A BIBLIOGRAPHY ON MULTICUT AND INTEGER MULTIFLOW PROBLEMS

CÉDRIC BENTZ[1], MARIE-CHRISTINE COSTA[1], LUCAS LÉTOCART[2] AND FRÉDÉRIC ROUPIN[3]

(1) *CEDRIC, CNAM, 292 rue St-Martin 75141 Paris cedex 03, France.*
(2) *LIPN, Université Paris 13, Institut Galilée, avenue J.-B. Clément 93430 Villetaneuse, France.*
(3) *CEDRIC, CNAM-IIE, 18 allée Jean Rostand 91025 Evry cedex, France.*
(e-mails: {cedric.bentz,costa}@cnam.fr, letocart@lipn.univ-paris13.fr, roupin@iie.cnam.fr)

Abstract. We present a bibliography about the maximum integral multiflow and the minimum multicut problems and their subproblems, such as the multiterminal cut and the unsplittable flow problems. Some references also concern related problems, such as the integer multiflow problem with demands or the minimum cost multiflow problem. Papers are relative to unrestricted graphs as well as special graphs (trees, meshes, rings, trees of rings, bipartite and planar graphs, ...) Most of the results are very recent, and a table summarizes the most important ones.

Keywords: integer multicommodity flow, minimum multicut, multiway cut, complexity, approximation.

Remark 1. We try to keep this bibliography as complete as possible. Please let us know any comment or suggestion, mistake or missing paper, by sending us an e-mail (costa@cnam.fr or cedric.bentz@cnam.fr).

Remark 2. IMFP and IMCP respectively denote the maximum integer multiflow problem and the minimum multicut problem. For the exact definition of all the terms used in Table 1, see [1].

Remark 3. Each paper belongs to exactly one of the five categories used in this bibliography:

- Existence of disjoint paths and integer multiflow problems with demands (decision problems): see [2] to [25];
- IMFP and subproblems: see [26] to [69];
- IMCP and subproblems: see [70] to [98];
- Both IMFP and IMCP: see [99] to [112];
- Minimum cost integer multiflow: see [113] to [117].

Almost half of the references listed in this document can be found in [1].

References

[1] M.-C. Costa, L. Létocart and F. Roupin. Minimal multicut and maximal integer multiflow: A survey. EJOR 162 (2005) 55–69.

Integer multiflow with demands

[2] C. Chekuri, M. Mydlarz and F.B. Shepherd. Demand Flow in a Tree and Packing Integer Programs. Proceedings ICALP 03 (2003).

[3] S. Cosares and I. Saniee. An optimization problem related to balancing loads on SONET rings. Telecommunication Systems 3 (1994) 165–181.

[4] S. Even, A. Itai and A. Shamir. On the complexity of timetable and multicommodity flow problems. SIAM J. Comp. 5 (1976) 691–703.

[5] S. Fortune, J. Hopcroft and J. Wyllie. The directed subgraph homeomorphism problem. Theoretical Computer Science 10 (1980) 111–121.

[6] A. Frank. Disjoint paths in a rectilinear grid. Combinatorica 2 (1982) 361–371.

[7] A. Frank. Edge-disjoint paths in planar graphs. J. Comb. Theory, Series B 39 (1985) 164–178.

[8] A. Frank. Packing paths, circuits and cuts-a survey. In B. Korte, L. Lovász, H.J. Prömel and A. Schrijver (eds). *Paths, Flows and VLSI-Layout.* Algorithms and combinatorics 9 (1990) 47–100. Springer-Verlag. Berlin.

[9] A. Frank and Z. Szigeti. A note on packing paths in planar graphs. Math. Programming 70 (1995) 201–210.

[10] A. Frank, B. Shepherd, V. Tandon and Z. Végh. Node-capacitated ring routing. EGRES Technical report 2001-09 (2001).

[11] M. Kaufmann and K. Mehlhorn. Routing Problems in Grid Graphs. In B. Korte, L. Lovász, H. J. Prömel and A. Schrijver (eds). *Paths, Flows and VLSI-Layout.* Algorithms and combinatorics 9 (1990) 165–184. Springer-Verlag. Berlin.

[12] E. Korach and M. Penn. Tight integral duality gap in the Chinese postman problem. Mathematical Programming 55 (1992) 183–191.

[13] M.E. Kramer and J. van Leeuwen. The complexity of wire routing and finding the minimum area layouts for arbitrary VLSI circuits. Advances in computing research 2: VLSI theory, F.P. Preparata (ed.), JAI Press, London (1984) 129–146.

[14] P. Kubat, A. Shulman, R. Vachani and J. Ward. Multicommodity flows in ring networks. INFORMS Journal on Computing 8 (1996) 235–242.

[15] M.V. Lomonosov. Multiflow feasibility depending on cuts. Graph Theory Newsl. 9 (1979) 4.

[16] D. Marx. Eulerian disjoint paths problem in grid graphs is NP-complete. Discrete Applied Mathematics 143 (2004) 336–341.

[17] M. Middendorf and F. Pfeiffer. On the complexity of the disjoint path problem. Combinatorica 13 (1993) 97–107.

[18] T. Nishizeki, J. Vygen and X. Zhou. The edge-disjoint paths problem is NP-complete for series-parallel graphs. Discrete Applied Mathematics 115 (2001) 177–186.

[19] H. Okamura and P.D. Seymour. Multicommodity flows in planar graphs. Journal of Combinatorial Theory, Series B 31 (1981) 75–81.

[20] N. Robertson and P.D. Seymour. Graphs minors XIII: The disjoint paths problem. J. Comb. Theory, Ser. B, 63 (1995) 65–110.

[21] A. Schrijver. Homotopic routing methods. In B. Korte, L. Lovász, H.J. Prömel and A. Schrijver. *Paths, Flows and VLSI-Layout.* Algorithms and combinatorics 9 (1990) 329–371. Springer-Verlag. Berlin.

[22] A. Sebö. Integer plane multicommodity with a fixed number of demands. J. Combin. Theory, Ser. B, 59 (1993) 163–171.

[23] A. Slivkins. Parameterized Tractability of Edge-Disjoint Paths on Directed Acyclic Graphs. ESA 03 (2003).

[24] A. Srivastav and P. Stangier. Integer multicommodity flows with reduced demands. Technical Report 93–146, Köln University, Germany (1993).

[25] K. Weihe. Edge-disjoint routing in plane switch graphs in linear time. Proceedings FOCS 99 (1999) 330–340. New York City.

Maximum integer multiflow

[26] U. Adamy, C. Ambuehl, R. Sai Anand and T. Erlebach. Call Control in Rings. Proceedings ICALP, Lecture Notes in Computer Science 2380 (2002) 788–799.

[27] A. Aggarwal, J. Kleinberg and D.P. Williamson. Node-disjoint paths on the mesh and a new trade-off in VLSI layout. Proceedings STOC 96 (1996).
[28] R. Sai Anand and T. Erlebach. Routing and call control algorithm for ring networks. TIK-Report 171 (2003).
[29] M. Andrews and L. Zhang. Hardness of the undirected edge-disjoint paths problem. Proceedings STOC 05 (2005).
[30] A. Baveja and A. Srinisvasan. Approximation algorithms for disjoint paths and related routing and packing problems. Mathematics of Operations Research 25 (2000) 255–280.
[31] L. Brunetta, M. Conforti and M. Fischetti. A polyhedral approach to an integer multicommodity flow problem. Discrete applied Mathematics 101 (2000) 13–26.
[32] P. Carmi, T. Erlebach and Y. Okamoto. Greedy edge-disjoint paths in complete graphs. TIK-Report 155 (2003).
[33] A. Chakrabarti, C. Chekuri, A. Gupta and A. Kumar. Approximation Algorithms for the Unsplittable Flow Problem. Proceedings of the 5th International Workshop on Approximation Algorithms for Combinatorial Optimization, Lecture Notes In Computer Science (2002) 51–66.
[34] W.-T. Chan, F.Y.L. Chin and H.F. Ting. A faster algorithm for finding disjoint paths in grids. Proceedings of the 10th International Symposium on Algorithms and Computation *ISAAC'99* (1999) 393–402. India. Springer-Verlag.
[35] W.-T. Chan and F.Y.L. Chin. Efficient algorithms for finding the maximum number of disjoint paths in grids. Journal of Algorithms 34 (2000) 337–369. Academic Press.
[36] C. Chekuri and S. Khanna. Edge disjoint paths revisited. Proceedings SODA 03 (2003) 628–637.
[37] C. Chekuri, S. Khanna and F.B. Shepherd. The All-Or-Nothing Multicommodity Flow Problem. Proceedings STOC 04 (2004).
[38] C. Chekuri, S. Khanna and F.B. Shepherd. Edge-disjoint paths in planar graphs. Proceedings FOCS 04 (2004).
[39] C. Chekuri, S. Khanna and F.B. Shepherd. Multicommodity Flow, Well-linked Terminals, and Routing Problems. Proceedings STOC 05 (2005).
[40] C. Chekuri, S. Khanna and F.B. Shepherd. An $O(\sqrt{n})$-approximation and integrality gap for EDP and UFP in undirected graphs and DAGs. Working paper. Submitted to ORL (2005).
[41] C. Chekuri, S. Khanna and F.B. Shepherd. Edge-Disjoint Paths in Planar Graphs with Constant Congestion. Working paper. Submitted (2005).
[42] T. Erlebach and K. Jansen. Call Scheduling in Trees, Rings and Meshes. Proceedings of the 30th Hawaii International Conference on System Sciences (HICSS-30), Vol. 1. IEEE Computer Society Press (1997) 221–222.
[43] T. Erlebach and K. Jansen. The maximum edge-disjoint paths problem in bidirected trees. SIAM J. on Discrete Mathematics 14 (2001) 326–355.
[44] T. Erlebach. Approximation algorithms and complexity results for path problems in trees of rings. TIK-Report 109 (2001).
[45] T. Erlebach. Call admission control for advance reservation requests with alternatives. TIK-Report 142 (2002).
[46] T. Erlebach, A. Pagourtzis, K. Potika and S. Stefanakos. Resource allocation problems in multifiber WDM tree networks. TIK-Report 178 (2003).
[47] A. Frank, A. Karzanov and A. Sebö. On integer multiflow maximization. SIAM J. Discrete Mathematics 10 (1997) 158–170.
[48] V. Guruswani, S. Khanna, R. Rajaraman, B. Shepherd and M. Yannakakis. Near-optimal hardness results and approximation algorithms for edge-disjoint paths and related problems. Proceedings STOC 99 (1999) 19–28.
[49] T.C. Hu. Multicommodity network flows. Oper. Res. 11 (1963) 344–360.
[50] J. Kleinberg. Approximation algorithms for disjoint paths problems. PhD thesis, MIT, Cambridge, MA (1996).
[51] J. Kleinberg and É. Tardos. Disjoint paths in densely embedded graphs. Proc. 36th IEEE FOCS (1995).
[52] J. Kleinberg and É. Tardos. Approximations for the disjoint paths problem in high-diameter planar networks. Proc. 27th ACM STOC (1995) 26–35.
[53] J. Kleinberg. An approximation algorithm for the disjoint paths problem in even-degree planar graphs. Proceedings FOCS'05 (2005).

[54] S. Kolliopoulos and C. Stein. Approximating disjoint-paths using greedy algorithms and packing integer programs. Proceedings IPCO 98 (1998).
[55] S.G. Kolliopoulos and C. Stein. Approximation Algorithms for Single-Source Unsplittable Flow. SIAM J. Computing 31 (2002) 919–946.
[56] S.G. Kolliopoulos and C. Stein. Approximating Disjoint-Path Problems using Packing Integer Programs. Mathematical Programming, series A (99) 63–87 (2004).
[57] P. Kolman. A note on the greedy algorithm for the unsplittable flow problem. Information Processing Letters 88 (2003) 101–105.
[58] P. Kolman and C. Scheideler. Improved bounds for the unsplittable flow problem. Proceedings SODA 02 (2002).
[59] E. Korach. Packing of T-cuts, and other aspects of dual integrality. Ph.D. Thesis, University of Waterloo (1982).
[60] E. Korach and M. Penn. A fast algorithm for maximum integral two-commodity flow in planar graphs. Discrete Applied Mathematics 47 (1993) 77–83.
[61] L. Liu and P.J. Wan. Maximal throughput in wavelength-routed optical networks. In "Multichannel Optical Networks: Theory and Practice", volume 46 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science (1998, AMS) 15–26.
[62] B. Ma and L. Wang. On the inapproximability of disjoint paths and minimum steiner forest with bandwith constraints. Journal of Computer and System Sciences 60 (2000) 1–12.
[63] D. Peleg and E. Upfal. Disjoint paths on expander graphs. Combinatorica 9 (1989) 289–313.
[64] Y. Rabani. Path coloring on the mesh. Proceedings FOCS 96 (1996) 400–410.
[65] P. Raghavan. Probabilistic construction of deterministic algorithms: approximating packing integer programs. Journal of Computer and System Sciences 37 (1988) 130–143.
[66] S. Rajagopalan. Two commodity flows. Oper. Res. Letters 15 (1994) 151–156.
[67] A. Srinivasan. Improved approximations for edge-disjoint paths, unsplittable flow, and related routing problems. Proceedings FOCS 97 (1997) 416–425.
[68] A. Srivastav and P. Stangier. On complexity, representation and approximation of integral multicommodity flows. Discrete Applied Mathematics 99 (2000) 183–208.
[69] K.R. Varadarajan and G. Venkataraman. Graph decomposition and a greedy algorithm for edge-disjoint paths. Proceedings SODA 04 (2004) 379–380.

Minimum multicut

[70] A. Agarwal, M. Charikar, K. Makarychev and Y. Makarychev. $O(\sqrt{\log n})$-approximation algorithms for Min UnCut, Min 2CNF Deletion, and directed cut problems. Proceedings STOC 05 (2005).
[71] D. Bertsimas, C.-P. Teo and R. Vohra. Analysis of LP relaxations for multiway and multicut problems. Networks 34 (1999) 102-114.
[72] G. Călinescu, H. Karloff, and Y. Rabani. An improved approximation algorithm for Multiway Cut. Proceedings 30th STOC (1998) 48–52.
[73] G. Călinescu, C.G. Fernandes and B. Reed. Multicuts in unweighted graphs and digraphs with bounded degree and bounded tree-width. Journal of Algorithms 48 (2003) 333–359. Academic Press.
[74] S. Chawla, R. Krauthgamer, R. Kumar, Y. Rabani and D. Sivakumar. On the hardness of approximating multicut and sparsest-cut. Proceedings CCC 05 (2005).
[75] D.Z. Chen and X. Wu. Efficient algorithms for k-terminal cuts on planar graphs. Algorithmica 38 (2004) 299–316.
[76] J. Cheriyan, H. Karloff and Y. Rabani. Approximating directed multicuts. Proceedings FOCS 01 (2001) 320–328.
[77] S. Chopra and M.R. Rao. On the multiway cut polyhedron. Networks 21 (1991) 51–89.
[78] W.H. Cunningham. The optimal multiterminal cut problem. DIMACS Series in Disc. Math. and Theor. Comput. Sci. 5 (1991) 105–120.
[79] E. Dahlhaus, D.S. Johnson, C.H. Papadimitriou, P.D. Seymour and M. Yannakakis. The complexity of multiterminal cuts. SIAM J. Comput. 23 (1994) 864–894.
[80] R. Easley and D. Hartvigsen. Crossing Properties of Multiterminal Cuts. Networks 34 (1999) 215–220.
[81] P.L. Erdös and L.A. Székely. On weighted multiway cuts in trees. Math. Programming 65 (1994) 93–105.

[82] P.L. Erdös, A. Frank and L.A. Székely. Minimum multiway cuts in trees. Discrete Applied Mathematics 87 (1998) 67–76.
[83] N. Garg, V.V. Vazirani and M. Yannakakis. Approximate max-flow min-(multi)cut theorems and their applications. SIAM J. Comput. 25 (1996) 235–251.
[84] O. Goldschmidt, and D.S. Hochbaum, A polynomial algorithm for the k-cut problem for fixed k. Mathematics of operations research 19 (1994).
[85] J. Guo and R. Niedermeier. Fixed-parameter tractability and data reduction for multicut in trees. Networks 46 (2005) 124 – 135.
[86] A. Gupta. Improved results for directed multicut. Proceedings SODA 03 (2003) 454–455.
[87] M.T. Hajiaghayi and F.T. Leighton. On the max-flow min-cut ratio for directed multicommodity flows. Technical Report MIT-LCS-TR-910 (2003). Submitted to Theoretical Computer Science.
[88] D. Hartvigsen. The planar multiterminal cut problem. Discrete Applied Mathematics 85 (1998) 203–222.
[89] D. Karger, P. Klein, M. Thorup, C. Stein and N. Young. Better Rounding Algorithms for a Geometric Embedding Relaxation of Minimum Multiway Cut. Proceedings STOC 99 (1999).
[90] Y. Kortsarts, G. Kortsarz and Z. Nutov. Greedy approximation algorithms for directed multicuts. Networks 45 (2005) 214 – 217.
[91] F.T. Leighton and S. Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. J. ACM 46 (1999) 787–832.
[92] L. Létocart and F. Roupin. A semidefinite approach to solve multicut in trees. J. Opt. (2002). Montréal.
[93] M.S. Levine. Fast randomized algorithms for computing minimum {3,4,5,6}-way cuts. Proceedings SODA 00 (2000) 735–742.
[94] J. Naor and L. Zosin. A 2-approximation algorithm for the directed multiway cut problem. Proceedings 38th FOCS (1997) 548–553.
[95] P.D. Seymour. A two-commodity cut survey. Discrete Mathematics 23 (1978) 177–181.
[96] D.B. Shmoys. Cut problems and their application to divide-and-conquer. In D. Hochbaum. *Approximation algorithms for NP-hard problems.* (1997) 192–234. PWS Publishing Company. Boston.
[97] É. Tardos and V.V. Vazirani. Improved bounds for the max-flow min-multicut ratio for planar and $K_{r,r}$-free graphs. Inform. Proc. Lett. 47 (1993) 77–80.
[98] W.-C. Yeh. A simple algorithm for the planar multiway cut problem. J. Algorithms 39 (2001) 68–77.

IMFP and IMCP

[99] R. Ahuja, T. Magnanti and J. Orlin. *Networks Flows, Theory, Algorithms, Applications*. Prentice Hall, Englewood Cliffs, New Jersey (1993).
[100] C. Bentz. Edge disjoint paths and max integral multiflow/min multicut theorems in planar graphs. Proceedings ICGT'05. Electronic Notes in Discrete Mathematics 22 (2005), pp. 55-60.
[101] C. Bentz. Edge disjoint paths and multicut problems in graphs generalizing the trees. Technical Report No 948, CEDRIC, 2005. Submitted.
[102] C. Bentz. The maximum integer multiterminal flow problem in directed graphs. Technical Report No 961, CEDRIC, 2005. Submitted.
[103] C. Bentz, M.-C. Costa and F. Roupin. Maximum integer multiflow and minimum multicut problems in two-sided uniform grid graphs. To appear in Journal of Discrete Algorithms (2005).
[104] M-C. Costa. Polynomial algorithms to solve the multiway cut and integer flow problems on trees. ECCO XV, Lugano. CEDRIC report (2002).
[105] M.-C. Costa, L. Létocart and F. Roupin. A greedy algorithm for multicut and integral multiflow in rooted trees. Operations Research Letters Operation Research Letters 31 (2003) 21–27.
[106] N. Garg, V.V. Vazirani and M. Yannakakis. Multiway cuts in directed and node weighted graphs. In Proceedings ICALP (1994) 487–498.
[107] N. Garg, V.V. Vazirani and M. Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees. Algorithmica 18 (1997) 3–20.
[108] T.C. Hu. Two-commodity cut-packing problem. Discrete Mathematics 4 (1973) 108–109.
[109] L. Létocart. Problèmes de multicoupe et multiflot en nombres entiers (in French). Thèse de doctorat en informatique, CEDRIC-CNAM. (2002).
[110] K. Obata. Approximate max-integral-flow/min-multicut theorems. Proceedings STOC 04 (2004).

[111] B. Rothschild and A. Whinston. On two-commodity network flows. Oper. Res. 14 (1966) 377–387.
[112] A. Srinivasan. Improved approximation guarantees for packing and covering integer problems. DIMACS Technical report 95-37, National University of Singapore (1995).

Minimum cost integer multiflow

[113] C. Barnhart, C. Hane and P. Vance. Using Branch-and-Price to solve Origin-Destination Integer Multicommodity Flow Problem. Operations Research 32 (1998) 208–220.
[114] M.-C. Costa, A. Hertz and M. Mittaz. Bounds and Heuristics for the Shortest Capacitated Paths Problem. Journal of Heuristics 8 (2002) 449–466.
[115] M.-C. Costa, F-R. Monclar and M. Zrikem. Variable Neighborhood Search for the Optimization of Cable Layout Problem. Journal of Intelligent Manufacturing 13 (2002) 353–365.
[116] A. Löbel. Vehicle scheduling in public transit and lagrangian pricing. ZIB Berlin (1997). Preprint SC 97-16.
[117] J.B. Orlin. A faster strongly polynomial minimum cost flow algorithm. Oper. Res. 41 (1993) 338–349.

	IMFP	IMCP	UnSplitFlow	CapPath	EdgeDisjPath	Multiterm. Cut	Multiterm. Flow
Undirected Graphs	NP-Hard if $K = 2$ [4] NP-Hard to approx. within $(\log m)^{\frac{1}{3}-\varepsilon}$ [29] $O(1)-$approx. algo. if capacities are $\Omega(\log n)$ [65]	Max SNP-Hard [79] $O(\log K)-$approx. algo. [83] Not in APX if Unique Games Conjecture is true [74]	NP-Hard to approx. within $(\log m)^{\frac{1}{3}-\varepsilon}$ [29] $O\left(\sqrt{n}\right)-$approx. algo. [40]	NP-Hard if $K = 2$ [4] NP-Hard to approx. within $(\log m)^{\frac{1}{3}-\varepsilon}$ [29] $O\left(\sqrt{n}\right)-$approx. algo. [40]	NP-Hard if $K = 2$ [4] NP-Hard to approx. within $(\log m)^{\frac{1}{3}-\varepsilon}$ [29] $O\left(\sqrt{n}\right)-$approx. algo. [40]	Max SNP-Hard for $K = 3$ [79] 1.3438−approx. algo. [89]	Polyn. if inner degrees are even [47] $\log K-$approx. algo. [106]
Directed Graphs	NP-Hard if $K = 2$ [4] NP-Hard to approx. within $m^{\frac{1}{2}-\varepsilon}$[48]	Max SNP-Hard [79] $C^* \leq 39F^{*2} \ln(K+1)$ and $C^* \leq 108F^{*3}$ [76] $O(\sqrt{n})-$approx. algo. [86]	NP-Hard if $K = 2$ [5] $O\left(\sqrt{n}\right)-$approx. algo. [40] NP-Hard to approx. within $m^{\frac{1}{2}-\varepsilon}$[48]	NP-Hard if $K = 2$ [4] $O\left(\sqrt{n}\right)-$approx. algo. [40] NP-Hard to approx. within $m^{\frac{1}{2}-\varepsilon}$[48]	NP-Hard if $K = 2$ [4] $O\left(\sqrt{n}\right)-$approx. algo. [40] NP-Hard to approx. within $m^{\frac{1}{2}-\varepsilon}$[48]	Max SNP-Hard for $K = 2$ [106] 2−approx. algo. [94] Polyn. if acyclic $O\left(n^3\right)$ [1]	NP-Hard if $K = 2$ [102] $2\log K-$approx. algo. [106] Polyn. if acyclic $O\left(n^3\right)$ [1]
Directed (Rooted) Trees	Polyn. $O(K^2 \log n + n^2 \log^2 n)$ [105] $(O(\min(K, n)n))$	Polyn. [105] $(O(\min(Kn, n^2)))$	Polyn. [105] $(O(\min(Kn, n^2)))$	Polyn. [105] $(O(\min(Kn, n^2)))$	Polyn. [105] $(O(\min(Kn, n^2)))$	Polyn. $O\left(n^3\right)$ [1] $(O(n)$ [104])	Polyn. $O\left(n^3\right)$ [1] (O(height(T) K) [104])
Bidirected Trees	Max SNP-Hard [43]		Max SNP-Hard [43]	Max SNP-Hard [43]	Max SNP-Hard $(\frac{5}{3}+\varepsilon)$−approx. algo. [43]		
Undirected Trees (Gr. with cyclomatic numb. γ)	Max SNP-Hard 2−approx. algo. [107] $(2(\gamma+1)-$ approx. [100])	Max SNP-Hard 2−approx. algo. [107] $(2(\gamma+1)-$ appr. [100])	Max SNP-Hard 2−approx. algo. [107] $(2(\gamma+1)-$ appr. [100])	Max SNP-Hard 2−approx. algo. [107] $(2(\gamma+1)-$ appr. [100])	Polyn. [107] (Polyn. for fixed γ [100])	Polyn. O(n) [104]	Polyn. $O\left(n^2\right)$ [104]
Augmented Bipartite Graphs	NP-Hard for $K \geq 3$ [1] Open for $K = 2$						
Planar Graphs	Max SNP-Hard [107] $O(\log n)/O(1)-$approx. algo. if capacities $\geq 2/4$ [38, 39, 41] $O(\log^2 n)-$approx. if even degrees [53]	Max SNP-Hard [107] $O(1)-$approx. algo. [97] Polyn. for fixed K [101]	Max SNP-Hard [107]	Max SNP-Hard [107]	Max SNP-Hard [107] $4k-$approx. algo. for k-edge-outerplanar graphs [100]	NP-Hard Polyn. for fixed K [79]	
Rings		Polyn. [1]		Polyn. [26]	Polyn. [61]	Polyn. [1]	Polyn. [1]
Trees Of Rings	Max SNP-Hard [44] 4−approx. algo. [101]	Max SNP-Hard [107] 4−approx. algo. [101]	Max SNP-Hard [44] 4−approx. algo. [101]	Max SNP-Hard [44] 4−approx. algo. [101]	Max SNP-Hard 3−approx. algo. [44]	Polyn. [75]	
Meshes (Grids)	NP-hard [13]	NP-hard [103]	NP-Hard [13]	NP-hard [13]	NP-hard [13] $O(1)-$approx. algo. [51]		

Table 1. Main results for $IMFP$, $IMCP$ and their subproblems.