarrow c_{3r+2}$$

and in the other direction

$$(c_{2r+3}, c_{2r+2}) \longrightarrow c_{3r+3}$$

$$\cdots$$

$$(c_{3r+3}, c_{3r+2}) \longrightarrow c_{4r+3}.$$

Thus for every edge $e \in E(G)$ we have $V(C_e) \subset \mathcal{C}'$. Since $|V(C_e)| = 6r+4$, if we define $\mathcal{C}$ as the trace of $\mathcal{C}'$ on $V(G)$

$$\mathcal{C} = \mathcal{C}' \cap V(G),$$

the cardinal of $\mathcal{C}$ is at most

$$|\mathcal{C}| \leq \left|\mathcal{C}'\right| - (6r+4)m \leq k.$$

To conclude, note that for every edge $xy \in E(G)$, we have in C_e

$$B(p_{r+1}, r) \Delta \left(B(p_r, r) \cup B(p_{r+1}, r)\right) = \{x, y\}$$

and so we must have $x \in \mathcal{C}$ or $y \in \mathcal{C}$; thus $\mathcal{C}$ is a vertex cover of G. This ends the proof of Proposition 10. $\square$

In conclusion, Propositions 2–5 together with Proposition 10 cover all the cases in Theorem 1 and give a complete proof for it: we have therefore shown that the problem Min $(r, \leq l)$-ID-code in Π_3 is NP-complete for $l \in \{1, 2\}$ and all $r \geq 1$.

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