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Critical edges for the assignment problem: Complexity and exact resolution

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ABSTRACT

This paper investigates two problems related to the determination of critical edges for the minimum cost assignment problem. Given a complete bipartite balanced graph with n vertices on each part and with costs on its edges, k Most Vital Edges Assignment consists of determining a set of k edges whose removal results in the largest increase in the cost of a minimum cost assignment. A dual problem, Min Edge Blocker Assignment, consists of removing a subset of edges of minimum cardinality such that the cost of a minimum cost assignment in the remaining graph is larger than or equal to a specified threshold. We show that k Most Vital Edges Assignment is NP-hard to approximate within a factor $c < 2$ and Min Edge Blocker Assignment is NP-hard to approximate within a factor 1.36. We also provide an exact algorithm for k Most Vital Edges Assignment that runs in $O(n^{k+2})$. This algorithm can also be used to solve exactly Min Edge Blocker Assignment.

1. Introduction

In many applications involving the use of communication or transportation networks, we often need to identify critical infrastructures. By critical infrastructure we mean a set of lines/nodes whose damage causes the largest inconvenience within the network. Modeling the network by a weighted graph, where weights represent costs, identifying a vulnerable infrastructure amounts to finding a subset of edges/nodes whose removal from the graph causes the largest cost increase. In the literature this problem is referred to as the *k most vital edges/nodes* problem. A dual problem consists of determining a set of edges/nodes of minimum cardinality whose removal causes the cost within the residual network to become larger than a given threshold. In the literature this problem is referred to as the *min edge/node blocker* problem. In this paper the k most vital edges and min edge blocker versions for the assignment problem are investigated.

The k most vital edges/nodes and min edge/node blocker versions have been studied for various problems including shortest path, spanning tree, maximum flow, independent set, vertex cover, p-median, p-center and maximum matching. The k most vital arcs problem with respect to shortest path was proved NP-hard in [2]. Later, k most vital arcs/nodes shortest path and min arc/node blocker shortest path were proved to be not 2-approximable and not 1.36-approximable, respectively, if $P \neq NP$ [8]. No positive result is known about the approximation of these problems. For minimum spanning tree, k most vital edges is NP-hard and $O(\log k)$-approximable [6] while several efficient exact algorithms have been proposed [10,4]. It is proved in [15] that k most vital arcs maximum flow is NP-hard. It is shown in [3] that k most vital nodes and min node blocker with respect to independent set and vertex cover for bipartite graphs remain polynomial time solvable on unweighted graphs and become NP-hard for weighted graphs. It is shown in [5] that k most vital edges p-median and k most vital edges p-center are NP-hard to approximate within a factor $\frac{7}{5} - \epsilon$ and $\frac{4}{3} - \epsilon$ respectively, for any $\epsilon > 0$, while k most vital nodes p-median and k most vital nodes p-center are NP-hard to approximate within a factor $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$. The blocker versions of these four problems are NP-hard to approximate within a factor 1.36 [5]. For maximum matching, k most vital nodes was shown polynomial time solvable for unweighted bipartite graphs and NP-hard for bipartite graphs when edge weights are bounded by a constant [16]. Moreover, min edge blocker maximum matching is NP-hard even for unweighted bipartite graphs [17], but polynomial for grids and trees [14].

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After introducing some preliminaries in Section 2, we prove in Section 3 that k Most Vital Edges Assignment and Min Edge Blocker Assignment are *NP*-hard to approximate within a constant factor. An exact algorithm is presented in Section 4 for both problems. Conclusions are provided in Section 5.

2. Basic concepts and preliminary results

Given a directed or an undirected graph $G = (V, E)$, we denote by $G - E'$ the graph obtained from G by removing a subset $E' \subseteq E$ of arcs or edges. Moreover, for any $V' \subseteq V$, $\Gamma(V')$ denotes the set of vertices which are adjacent to V'.

Given a complete bipartite graph $G = (V, E)$ with a bipartition $V = V_1 \cup V_2$ where $|V_1| = |V_2| = n$ and costs c_{ij} associated with each edge $(i, j) \in E$, the assignment problem consists of determining a perfect matching of minimum total cost. Let a^* denote a minimum cost assignment in G.

We consider in this paper the k most vital edges and min edge blocker versions of the assignment problem. These problems are defined respectively as follows.

k Most Vital Edges Assignment

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, where each edge $(i, j) \in E$ has a cost c_{ij}, and an integer k.

Output: A subset $S^* \subseteq E$, with $|S^*| = k$, such that the minimum cost of an assignment in $G - S^*$ is maximum.

Min Edge Blocker Assignment

Input: A complete bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and $|V_1| = |V_2| = n$, where each edge $(i, j) \in E$ has a cost c_{ij}, and an integer U.

Output: A subset $S^* \subseteq E$ of minimum cardinality such that the minimum cost of an assignment in $G - S^*$ is at least U.

Given an optimization problem and an instance I of this problem, we denote by $|I|$ the size of I, by $opt(I)$ the optimum value of I and by $val(I, S)$ the value of a feasible solution S of I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max\left\{\frac{val(I,S)}{opt(I)}, \frac{opt(I)}{val(I,S)}\right\}$. The *error* of S, $\varepsilon(I, S)$, is defined by $\varepsilon(I, S) = r(I, S) - 1$.

For a function f, an algorithm is an $f(n)$-*approximation*, if for every instance I of the problem, it returns a solution S such that $r(I, S) \leq f(|I|)$.

The notion of a *gap*-reduction was introduced in [1] by Arora and Lund. In this paper we use a *gap*-reduction between two minimization problems. A minimization problem Π is called *gap-reducible* to a minimization problem Π' with parameters (c, ρ) and (c', ρ'), if there exists a polynomial time computable function f such that f maps an instance I of Π to an instance I' of Π', while satisfying the following properties.

- If $opt(I) \leq c$ then $opt(I') \leq c'$.
- If $opt(I) > c\rho$ then $opt(I') > c'\rho'$.

Parameters c and ρ are functions of $|I|$ and parameters c' and ρ' are functions of $|I'|$. Also, we have $\rho, \rho' \geq 1$.

The interest of a *gap*-reduction is that if Π is not approximable within a factor ρ then Π' is not approximable within a factor ρ'.

The notion of an E-reduction (*error-preserving* reduction) was introduced by Khanna et al. [9]. A problem Π is called *E-reducible* to a problem Π', if there exist polynomial time computable functions f, g and a constant β such that

- f maps an instance I of Π to an instance I' of Π' such that $opt(I)$ and $opt(I')$ are related by a polynomial factor, i.e. there exists a polynomial $p(n)$ such that $opt(I') \leq p(|I|)opt(I)$,
- g maps solutions S' of I' to solutions S of I such that $\varepsilon(I, S) \leq \beta\varepsilon(I', S')$.

An important property of an E-reduction is that it can be applied uniformly to all levels of approximability; that is, if Π is E-reducible to Π' and Π' belongs to $\mathcal{C}$ then Π belongs to $\mathcal{C}$ as well, where $\mathcal{C}$ is a class of optimization problems with any kind of approximation guarantee (see [9] for more details).

To conclude this section, we give a preliminary result concerning our problems.

Lemma 1. *Given a complete bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$, for any subset $S \subset E$ with $|S| \leq n - 1$, $G - S$ contains an assignment.*

Proof. We show that the sufficient condition of Hall's theorem is satisfied, i.e. that $|\Gamma(A)| \geq |A|$ for all $A \subset V_1$, which means that we can match V_1 in V_2, thus obtaining an assignment. In order to reduce $|\Gamma(A)|$ by one unit, S must contain $|A|$ edges incident to the same node of V_2. Thus, after removing edges of S, A loses at most $\lfloor\frac{|S|}{|A|}\rfloor$ neighbors in V_2. Then, we have $|\Gamma(A)| \geq n - \lfloor\frac{|S|}{|A|}\rfloor$. If $|A| = n$, we have $|\Gamma(A)| \geq n$ and then $|\Gamma(A)| \geq |A|$. If $|A| \leq n - 1$, we have $|\Gamma(A)| \geq n - \frac{|S|}{|A|} \geq n - \frac{n-1}{|A|} = \frac{(|A|-1)\,n+1}{|A|} \geq \frac{(|A|-1)(|A|+1)+1}{|A|} = |A|$. □

Observe that there exists a subset S of edges, with $|S| \geq n$, such that no assignment exists in $G - S$. Indeed, if we select in Sn edges incident to the same node v, then in $G - S$ node v becomes isolated and cannot be assigned.

Therefore, we suppose in the following that $k \leq n - 1$ for k Most Vital Edges Assignment and that $|S^*| \leq n$ for any optimal solution S^* for Min Edge Blocker Assignment.

Observe finally that in order to have a chance to increase the value of a minimum cost assignment in $G - S^*$, S^* must contain at least one edge of a^* so as to eliminate a^* as an optimal solution.

3. Complexity

We study in this section the complexity of k Most Vital Edges Assignment and Min Edge Blocker Assignment. We show that each of these two problems is not approximable within a ratio that is better than a certain constant, unless $P = NP$.

Hoffman and Markowitz [7] describe a polynomial reduction from the shortest path problem to the assignment problem. We extend this reduction in order to prove our inapproximability results. For this, we propose reductions from k Most Vital Arcs Shortest Path and Min Arc Blocker Shortest Path defined as follows:

k Most Vital Arcs Shortest Path

Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$, the length ℓ_{ij} for each arc $(i, j) \in A$, and an integer k.

Output: A subset $A' \subseteq A$, with $|A'| = k$, such that the minimum length of a path from s to t in $G - A'$ is maximum.

For an instance of k Most Vital Arcs Shortest Path formed by a graph G, we consider that $k \leq \lambda_{s,t}(G) - 1$, where $\lambda_{s,t}(G)$ is the cardinality of an $s - t$ minimum cut in G. Otherwise, taking all arcs of an $s - t$ minimum cut among the k arcs to be removed would lead to a solution with infinite value.

Min Arc Blocker Shortest Path

Input: A directed graph $G = (V, A)$, two vertices $s, t \in V$, the length ℓ_{ij} for each arc $(i, j) \in A$, and an integer U.

Output: A subset $A' \subseteq A$ of minimum cardinality such that the minimum length of a path from s to t in $G - A'$ is at least U.

An optimal solution A' of an instance of Min Arc Blocker Shortest Path formed by a graph G is such that $|A'| \leq \lambda_{s,t}(G)$.

We define in the following the construction used in our reductions.

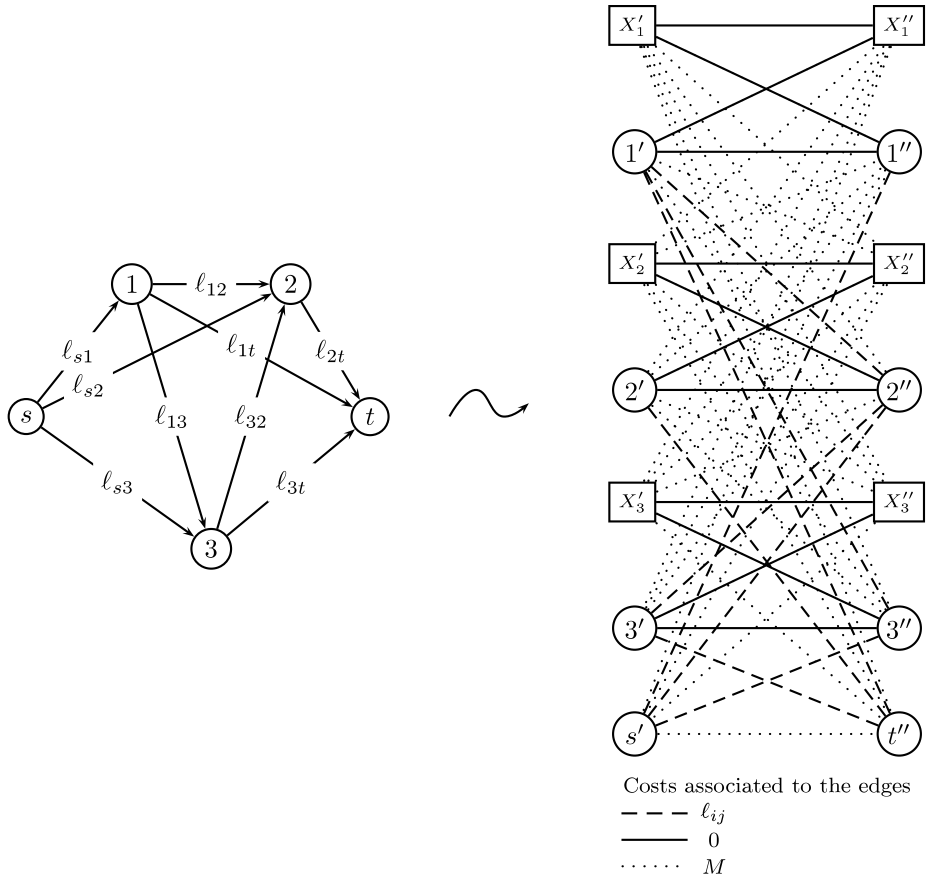

Fig. 1. Construction of $\widetilde{G}$ from G.

Consider an instance of the shortest path problem: a directed graph $G = (V, A)$ with $|V| = n$ including two vertices $s, t \in V$ corresponding to the origin and destination nodes respectively, and the length ℓ_{ij} for each arc $(i, j) \in A$. We construct an instance $\widetilde{G} = (W, E)$ of the assignment problem with bipartition $W = V' \cup V''$ (see Fig. 1). For each vertex $i \in V \setminus \{s, t\}$ we associate two vertices $i' \in V'$ and $i'' \in V''$, and we add vertex s' to V' and vertex t'' to V''. We create, for each arc $(i, j) \in A$, an edge (i', j'') in E of cost ℓ_{ij} and, for each vertex $i \in V \setminus \{s, t\}$, an edge (i', i'') in E of cost 0. To complete the construction of $\widetilde{G}$, we consider a complete bipartite graph $K^i = (X_i, Y_i)$ for each $i \in V \setminus \{s, t\}$ with $X_i = X_i' \cup X_i''$, where $X_i' = \{x'_{i1}, \ldots, x'_{i(n-1)}\}$ and $X_i'' = \{x''_{i1}, \ldots, x''_{i(n-1)}\}$, and a cost 0 associated to each edge of Y_i. We add the edges $(i', x''_{i\ell})$ and $(x'_{i\ell}, i'')$ of cost 0 for each $i \in V \setminus \{s, t\}$ and $\ell = 1, \ldots, n-1$. Hence, we have $|V'| = |V''| = 1 + n(n-2)$. Finally, in order to obtain a *complete* bipartite graph $\widetilde{G}$, we add *dummy* edges of cost $M = \sum_{(i,j)\in A} \ell_{ij} + 1$.

We denote by $\mathcal{P}$ the set of all simple paths from s to t in G, by $\mathcal{A}$ the set of all feasible assignments in $\widetilde{G}$ and by $\mathcal{A}' \subseteq \mathcal{A}$ the set of all feasible assignments in $\widetilde{G}$ that do not include any dummy edge of cost M.

The following constructions describe a transformation from a path in $\mathcal{P}$ to an assignment in $\mathcal{A}'$ and its converse transformation.

1. For each simple path p in $\mathcal{P}$ we associate a unique assignment a^p in $\mathcal{A}'$ in the following way: we include in a^p, the edge $(i', j'') \in E$ for each arc $(i, j) \in p$, the edges $(i', i'') \in E$ if vertex i does not belong to path p and the edges $(x'_{i\ell}, x''_{i\ell})$ for $\ell = 1, \ldots, n-1$, $i \in V \setminus \{s, t\}$. Clearly, the cost of a^p is the same as the length of p.
2. Each assignment a in $\mathcal{A}'$ contains a subset of edges (s', i_1''), $(i_1', i_2''), \ldots, (i'_{b-1}, i_b''), (i_b', t'')$ corresponding to a unique simple path $p^a = (s, i_1, i_2, \ldots, i_b, t)$ in $\mathcal{P}$. Indeed, each a in $\mathcal{A}'$ necessarily contains an edge of type (s', i''). Moreover, if edges $(s', i_1''), (i_1', i_2''), \ldots, (i'_{c-1}, i_c'')$ belong to a then there exists $k \in V \setminus \{i_1, i_2, \ldots, i_c\}$ such that (i_c', k'') belongs to a. Clearly $k \in \{i_1, i_2, \ldots, i_c\}$ is impossible, but also $(i_c', x''_{i_c\ell})$ since otherwise a must contain a dummy edge incident to one vertex of X'_{i_c}. Assignment a can also contain a set of edges of type (i', i'') or $(i', x''_{i\ell})$ or $(x'_{i\ell}, i'')$ or $(x'_{i\ell}, x''_{ij})$ and possibly a set of edges corresponding to arcs forming circuits in G.
In general, the cost of a is equal to the length of p^a plus the lengths of the circuits corresponding to the cycles described by a. However, when a is a minimum cost assignment, the cost of a coincides with the length of p^a, since the cycles described by a can only have a cost 0 (otherwise all vertices i of these cycles could be replaced by edges (i', i'') with cost 0).

Given a subset S of arcs from G, the subset of edges associated to S in $\widetilde{G}$, denoted by $\widetilde{Im}(S)$, is defined by $\widetilde{Im}(S) = \{(i', j'') \in E : (i, j) \in S\}$. We have $|\widetilde{Im}(S)| = |S|$.

Given a subset $\widetilde{S}$ of edges from $\widetilde{G}$, the subset of arcs associated to $\widetilde{S}$ in G, denoted by $Im(\widetilde{S})$, is defined by $Im(\widetilde{S}) = \{(i, j) \in A : (i', j'') \in \widetilde{S}, i \neq j, c_{i'j''} \neq M\}$. We have $|Im(\widetilde{S})| \leq |\widetilde{S}|$.

Observe that for any subset S of arcs we have $Im(\widetilde{Im}(S)) = S$.

In the following, we present two preliminary results. The first one characterizes a minimum cost assignment generated by deleting a subset of edges and the second one allows us to establish the non-approximability results for k Most Vital Edges Assignment and Min Edge Blocker Assignment.

Lemma 2. *For any subset $\widetilde{S} \subset E$ of cardinality k, with $k \leq \lambda_{s,t}(G) - 1$, any minimum cost assignment in $\widetilde{G} - \widetilde{S}$ does not contain any dummy edge of cost M.*

Proof. By removing the subset of edges $\widetilde{S}$ of E of cardinality k, the subset of arcs $Im(\widetilde{S})$ contains at most k arcs of G. Since $k \leq \lambda_{s,t}(G)-1$ then there exists at least one path from s to t in $G-Im(\widetilde{S})$. Denote by p a shortest path from s to t in $G-Im(\widetilde{S})$. If no edge of a^p belongs to $\widetilde{S}$, then the result is established since a^p is an assignment in $\widetilde{G}-\widetilde{S}$ of cost less than M. Otherwise, consider the nonempty set of edges $a^p \cap \widetilde{S}$. These edges belong either to complete bipartite subgraphs K_i' induced by $X_i' \cup X_i''$ when $i \in V(p) \setminus \{s,t\}$ or to complete bipartite subgraphs K_i'' induced by $X_i' \cup X_i'' \cup \{i', i''\}$ when $i \in V \setminus V(p)$. All these subgraphs contain only edges of cost 0. Moreover, subgraphs K_i' contain $n-1$ vertices on each part while subgraphs K_i'' contain n vertices on each part. Since $|\widetilde{S}| \leq n-2$, we can apply Lemma 1 to all relevant subgraphs K_i' and K_i'' and derive an assignment a' with the same cost as a^p (and thus without dummy edges) but without edges belonging to $\widetilde{S}$. Since a' has a cost less than M, it is also the case for any minimum cost assignment in $\widetilde{G}-\widetilde{S}$ which thus does not contain dummy edges. $\square$

Lemma 3. (i) *Let S be a subset of k arcs of G, with $k \leq \lambda_{s,t}(G)-1$, and p be a shortest path from s to t in $G-S$. There exists a subset $\widetilde{S} = \widetilde{Im}(S)$ of k edges of $\widetilde{G}$ such that the assignment a^p is a minimum cost assignment in $\widetilde{G}-\widetilde{S}$ and the cost of a^p is the same as the length of p.*
(ii) *Let $\widetilde{S}$ be a subset of k edges of $\widetilde{G}$, with $k \leq \lambda_{s,t}(G)-1$, and a be a minimum cost assignment in $\widetilde{G}-\widetilde{S}$. There exists a subset $S' \supseteq Im(\widetilde{S})$ of k arcs such that the path p^a is a shortest path from s to t in $G-S'$ and its length is the same as the cost of a.*

Proof. (i) The existence of an assignment a of cost lower than that of a^p in $\widetilde{G}-\widetilde{Im}(S)$ would imply that there exists in $G-Im(\widetilde{Im}(S)) = G-S$ a path p^a of length strictly less than that of p. Hence, a^p is a minimum cost assignment in $\widetilde{G}-\widetilde{S}$ and its cost is the same as the length of p.

(ii) According to Lemma 2, a contains no dummy edge of cost M. Let $S' = Im(\widetilde{S}) \cup S''$, where S'' is any subset of $k-|Im(\widetilde{S})|$ arcs not belonging to p^a. The length of p^a is the same as the cost of a. We show in the following that p^a is a shortest path from s to t in $G-S'$.

Suppose that there exists a path p from s to t in $G-S'$ of length strictly less than that of p^a. Let a^p be the assignment corresponding to p in $\widetilde{G}-\widetilde{Im}(S')$. By construction, a^p contains no dummy edge. If a^p contains no edge of $\widetilde{S}$ then a^p is an assignment in $\widetilde{G}-\widetilde{S}$ of cost strictly less than that of a, which contradicts the optimality of a in $\widetilde{G}-\widetilde{S}$. Otherwise, a^p can contain only edges of $\widetilde{S}$ of type (i', i''), $i = 1, \ldots, n-2$, or $(x'_{i\ell}, x''_{i\ell})$, $\ell = 1, \ldots, n-1$. Then, we can exhibit an assignment a' from a^p in $\widetilde{G}-\widetilde{Im}(S')$ which contains no edge of $\widetilde{S}$ and with the same cost as that of a^p, as shown in the proof of Lemma 2. Hence, a' is an assignment in $\widetilde{G}-\widetilde{S}$ of cost strictly less than that of a, contradicting again the optimality of a in $\widetilde{G}-\widetilde{S}$. Therefore, p^a is a shortest path from s to t in $G-S'$. $\square$

We are now in a position to give our two main inapproximability results.

Theorem 1. *k Most Vital Edges Assignment is NP-hard to approximate within a factor $2-\epsilon$, for any $\epsilon > 0$.*

Proof. We construct an E-reduction from k Most Vital Arcs Shortest Path which is shown to be *NP*-hard to approximate within a factor $2-\epsilon$, for any $\epsilon > 0$ [8]. This establishes that k Most Vital Edges Assignment is also *NP*-hard to approximate within a factor $2-\epsilon$, for any $\epsilon > 0$.

Let I be an instance of k Most Vital Arcs Shortest Path consisting of a graph $G = (V, A)$. We use the previous construction to define from I an instance $\widetilde{I}$ of k Most Vital Edges Assignment formed by the graph $\widetilde{G} = (W, E)$.

Consider an optimal solution $S \subset A$ for I, with $|S| = k$, and denote by p a path of minimum length from s to t in $G-S$. When removing from $\widetilde{G}$ the subset of edges $\widetilde{Im}(S)$, the assignment a^p is, according to Lemma 3(i), a minimum cost assignment in $\widetilde{G}-\widetilde{Im}(S)$. Thus, $opt(\widetilde{I}) \geq opt(I)$.

Consider now a solution $\widetilde{S} \subset E$ of $\widetilde{I}$, with $|\widetilde{S}| = k$, and denote by a a minimum cost assignment in $\widetilde{G}-\widetilde{S}$. Consider the subset of arcs $Im(\widetilde{S})$ and let p^a be the path from s to t in $G-Im(\widetilde{S})$ corresponding to a. Let S be a subset of k arcs consisting of $Im(\widetilde{S})$ possibly completed by any subset of $k-|Im(\widetilde{S})|$ arcs not belonging to p^a. According to Lemma 3(ii), p^a is a path of minimum length in $G-S$ whose length is equal to the cost of a. Hence, $val(I, S) = val(\widetilde{I}, \widetilde{S})$. In particular, if $\widetilde{S}$ is an optimal solution of $\widetilde{I}$, then $opt(\widetilde{I}) = val(I, S) \leq opt(I)$.

Therefore, we have $opt(I) = opt(\widetilde{I})$ and the error of the two solutions S and $\widetilde{S}$ are equal $\varepsilon(I, S) = \varepsilon(\widetilde{I}, \widetilde{S})$. $\square$

We prove now an inapproximability result for Min Arc Blocker Assignment. Unlike for k Most Vital Edges Assignment, using our construction, it seems difficult to build an E-reduction which imposes conditions on all feasible solutions (in particular for those in $\widetilde{G}$ of size more than $\lambda_{s,t}(G)$ that do not give necessarily a feasible solution in G). Thus, we resort to a gap-reduction which imposes conditions on optimal solutions only.

Theorem 2. *Min Edge Blocker Assignment is NP-hard to approximate within a factor 1.36.*

Proof. We construct a gap-reduction from Min Arc Blocker Shortest Path which is known to be *NP*-hard to approximate within a factor 1.36 even for graphs G such that the optimum value is less than $\lambda_{s,t}(G)$ [8].

Let I be an instance of Min Arc Blocker Shortest Path consisting of a graph $G = (V, A)$ and a positive integer U. We use the previous construction to define from I an instance $\widetilde{I}$ of Min Edge Blocker Assignment formed by the graph $\widetilde{G} = (W, E)$ and U.

Consider an optimal solution $S \subset A$ for I, and denote by p a path of minimum length in $G-S$ from s to t. Since $|S| \leq \lambda_{s,t}(G)-1$, according to Lemma 3(i), the assignment a^p is a minimum cost assignment in $\widetilde{G}-\widetilde{Im}(S)$ of cost equal to the length of p, which is at least U. Thus, we have $opt(\widetilde{I}) \leq opt(I) \leq \lambda_{s,t}(G)-1$.

Let $\widetilde{S} \subset E$ be an optimal solution of $\widetilde{I}$, and denote by a an assignment of minimum cost in $\widetilde{G}-\widetilde{S}$. Assignment a is such that its cost is at least U. According to Lemma 3(ii), there exists a subset S' of $|\widetilde{S}|$ arcs such that the path p^a is a shortest path in $G-S'$ and its length is the same as the cost of a. The length of p^a is then greater than or equal to U. Hence, $opt(I) \leq |S'| = opt(\widetilde{I})$. Thus $opt(\widetilde{I}) = opt(I)$, showing that $opt(I) \leq c$ implies $opt(\widetilde{I}) \leq c$ and $opt(I) > c\rho$ implies $opt(\widetilde{I}) > c\rho$ which establishes that Min Edge Blocker Assignment is also *NP*-hard to approximate within a factor 1.36. $\square$

4. Exact resolution

We propose in this section an exact algorithm for solving k Most Vital Edges Assignment and Min Edge Blocker Assignment. Consider $G = (V_1 \cup V_2, E)$ a complete bipartite graph with $|V_1| = |V_2| = n$ and a cost is associated to each edge of E. Denote by a^* a minimum cost assignment in G.

An approach to solve 1 Most Vital Edge Assignment is to delete one by one each of the n edges belonging to a^*, determine the minimum cost assignments on the n resulting partial graphs, and retain the deleted edge which leads to a largest minimum cost assignment. This approach is very similar to the scheme developed by Murty [12] for ranking the assignments in increasing cost order, except that in Murty's approach a minimum cost assignment is selected among the n candidate assignments. In this context, Miller et al. [11] and Pedersen et al. [13] showed that the n assignments can be found efficiently using reoptimization. Indeed, given an edge $e = (y, z) \in a^*$, a minimum cost assignment a_e in $G - \{e\}$ can be found using Dijkstra's algorithm in $O(n^2)$ by solving a single shortest path problem between y and z where arcs are valued by (nonnegative) reduced costs. Therefore, the time complexity for finding all assignments a_e for all edges $e \in a^*$ is $O(n^3)$. Thus, we obtain the following result.

Theorem 3. 1 Most Vital Edge Assignment *can be solved in $O(n^3)$ for complete bipartite graphs with n vertices in each part.*

In the following, we are interested in the exact resolution of k Most Vital Edges Assignment. Taking advantage of the fact that optimal solutions must contain at least one edge of a^*, a naive approach would be to remove each edge $e \in a^*$, consider all possible combinations of $k - 1$ edges to delete from the $n^2 - 1$ remaining edges and determine a minimum cost assignment in the resulting partial graphs. An optimal solution is a subset of removed edges which leads to the largest minimum cost assignment. Hence, a naive approach for solving k Most Vital Edges Assignment would require $n \binom{n^2-1}{k-1} O(n^3) = O(n^{2k+2})$ time. A more efficient algorithm can be obtained through the following result.

Theorem 4. k Most Vital Edges Assignment *can be solved in $O(n^{k+2})$ time for complete bipartite graphs with n nodes in each part and for general k.*

Proof. Consider a minimum cost assignment a^* in G. Obviously, a set S^* of k most vital edges must contain at least one edge e in a^*. Consider now a minimum cost assignment b^* in $G - \{e\}$. If $k \geq 2$, then S^* must contain at least one edge of b^*, and so on. Hence, by simply enumerating all possibilities to choose an edge in a^*, then one in b^* and so on, one can find an optimal solution by looking at $O(n^k)$ possible subsets of removed edges. At each step, we compute a minimum cost assignment in time $O(n^2)$, as for example when determining b^* in $G - \{e\}$ starting from a^*. Therefore, we compute in this way $n + n^2 + \cdots + n^k$ minimum cost assignments, resulting in a time $O(n^{k+2})$. □

This algorithm can be implemented by developing a search tree with $k + 1$ levels. The root node at level 0 corresponds to the optimal assignment a^* and each node at level i ($i = 1, \ldots, k$) represents a tentative selection of i edges which could be part of the k most vital edges. A refined implementation, avoiding the repetition of tentative selections but still in $O(n^{k+2})$, can be obtained using a branching scheme similar to the one used by Murty [12]. Moreover, observe that solving k Most Vital Edges Assignment in this way (developing a complete or reduced search tree) allows the determination of an optimal solution for i Most Vital Edges Assignment by simply scanning all nodes of level i and retaining a node corresponding to the largest minimum cost assignment ($i = 1, \ldots, k$).

We show now how to solve Min Edge Blocker Assignment. If the minimum cost of an assignment is at least U then the optimal cardinality is 0. Otherwise, we search for the smallest level i, $1 \leq i \leq n - 1$ such that the optimum value of i Most Vital Edges Assignment is at least U. If such an i does not exists, then any subset of n edges incident to a vertex is optimal. Thus, considering that we need to develop our search tree until level $n - 1$ at most, we can solve Min Edge Blocker Assignment in $O(n^{n+1})$.

5. Conclusions

We established in this paper negative results concerning the approximation of k most vital edges and min edge blocker versions of the assignment problem.

It is remarkable that all the proofs of NP-hardness or inapproximability previously used up to now for k most vital edges and min edge blocker versions of classical optimization problems are based on reductions from standard problems like vertex cover, clique, independent set, or min k cut. Our proofs are the first ones using reductions from a k most vital edges and min edge blocker version of a classical optimization problem, namely shortest path. A main advantage of our E-reduction is to preserve the value of solutions and therefore approximation properties between these versions of shortest path and assignment. Thus, a polynomial time approximation algorithm for k Most Vital Edges Assignment would imply a polynomial time approximation algorithm with the same approximation ratio for the corresponding versions of shortest path. A gap-reduction only preserves inapproximability results. Thus, any stronger inapproximability result for k most vital edges and min edge blocker shortest path would give rise to the same result for the corresponding versions of assignment.

Concerning positive results, we proposed exact algorithms, in $O(n^{k+2})$ for k Most Vital Edges Assignment and in $O(n^{n+1})$ for Min Edge Blocker Assignment. An interesting open question is to try to establish approximation algorithms or better exact algorithms for these problems.

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