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Exact and approximate resolution of integral multiflow and multicut problems: algorithms and complexity

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Abstract This is a summary of the author's PhD thesis supervised by Marie-Christine Costa and Frédéric Roupin and defended on 20 November 2006 at the Conservatoire National des Arts et Métiers in Paris (France). The thesis is written in French and is available upon request from the author. This work deals with two well-known optimization problems from graph theory: the maximum integral multiflow and the minimum multicut problems. The main contributions of this thesis concern the polynomial-time solvability and the approximation of these two problems (and of some of their variants) in classical classes of graphs: bounded tree-width graphs, planar graphs and grids, digraphs, etc.

Keywords: Multicuts, Integral Multiflows, Polynomial-Time Solvability, Polynomial Approximation, Combinatorial Optimization, Graph Theory

MSC classification: 05C85, 68Q17, 90C27

1 Introduction

In this thesis, we study the minimum multicut (MCM) and maximum integral multiflow (MIMF) problems. These problems, arising in routing and in networks or VLSI design, are $\mathcal{NP}$-hard generalizations of the classical minimum cut and maximum flow problems (see Ford and Fulkerson, 1956). For both problems, we are given an edge-weighted graph (network) and a list of source-sink pairs of vertices. MCM consists in selecting a set of edges of minimum weight, the removal of which leaves no path between the ith source and the ith sink, for each i. MIMF consists in routing the maximum number of flow units (each unit being routed from a source to the corresponding sink) while enforcing the capacity constraints on the edges.

The thesis is divided into three parts. In the first part, we study the complexity and approximation of MIMF and of MCM in bounded tree-width graphs. In the second part, we give polynomial-time algorithms and complexity results for MCM and MIMF in grids and for MCM in planar graphs. Eventually, in the last part, we propose and test on randomly generated instances a new heuristic for MIMF in general undirected graphs. We also study the complexity and approximation of a variant (special case) of MIMF in digraphs.

2 Main contributions

The complexity and approximation of MCM and MIMF have been studied a lot in previous work: both problems are APX-hard (and, thus, admit no PTAS unless $\mathcal{P} = \mathcal{NP}$) even in trees, or with only three source-sink pairs (see Dahlhaus et al., 1994; Garg et al., 1997; Guruswami et al., 2003). It is also worth noticing that MCM and MIMF are linked by a strong relationship: namely, the continuous relaxations of their linear programming formulations are dual (see Garg et al., 1996).

However, although logarithmic-factor (and even constant-factor in planar graphs) approximation algorithms are known for MCM (see Garg et al., 1996; Tardos and Vazirani, 1993), no algorithm with a non-polynomial approximation factor is known for MIMF, even in planar graphs. Furthermore, no algorithm using the rounding of continuous solutions (computed by linear programming) to obtain integral solutions for MIMF can provide better factors (see Garg et al., 1997), while the approximation algorithms for MCM are all based on this idea. We give in this thesis a constant-factor approximation algorithm for MIMF in a class of unweighted planar graphs of bounded (but arbitrary) tree-width. These planar graphs generalize trees (for which a 2-approximation algorithm is known) and cacti. This result is detailed in Bentz (2005).

Moreover, while MIMF is polynomial-time solvable in chains and in stars (see Garg et al., 1997), it is not known to be tractable in other traditional classes of graphs. We prove it to be tractable in rings (cycles): more generally, our approach shows the polynomial-time solvability of a class of linear programs whose 0-1 constraint matrices satisfy the *circular 1's property.*

We also consider a *multiway* variant of MIMF where, given a list of *terminal* vertices $t_1, \ldots, t_p$, the source-sink pairs are (t_i, t_j) for $i \neq j$. This problem is polynomial-time solvable in undirected graphs (see Keijsper et al., 2006), so we study the directed case. We introduce a new parameter k_S and give a complete characterization of the tractable and intractable cases of this problem with respect to k_S; then, we design a $2\log_2(k_S+2)$-approximation algorithm. This work, presented in Bentz (2007), improves and unifies previous results obtained in Costa et al. (2005) and in Garg et al. (1994).

Concerning MCM, we first show the $\mathcal{NP}$-hardness (or APX-hardness) of this problem in several cases close to known tractable cases, and in particular in directed acyclic graphs (even if the underlying undirected graph is a cactus) and in some other digraphs with bounded (directed) tree-width.

We also study this problem in planar graphs. Dahlhaus et al. (1994) gave many results concerning the multiway version of MCM in planar graphs. This variant is $\mathcal{NP}$-hard in these graphs, but it becomes tractable if the number of terminals (and thus of terminal pairs) is bounded (while it is $\mathcal{NP}$-hard in unrestricted graphs, even with only three terminals or three terminal pairs) or if all the terminals lie on the outer face (see Chen and Wu, 2004). In comparison, MCM is tractable in trees if the number of source-sink pairs is bounded (and APX-hard otherwise). We give a polynomial-time algorithm for MCM in planar graphs where all the sources and sinks are on the outer face and the number of source-sink pairs is bounded. An interesting aspect of our algorithm is that it makes use of a reduction from MCM to the above-mentioned variant that preserves planarity (no such reduction being previously known).

Eventually, we study MCM and MIMF in grids. There have been a lot of papers about paths problems in grids and in planar graphs (due to the numerous applications in VLSI design), but they mainly considered decision versions of these problems (see Korte et al., 1990). We first consider the problem of determining a routing with minimum total length: we give a greedy polynomial-time algorithm to solve this problem in particular grids called *dense channels*. This work, presented in Bentz et al. (2007a), extends previous results of Formann et al. (1993). Then, we show that MCM and MIMF are polynomial-time solvable in a class of grids with uniform capacity: the algorithms we design are based on a duality relationship and on previous results concerning decision problems associated with MIMF (see Frank, 1982; Okamura and Seymour, 1981). As a by-product, we prove an approximate min-max theorem on the optimal values of MCM and MIMF. We also give complexity results concerning MCM in several types of grids. All these results are reported in Bentz et al. (2007b).

To conclude, we would like to point out that tables summarizing the main results currently known for MCM and MIMF are included in the appendix of the thesis, as well as ten open problems in this field of research which we consider as the most important ones.

References

Bentz C (2005) Edge disjoint paths and max integral multiflow/min multicut theorems in planar graphs, *in: Proceedings ICGT'05, Hyères, Electronic Notes in Discrete Mathematics* 22: 55–60

Bentz C (2007) The maximum integer multiterminal flow problem in directed graphs, *Operations Research Letters* 35: 195–200

Bentz C, Costa MC, Picouleau C, Zrikem M (2007a) The shortest multipaths problem in a capacitated dense channel, *European Journal of Operational Research* 178: 926–931

Bentz C, Costa MC, Roupin F (2007b) Maximum integer multiflow and minimum multicut problems in two-sided uniform grid graphs, *Journal of Discrete Algorithms* 5: 36–54

Chen D, Wu X (2004) Efficient algorithms for k-terminal cuts on planar graphs, *Algorithmica* 38: 299–316

Costa MC, Létocart L, Roupin F (2005) Minimal multicut and maximal integer multiflow: A survey, *European Journal of Operational Research* 162: 55–69

Dahlhaus E, Johnson D, Papadimitriou C, Seymour P, Yannakakis M (1994) The complexity of multiterminal cuts, *SIAM Journal on Computing* 23: 864–894

Ford L, Fulkerson D (1956) Maximal Flow Through a Network, *Canadian Journal of Mathematics* 8: 339–404

Formann M, Wagner D, Wagner F (1993) Routing through a dense channel with minimum total wire length, *Journal of Algorithms* 15: 267–283

Frank A (1982) Disjoint paths in a rectilinear grid, *Combinatorica* 2: 361–371

Garg N, Vazirani V, Yannakakis M (1994) Multiway cuts in directed and node weighted graphs, *in: Proceedings ICALP, Lecture Notes in Computer Science* 820: 487–498

Garg N, Vazirani V, Yannakakis M (1996) Approximate max-flow min-(multi)cut theorems and their applications, *SIAM Journal on Computing* 25: 235–251

Garg N, Vazirani V, Yannakakis M (1997) Primal-dual approximation algorithms for integral flow and multicut in trees, *Algorithmica* 18: 3–20

Guruswami V, Khanna S, Rajaraman R, Shepherd B., Yannakakis M (2003) Near-optimal hardness results and approximation algorithms for edge-disjoint paths and related problems, *Journal of Computer and System Sciences* 67: 473–496

Keijsper JCM, Pendavingh RA, Stougie L (2006) A linear programming formulation of Mader's edge-disjoint paths problem, *Journal of Combinatorial Theory, Series B* 96: 159–163

Korte B, Lovász L, Prömel HJ, Schrijver A (eds) (1990) *Paths, Flows and VLSI-Layout, Algorithms and combinatorics* 9, *Springer-Verlag, Berlin*

Okamura H, Seymour P (1981) Multicommodity flows in planar graphs, *Journal of Combinatorial Theory, Series B* 31: 75–81

Tardos E, Vazirani V (1993) Improved bounds for the max-flow min-multicut ratio for planar and $K_{r,r}$-free graphs, *Information Processing Letters* 47: 77-80