MULTICUTS AND INTEGRAL MULTIFLOWS IN RINGS

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Abstract. We show how to solve in polynomial time the multicut and the maximum integral multiflow problems in rings. Moreover, we give linear-time procedures to solve both problems in rings with uniform capacities.

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1. Introduction

Let G be a graph, with a positive integral capacity (or weight) u_e on each edge e, and let $\mathfrak{L}$ be a list of K pairs of terminal vertices $\{s_k \in G, t_k \in G\}$ $(s_k \neq t_k)$, $k \in \{1, \ldots, K\}$. The multicut problem MCP consists in finding a minimum weight set of edges whose removal leaves no chain between s_k and t_k for each pair $\{s_k, t_k\}$ of $\mathfrak{L}$. Associate a commodity with each pair $\{s_k, t_k\}$: the maximum integral multiflow problem IMFP consists in maximizing the sum over all commodities of the integral flow corresponding to a commodity subject to capacity and flow conservation requirements. Both problems can be defined similarly in digraphs, by replacing "edge" by "arc" and "chain" by "path". For $K = 1$ the problems are the classical min cut-max flow problems solvable in polynomial time, but both problems are known to be $\mathcal{NP}$-hard and APX-hard for arbitrary K, even in trees [9].

A ring $\mathfrak{R}$ is a connected graph where all vertices have degree 2. Such structures are encountered for instance in telecommunications because of the deployment of fiber equipment (SONET: Synchronous Optical Networks [4, 13, 14]). In an undirected ring the flow routed from s_k to t_k can be split into two parts. One part is routed in a clockwise direction and the other in a counterclockwise direction. In order to separate s_k from t_k, the multicut must contain at least one edge between s_k and t_k in each direction. In a directed ring the flow from s_k to t_k is routed entirely in a clockwise (for instance) direction; the multicut must contain at least one arc of the unique path from s_k to t_k. We will see in Section 2 that, in rings, an undirected instance of IMFP or MCP can be reduced to a directed instance of the same problem.

Consider a directed ring $\mathfrak{R} = (V, A)$. Let p_k be the only path from s_k to t_k, let f_k be the flow routed on p_k, $k \in \{1, \ldots, K\}$, and let c_a, $a \in A$, be a binary variable such that $c_a = 1$ if arc a belongs to the cut, and $c_a = 0$ otherwise. MCP and IMFP can be stated as two integer linear programs:

$$(P-IMFP)\quad \left| \begin{array}{lll} \max & \displaystyle\sum_{k=1}^{K} f_k & \\ \text{s. t.} & \displaystyle\sum_{k \text{ s.t. } a \in p_k} f_k \leq u_a & \forall a \in A \\ & f_k \in \mathbb{N} & \forall k \in \{1, \ldots, K\} \end{array} \right. \tag{1}$$

$$(P-MCP)\quad \left| \begin{array}{lll} \min & \displaystyle\sum_{a \in A} u_a c_a & \\ \text{s. t.} & \displaystyle\sum_{a \in p_k} c_a \geq 1 & \forall k \in \{1, \ldots, K\} \\ & c_a \in \{0, 1\} & \forall a \in A \end{array} \right. \tag{2}$$

Note that the continuous relaxations of the linear programming formulations of MCP and IMFP are dual; this is also the case in unrestricted graphs [8].

Except in some special cases such as directed trees [5], there is in general a gap between the optimal values of MCP and IMFP. This is also the case in rings. An example is given by a directed ring with 3 vertices v_1, v_2, v_3, 3 arcs of weight/capacity 5 and 3 pairs $\{s_k, t_k\}$ such that $s_1 = t_2 = v_1$, $s_2 = t_3 = v_2$ and $s_3 = t_1 = v_3$. The optimal values of MCP and IMFP in this instance are 10 and 7 respectively.

Let $\mathfrak{R}$ be a ring and let n be the number of vertices and edges (or arcs) of $\mathfrak{R}$. If we consider the special case of IMFP where all the edge capacities are equal to 1 we get MEDP, the maximum edge disjoint paths problem, which is polynomial-time solvable in rings [15]. Several authors also consider the multicommodity flow problem with *demands*: one wishes to send d_k units of flow from s_k to t_k, $k \in$

$\{1, \ldots, K\}$, and the problem is to decide whether this is possible or not. This problem can be solved in $O\left(n^3\right)$ time in bidirected rings with uniform capacities [14] and in $O\left(n^2\right)$ time in undirected rings [11]. The case where the capacities are on the nodes is also polynomial-time solvable [7]. However, when we require that the flows are unsplittable, i.e., that each demand must be routed entirely in a clockwise or counterclockwise direction, the edge-capacitated problem was shown to be $\mathcal{NP}$-hard [4]. A related problem is the ring loading problem, where one seeks to minimize the maximum load on the edges while enforcing all the demands. This problem is $\mathcal{NP}$-hard when the demands cannot be split [13], but can be solved by a linear-time algorithm when demand splitting is allowed [16]. It should be noticed that these problems with demands cannot be reduced to the corresponding maximization problems without losing the ring structure [6].

In this paper, we first propose in Section 2 some preliminary reduction rules. Then, in Section 3, we show that MCP can be solved in rings by using a polynomial algorithm for chain networks [10], and, using some ideas from [2, 3], we derive a polynomial algorithm solving IMFP in rings; we also prove that the integrality gap for this problem is strictly smaller than 1. In Section 4 we propose $O(n)$ algorithms to solve IMFP and MCP in uniform (reduced) rings, i.e., in rings where all the capacities are equal and where all the reductions have been made.

2. Simplifications and reductions

With any instance of MCP or IMFP in an undirected ring we can associate an equivalent instance in a directed ring by doubling the number of terminal pairs. With each pair $\{s_k, t_k\}$, $k \in \{1, \ldots, K\}$, we associate a new pair $\{s_{k+K}, t_{k+K}\}$ where s_{k+K} (resp. t_{k+K}) lies on the same vertex as t_k (resp. s_k). The path from s_{k+K} to t_{k+K} in the directed ring corresponds to the chain from s_k to t_k in a counterclockwise direction of the undirected ring. It is clear that to any solution of MCP or IMFP obtained in the directed (resp. undirected) ring corresponds a solution with the same value in the undirected (resp. directed) ring. Now, we show how to simplify a *directed* instance in such a way that the resulting instance has a source and/or a sink lying on each vertex, and contains only proper pairs, i.e., no path p_i is a subpath of p_j for $i \neq j$.

Contracting arcs. First, a path without terminals, except for its endpoints, can be replaced by a single arc, which is the lowest weighted arc of the path. Second, consider two adjacent arcs (u, v) and (v, w) such that there is only a source s_k lying on v. If the capacity of (u, v) is greater than or equal to the one of (v, w) then the arc (u, v) can be contracted into a single vertex $(u = v)$, since (v, w) can be selected in place of (u, v) in any minimal multicut and since (v, w) is more constraining for the flow than (u, v). Now, s_k lies on u. In the same way, if there is only a sink lying on v and if the capacity of (u, v) is smaller than or equal to the one of (v, w) then (v, w) can be contracted into a single vertex.

Suppressing pairs. Each pair $\{s_k, t_k\}$ corresponding to a path p_k which contains as a subpath (or which is equivalent to) a path p_j can be removed from the list $\mathfrak{L}$. Indeed, first, a multicut for $\mathfrak{L} \setminus \{s_k, t_k\}$ is a multicut for $\mathfrak{L}$ since if s_j is separated from t_j, so is s_k from t_k. Second, any flow unit routed from s_k to t_k can be re-routed from s_j to t_j. A set of pairs such that no pair can be removed is called a "proper set" of pairs (as in [2]). Note that a proper set of pairs contains at most n pairs, since two pairs in $\mathfrak{L}$ cannot have their sources lying on the same vertex.

The two reductions (contracting arcs and suppressing pairs) must be iterated recursively until no more reductions can be made. Since at least one arc or one pair is suppressed by each reduction, the ring can be reduced in $O(K + n)$ time. Note that the suppressing pairs reduction can be efficiently implemented by using a stack (each pair will be seen twice).

Therefore, for the remainder of the paper, it is no loss of generality to consider a reduced ring, denoted by $\mathfrak{R} = (V, A)$, directed in a clockwise direction, and to assume, on the one hand, that there is a source and/or a sink lying on each vertex and, on the other hand, that the list $\mathfrak{L}$ is a proper set of pairs. The vertices and the arcs are numbered from 1 to n, and the sources and the sinks are numbered from 1 to K, $K \leq n$, in a clockwise direction.

3. Polynomial algorithms for general rings

In this section, we give polynomial algorithms to solve MCP and IMFP.

Proposition 1. *MCP can be solved in $O\left(n^2\right)$ time in rings.*

Proof. The path p_i from s_i to t_i must contain a cut arc, for all i. Hence, we solve $O(n)$ MCP instances in paths: the j^{th} instance is obtained by deleting the j^{th} arc of p_k, the shortest path among the p_i's, and is solvable in $O(K+n)$ time [10] ($K \leq n$ here). The cost of the solution for the j^{th} instance includes the cost of the j^{th} deleted arc. We keep the best solution among the $O(n)$ solutions obtained: the initial reduction step runs in $O(K+n)$ time, so the overall complexity is $O(n^2)$. □

Now, we show that IMFP is polynomial-time solvable in rings. Our approach borrows ideas given in [2] to solve special cases of the Pre-routed Call Admission Control problem (PCAC) in rings. Roughly speaking, the unweighted case of PCAC in rings is a restricted version of IMFP in directed rings where the flow routed from s_k to t_k can only be either 0 or 1 for each k. It should be noticed that our approach is more direct than the one in [2], since it gives an explicit link between our problem and problems involving interval matrices, without the need of introducing new variables. We prove the following theorem:

Theorem 1. *IMFP can be solved in polynomial time in rings.*

Proof. We make a binary search on the value of $\sum_{i=1}^{K} f_i \leq K u_{max}$, where u_{max} is the maximum arc capacity of the ring. Once the value of this sum is fixed and equal to an integer F, we can add to $(P-IMFP)$ the constraint (Λ_0): $\sum_{i=1}^{K} f_i = F$. Note that there exists an integer multiflow of value F' for any integer $F' \leq F$ iff there exists an integer multiflow of value F. Our problem is now a decision problem: does there exist a feasible solution? Consider the constraint matrix of $(P-IMFP)$. There are two kinds of rows: either the 1's are consecutive or they are not, and in this case the 0's are consecutive (this property is known as the *circular 1's property*). For each constraint (Λ) involving nonconsecutive flows (i.e., where the 1's are not consecutive), we define a new constraint $(\Lambda') \leftarrow (\Lambda_0) - (\Lambda)$ and remove (Λ): in (Λ'), all the 1's are consecutive. Therefore, we obtain an equivalent problem whose $0-1$ constraint matrix verifies the consecutive 1's property (interval matrix) and thus is totally unimodular [1], with an integer vector on the right-hand side. Hence, there exists an integer solution iff there exists a fractional one. Since each interval matrix is a network matrix, the solution can be found in strongly polynomial time [12]. □

Note that we do not need to transform the rows where the 1's are already consecutive, while in [2, 3] some of these rows have to transformed (more precisely, a particular transformation was applied when both the 1's and the 0's are consecutive, the 0's preceding the 1's in the row). Also note that our approach implies that:

Corollary 1. *In rings, the absolute gap between the value of a maximum fractional multiflow and the value of a maximum integer multiflow is strictly smaller than* 1*.*

Proof. The existence of a fractional solution of value F^* implies the existence of a fractional solution of value $\lfloor F^* \rfloor$. Now, apply the transformation of constraints described in the proof of Theorem 1 with (Λ_0): $\sum_{i=1}^{K} f_i = \lfloor F^* \rfloor$. The resulting (equivalent) program has a totally unimodular constraint matrix with an integral vector on the right-hand side. This implies the existence of an integer solution of value $\lfloor F^* \rfloor$. □

4. The case of uniform rings

In this section, we propose $O(n)$ algorithms to solve MCP and IMFP in uniform (reduced) rings, i.e., where all the arcs have the same capacity, denoted by U. First note that, in a uniform ring, we can assume w.l.o.g. that there is exactly one source and one sink lying on each vertex: this results from the reductions of Section 2. Indeed, if there is only one terminal lying on a vertex v, then one of the two arcs incident to v is contracted. Hence, the number of terminal pairs K is now equal to the number of vertices n, and all the paths p_k have the same length, denoted by L. We number the arcs and the terminal pairs as previously: there are exactly L successive flows routed through each arc, assuming that f_1 follows f_n.

We make the following preliminary remark:

Remark 1. *The optimal values of the continuous relaxations of* $(P - IMFP)$ *and* $(P - MCP)$ *are both* $\frac{nU}{L}$ *in uniform rings with capacity* U*, since optimal solutions for these two problems are obtained by setting* $f_k = \frac{U}{L}$ *for all* $k \in \{1, \ldots, n\}$ *and* $c_a = \frac{1}{L}$ *for all* $a \in A$*, respectively.*

Theorem 2. *MCP can be solved in $O(n)$ time in uniforms rings.*

Proof. By summing the constraints (2) and using the integrality of the variables c_a, we get $\sum_{a \in A} c_a \geq \lceil \frac{n}{L} \rceil$, and so $\sum_{a \in A} u_a c_a \geq \lceil \frac{n}{L} \rceil U$. Eventually, one can define a multicut of value $\lceil \frac{n}{L} \rceil U$ (and thus optimal). Indeed, for $j \in \{1, \ldots, n\}$, set $c_{a_j} = 1$ if $j \in \{1 + pL, p \in \{0, \ldots, \lceil \frac{n}{L} \rceil - 1\}\}$, and $c_{a_j} = 0$ otherwise. This provides a feasible multicut with $\lceil \frac{n}{L} \rceil$ arcs in $O(n)$ time. □

Now consider the two integers $\beta = nU - L \lfloor \frac{nU}{L} \rfloor$ and $\delta = \lfloor \frac{nU}{L} \rfloor - n \lfloor \frac{U}{L} \rfloor$. Note that $0 \leq \beta < L$ and $0 \leq \delta < n$, because β is the remainder of the Euclidean division of nU by L, and δ is the remainder of the Euclidean division of $\lfloor \frac{nU}{L} \rfloor$ by n.

Algorithm 1 Max_integral_multiflow_uniform_rings

Ensure: An integral multiflow $\hat{f} = \left(\hat{f}_j, j \in \{1, \ldots, n\}\right)$ such that $\sum_{j=1}^{n} \hat{f}_j = \lfloor \frac{nU}{L} \rfloor$

for $j = 1$ to n **do**

  $\hat{f}_j := \lfloor \frac{U}{L} \rfloor$;

end for

$j := 1$;

for $i = 1$ to δ **do**

  $\hat{f}_j := \hat{f}_j + 1$;

  if $j + L < n + 1$ **then**

    $j := j + L$;

  else

    $j := j + L - n$;

  end if

end for

Theorem 3. *IMFP can be solved in $O(n)$ time in uniforms rings by applying Algorithm 1.*

Proof. First, let us show that no capacity constraint is violated by $\hat{f}$. In the second loop of Algorithm 1, we add a flow unit to the load of each arc at most $\lceil \frac{\delta L}{n} \rceil$ times (since we turn around a ring with n arcs by making δ leaps of length L). Since L flows cross each arc, the total amount of flow routed through each arc is at most $\lfloor \frac{U}{L} \rfloor L + \lceil \frac{\delta L}{n} \rceil \leq \lfloor \frac{U}{L} \rfloor L + \left\lceil \frac{\delta L}{n} + \frac{\beta}{n} \right\rceil = \lfloor \frac{U}{L} \rfloor L + \lceil \frac{L}{n} \lfloor \frac{nU}{L} \rfloor - L \lfloor \frac{U}{L} \rfloor + U - \frac{L}{n} \lfloor \frac{nU}{L} \rfloor \rceil =$

$\left\lfloor \frac{U}{L} \right\rfloor L + \lceil U - \left\lfloor \frac{U}{L} \right\rfloor L \rceil = \left\lfloor \frac{U}{L} \right\rfloor L + (U - \left\lfloor \frac{U}{L} \right\rfloor L)$, and thus at most U. This solution has value $\sum_i \hat{f}_i = n \left\lfloor \frac{U}{L} \right\rfloor + \delta = \left\lfloor \frac{nU}{L} \right\rfloor$, and thus, from Remark 1, is optimal. Eventually, the complexity of Algorithm 1 is $O(n)$, because $0 \leq \delta < n$. □

5. Concluding remarks

First, let us point out that the approach used in the proof of Theorem 1 can in fact be extended to solve in polynomial time any integer linear program such that:

- the $0-1$ constraint matrix satisfies the circular 1's property,
- all the coefficients in the objective function are equal and nonnegative,
- and all the constraints are packing-type (if we want to maximize the objective function) or cover-type constraints (if we want to minimize it).

In particular, one may apply the same approach to a "Ring-representable" Set Cover problem (in reference to the "Tree-representable" Set Cover problem considered in [9]). Second, note that generally the complexity of MCP and IMFP is affected by considering directed or undirected graphs [6], but we have shown that this is not the case in rings. Finally, in these graphs, the Multiterminal Flow and Cut problems can trivially be solved in $O(K+n)$ time. Recall that in these special cases of IMFP and MCP, we are given a set $\{r_1, r_2, \ldots, r_p\}$ of terminal vertices, and the terminal pairs are (r_j, r_k) for all $j \neq k$ in $\{1, \ldots, p\}$ (and hence $K = p(p-1)$). Here, the optimum value of both problems is equal to the sum of the weights of the arcs remaining in the ring after the reduction step.

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