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Open Problems

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This chapter contains open problems that were presented at the problem session of the GT04 conference, complemented by several submitted later. Comments and questions of a technical nature should be addressed to the poser of the problem.

Problem GT04-1: Superstrongly perfect graphs

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A graph G is *superstrongly perfect* if every induced subgraph H possesses a minimal dominating set that meets all the maximal complete subgraphs of H. Clearly, every strongly perfect graph is superstrongly perfect, but not conversely.

Problem. Characterize superstrongly perfect graphs.

Problem GT04-2: Eulerian Steinhaus graphs

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To every binary string $s = x_1x_2 \dots x_{n-1} \in \mathbb{F}_2^{n-1}$ is associated a simple graph $G(s)$ on the vertex set $\{0, 1, \dots, n-1\}$, whose adjacency matrix $M = (a_{i,j})_{0 \leq i,j \leq n-1} \in \mathcal{M}_n(\mathbb{F}_2)$ satisfies $a_{i,i} = 0$ for $i \geq 0$, $a_{0,i} = x_i$ for $i \geq 1$, and $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for $1 \leq i < j \leq n-1$. The graph $G(s)$ is called the *Steinhaus graph* associated to s.

Problem. Is it true that an Eulerian Steinhaus graph is completely determined by its vertex degree sequence?

It is known that $G(s)$ is connected unless s is the zero string $0\ldots0$. Denote $d_i = d_i(s)$ the degree of vertex i in $G(s)$, and

$$d(s) = (d_1, d_2, \ldots, d_n)$$

the vertex degree sequence of $G(s)$.

The example $s_1 = 010$, $s_2 = 100$ with $d(s_1) = d(s_2) = (1, 2, 2, 1)$ shows that $d(s)$ alone does not determine s or $G(s)$ in general.

A conjecture of Dymacek states that the only *regular* Steinhaus graphs are those corresponding to the binary strings 1, $0\ldots0$, and $110110\ldots110$, see [6]. In [2], we investigate *parity-regular* Steinhaus graphs $G(s)$, where the degrees $d_i(s)$ all have the same parity, even or odd. The even case corresponds to Eulerian Steinhaus graphs, except for $s = 0\ldots0$. Dymacek has shown that there are exactly $2^{\lfloor \frac{n-1}{3} \rfloor} - 1$ Eulerian Steinhaus graphs on n vertices [6].

In our study of parity-regular Steinhaus graphs, we came upon the observation that it is much harder in this context to find *collisions*, i.e., binary strings $s_1 \neq s_2 \in \mathbb{F}_2^{n-1}$ with $d(s_1) = d(s_2)$. The smallest collision in the parity-regular case occurs at $n = 26$ and is unique in this size:

$$s_1 = 0010101001111110010101001$$
$$s_2 = 1101010110000001101010111 = s_1 + 1111111111111111111111110$$

giving rise to the common degree sequence

$$d(s_1) = d(s_2)$$
$$= (13,15,9,9,13,13,17,11,9,19,9,9,11,11,9,9,19,9,11,17,13,13,9,9,15,13).$$

Up to $n \leq 50$ vertices, we have found a total of 29 collisions in the parity-regular case. They only occur if $n \equiv 2 \bmod 4$, namely at $n = 26$, 34, 38, 42, 46 and 50 (yes, 30 is missing), and they all satisfy $s_1 + s_2 = 11\ldots10$. We conjecture in [2] that these properties always hold.

Finally, in these 29 instances, all vertex degrees turn out to be odd. Thus, *up to $n \leq 50$ vertices, there are no collisions $s_1 \neq s_2$ where all degrees in $d(s_1) = d(s_2)$ are even.*

In other words, Eulerian Steinhaus graphs $G(s)$ are completely determined by their degree sequence $d(s) \in \mathbb{N}^n$ for $n \leq 50$. Does this remain true for $n \geq 51$?

Problem GT04-3: Edge-disjoint paths in planar graphs with a fixed number of terminal pairs

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Input: An undirected planar graph G, a list of k pairs of terminal vertices (source s_i, sink t_i), k being a fixed integer.

Problem. Find in $\bigcup_i P_i$ a maximum number of edge-disjoint paths (i.e., edge-disjoint in G), where, for each i, P_i is the set of elementary paths linking s_i to t_i in G.

If the graph is not planar, the problem is $\mathcal{NP}$-hard, even for $k = 2$ [7]. If k is not fixed, the problem is $\mathcal{NP}$-hard even in outerplanar graphs [9]. Moreover, in general graphs, the problem is tractable if the maximum degree is bounded or if we allow only one path between s_i and t_i for each i (in this case, one can solve the problem by solving a constant number of instances of the *edge-disjoint paths problem* and using the algorithm given in [13]). When $k = 2$ and adding the 2 edges (s_1, t_1) and (s_2, t_2) to G does not destroy planarity, the problem is polynomial-time solvable [11].

Problem GT04-4: Shortest alternating cycle

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The decision problem SAC (*Shortest Alternating Cycle*) is formally defined as follows:

Instance: A graph $G = (V, E)$ and a positive integer $L \leq |V|$.

Question: Is there a maximum matching M and an even cycle C with $|C| \leq L$ and $|C \cap M| = \frac{1}{2}|C|$?

Problem. Determine the complexity status of SAC.

The complexity status of SAC is unknown even if G is a 3-regular bipartite graph. Notice that the problem SAC becomes solvable in polynomial time if either a cycle C or a perfect matching M is given.

Problem GT04-5: Edge 3-coloration of K_{mn} with pre-specified colored degrees

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Let $G = (X, Y, E) = K_{mn}$ be the complete bipartite graph with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Let $L_1 = (a_1, \ldots, a_m)$, $L_2 = (b_1, \ldots, b_m)$, $R_1 = (c_1, \ldots, c_n)$ and $R_2 = (d_1, \ldots, d_n)$ be four sequences of nonnegative integers such that $a_i + b_i \leq n, i = 1, \ldots, m$ and $c_i + d_i \leq m, i = 1, \ldots, n$.