Decomposing Graphs under Degree Constraints

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ABSTRACT

We prove a conjecture of C. Thomassen: If s and t are non-negative integers, and if G is a graph with minimum degree $s + t + 1$, then the vertex set of G can be partitioned into two sets which induce subgraphs of minimum degree at least s and t, respectively.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple. For a graph G, we denote by $V(G), E(G), \delta(G)$ and $\chi(G)$ the vertex set, the edge set, the minimum degree, and the chromatic number of G, respectively. The degree of a vertex x with respect to G is denoted by $d_G(x)$. Let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G(X)$, i.e., $V(G(X)) = X$ and $E(G(X)) = \{e \in E(G) | e = xy \,\&\, x, y \in X\}$.

$(A_1, \ldots, A_k)$ is called a *partition* of a set V if $A_1, \ldots, A_k$ are pairwise disjoint, non-empty subsets of V such that their union is V.

Let G be a graph and $f : V(G) \to N$ be a function, where N is the set of non-negative integers. G is said to be *f-degenerate* if for every induced subgraph H of G there is a vertex $x \in V(H)$ such that $d_H(x) \leq f(x)$.

In this paper the following result is proved.

Theorem 1. Let G be a graph and $a, b : V(G) \to N$ two functions. Assume that $d_G(x) \geq a(x) + b(x) + 1$ for every vertex $x \in V(G)$. Then there is a partition (A, B) of $V(G)$ such that

(1) $d_{G(A)}(x) \geq a(x)$ for every vertex $x \in A$, and
(2) $d_{G(B)}(x) \geq b(x)$ for every vertex $x \in B$.

By induction, Theorem 1 implies immediately the following result.

Journal of Graph Theory Vol. 23, No. 3, 321–324 (1996)

CCC 0364-9024/96/030321-04

Corollary 2. Let G be a graph and let $f_1, \ldots, f_k : V(G) \to N$ be $k \geq 2$ functions. Assume that

$$d_G(x) \geq f_1(x) + \cdots + f_k(x) + k - 1$$

for every vertex $x \in V(G)$. Then there is a partition $(A_1, \ldots, A_k)$ of $V(G)$ such that, for all $i \in \{1, \ldots, k\}, d_{G(A_i)}(x) \geq f_i(x)$ for every vertex $x \in A_i$. ∎

Another consequence of Theorem 1 is the following result which has been conjectured by C. Thomassen (see [8] or [9]).

Corollary 3. Let G be a graph and $s, t \geq 0$ integers. If $\delta(G) \geq s + t + 1$, then there is a partition (A, B) of $V(G)$ such that $\delta(G(A)) \geq s$ and $\delta(G(B)) \geq t$. ∎

The complete graph K_{s+t+1} shows that $s+t+1$ cannot be replaced by $s+t$ in Corollary 3. A weaker version of Corollary 3 was first proved by C. Thomassen [8] and subsequently improved by R. Häggkvist, N. Alon and P. Hajnal [2] (with $2s + t - 3$ instead of $s + t + 1$ where $s, t \geq 3$). In 1966, L. Lovász [4] proved a counterpart to Corollary 3: if the maximum degree of a graph G is at most $s + t + 1$, then there is a partition (A, B) of $V(G)$ such that the subgraphs $G(A)$ and $G(B)$ have maximum degree at most s and t, respectively. For some interesting generalizations of this result the reader is referred to [1].

2. PROOF OF THEOREM 1

Let G be a graph and $a, b : V(G) \to N$ two functions such that

$$d_G(x) \geq a(x) + b(x) + 1$$

for every vertex $x \in V(G)$.

For $M \subseteq V(G)$ and $x \in M$, we briefly write $d_M(x)$ instead of $d_{G(M)}(x)$. A pair (A, B) is said to be *feasible* if A and B are disjoint, non-empty subsets of $V(G)$ such that

(1) $d_A(x) \geq a(x)$ for all $x \in A$, and

(2) $d_B(x) \geq b(x)$ for all $x \in B$.

We have to show that there is a feasible partition of $V(G)$. If $a(x) = 0$, or $b(x) = 0$, for some vertex x of G, then $(\{x\}, V(G) - \{x\})$, or $(V(G) - \{x\}, \{x\})$, is a feasible partition of $V(G)$. Therefore, in what follows, we assume that

$$a(x) \geq 1 \text{ and } b(x) \geq 1$$

for every vertex $x \in V(G)$.

The following simple observation has proved very useful.

Proposition 4. If there exists a feasible pair, then there exists a feasible partition of $V(G)$, too.

Proof. Let (A, B) be a feasible pair such that $A \cup B$ is maximal. We need only to show that $A \cup B = V(G)$. Suppose that this is not true, i.e., $C = V(G) - (A \cup B)$ is non-empty. Then the maximality of $A \cup B$ implies that $(A, B \cup C)$ is not feasible. Therefore, there is a vertex $x \in C$ such that $d_{B \cup C}(x) \leq b(x) - 1$. Since $d_G(x) \geq a(x) + b(x) + 1, x$ is joined to at least $a(x) + 2$ vertices in A. But then $(A \cup \{x\}, B)$ is a feasible pair, contradicting the maximality of $A \cup B$. This proves the proposition. ∎

Obviously, Proposition 4 remains valid under the weaker assumption that $d_G(x) \geq a(x) + b(x) - 1$ for all $x \in V(G)$.

We need some further notation. By an (a, b)-*partition* of $V(G)$ we mean a partition (A, B) of $V(G)$ such that $G(A)$ is a-degenerate and $G(B)$ is b-degenerate. Moreover, we define a weight $w(A, B)$ by

$$w(A, B) = |E(G(A))| + |E(G(B))| + \sum_{x \in A} b(x) + \sum_{x \in B} a(x).$$

For the proof of Theorem 1 we consider two possible cases.

Case 1. There is no (a, b)-partition of $V(G)$. Then we argue as follows. Among all non-empty subsets of $V(G)$ we select one, say A, such that

(i) $d_A(x) \geq a(x)$ for all $x \in A$, and
(ii) $|A|$ is minimum subject to (i).

Let $B = V(G) - A$. Since $V(G) - \{v\}$ satisfies (i), for each vertex v, A exists and is a proper subset of $V(G)$. Hence, B is non-empty. Because of (ii), for every non-empty proper subset A' of A there is a vertex $x \in A'$ such that $d_{A'}(x) \leq a(x) - 1$. This implies that $d_A(x) \leq a(x)$ for some $x \in A$. Consequently, $G(A)$ is a-degenerate. Clearly, $G(B)$ is not b-degenerate, since otherwise (A, B) were an (a, b)-partition of $V(G)$. Therefore, there is a non-empty subset B' of B such that $d_{B'}(x) > b(x)$ for all $x \in B'$. Then (A, B') is a feasible pair and, by Proposition 4, there is a feasible partition of $V(G)$.

Case 2. There is an (a, b)-partition of $V(G)$. Then let (A, B) be an (a, b)-partition of $V(G)$ such that $w(A, B)$ is maximum. $G(A)$ being a-degenerate, there is a vertex $x \in A$ such that $d_A(x) \leq a(x)$. Since $d_G(x) \geq a(x) + b(x) + 1, x$ is joined to at least $b(x) + 1$ vertices in B. This implies that $|B| \geq 2$. By symmetry we also have $|A| \geq 2$.

Next, we claim that there is a non-empty subset $\tilde{A} \subseteq A$ such that $d_{\tilde{A}}(x) \geq a(x)$ for all $x \in \tilde{A}$. Suppose, on the contrary, that this is not true. Then, clearly, for each $y \in B, G(A \cup \{y\})$ is a-degenerate. $G(B)$ being b-degenerate, there is a vertex $y' \in B$ such that $d_B(y') \leq b(y')$. Let $A' = A \cup \{y'\}$ and $B' = B - \{y'\}$. Obviously, B' is non-empty. Now, we easily conclude that (A', B') is an (a, b)-partition of $V(G)$. Since $d_G(y') \geq a(y') + b(y') + 1$ and $d_B(y') \leq b(y')$, we have $d_{A'}(y') \geq a(y') + 1$ and, therefore,

$$w(A', B') - w(A, B) = d_{A'}(y') - d_B(y') + b(y') - a(y') \geq 1,$$

contradicting the maximality of $w(A, B)$. This proves the claim. By symmetry there is a non-empty subset $\tilde{B} \subseteq B$ such that $d_{\tilde{B}}(x) \geq b(x)$ for all $x \in \tilde{B}$. Then $(\tilde{A}, \tilde{B})$ is a feasible pair and, by Proposition 4, there is a feasible partition of $V(G)$.

Thus, Theorem 1 is proved.

3. CONCLUDING REMARKS

For given integers $s, t \geq 1$, let $\mathcal{G}(s, t)$ denote the class of all graphs G such that there is no pair G_1, G_2 of vertex disjoint subgraphs of G with $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$. Corollary 3 implies that $\delta(G) \leq s + t$ for each graph $G \in \mathcal{G}(s, t)$. The complete graph K_{s+t+1} is an example for which the bound is attained. Another example, for $(s, t) = (1, 4)$, is the graph of the isocahedron, but we do not know whether $\mathcal{G}(s, t)$ contains a triangle-free graph with minimum degree $s + t$ for some pair (s, t).

In a forthcoming paper [7] the following result answering a question raised by Borodin, Kostochka and Toft (private communication) is proved.

For every graph $G \in \mathcal{G}(s,t), \chi(G) \leq s + t + 1$ where equality holds if and only if $G \supseteq K_{s+t+1}$.

That a much stronger result holds was conjectured by P. Erdös and L. Lovász in 1968 (see problem 5.12 in [3]), namely: if $\chi(G) \geq s + t + 1$, and if G does not contain a K_{s+t+1}, then there is a pair G_1, G_2 of vertex disjoint subgraphs of G satisfying $\chi(G_1) \geq s+1$ and $\chi(G_2) \geq t + 1$. Besides some few special cases (see [3] or [6]), this conjecture is still unsettled.

In 1983, C. Thomassen [8] and, independently, M. Szegedy proved that for each pair s, t of positive integers there exists a smallest number $f(s,t)$ such that the vertex set of each graph of connectivity at least $f(s,t)$ has a partition into two sets which induce subgraphs of connectivity at least s and t, respectively. The complete graph K_{s+t+1} shows that $f(s,t) \geq s + t + 1$, and C. Thomassen [9] conjectured that, in fact, $f(s,t) = s + t + 1$ for all pairs s, t. P. Hajnal [2] proved that $f(s,t) \leq 4s + 4t - 13$ for $s, t \geq 3$.

For a positive integer k, let $h(k)$ denote the smallest number such that every graph of minimum degree at least $h(k)$ contains a k-connected subgraph. W. Mader [5] proved that $2k - 2 \leq h(k) \leq 4k - 6$ for $k \geq 2$. In [8], C. Thomassen proved that if an $(s + t - 1)$-connected graph G contains two vertex disjoint subgraphs of connectivity at least s and t, respectively, then there is a partition (A, B) of $V(G)$ such that $G(A)$ is s-connected and $G(B)$ is t-connected. Combining Corollary 3 with the results of Mader and Thomassen, we obtain

$$f(s,t) \leq h(s) + h(t) + 1.$$

Therefore, an improvement of the upper bound for $h(k)$ would yield an improvement of the upper bound for $f(s,t)$.

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Received September 21, 1995