A type theory which is complete for Kreisel's modified realizability

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April 15, 1999

Abstract

We define a type theory with a strong elimination rule for existential quantification. As in Martin-Löf's type theory, the "axiom of choice" is thus derivable. Proofs are also annotated by realizers which are simply typed λ-terms. A new rule called "type extraction" which extract the type of a realizer allows us to derive the so-called "independance of premisses" schema. Consequently, any formula which is realizable in HA^ω, according to Kreisel's modified realizability, is derivable in this type theory.

Keywords: type theory, realizability, lambda-calculus, constructive logic.

1 Introduction

The well-known Brouwer-Heyting-Kolmogoroff semantics gave birth to Kleene's original recursive realizabitity (r-realizabitity) and to Kreisel's modified realizability (mr-realizabitity). While in the former realizers denote partial functions, in the latter they denote total functions. One can thus think that r-realizabitity is closer to provability than mr-realizability since terms extracted from proofs (thanks to the Curry-Howard isomorphism) denote total functions (functionals of Gödel's System T for proofs in finite-type arithmetic HA^ω, for instance). Surprisingly, it is not the case: there is a schema which is mr-realizable but not r-realizable (and thus not provable). To be more specific, Kreisel's mr-realizabitity is different from provability for two reasons:

1. the *axiom of choice* (which is actually a schema) AC:

$$\forall x : \sigma.\exists y : \tau.B \Rightarrow \exists f : \sigma \to \tau.\forall x : \sigma.B[f(x)/y]$$

   is mr-realizable but not provable in HA^ω.

2. the *independance of premisses* schema IP, where H is a Harrop formula in which y does not occur free (see definition 1.1.2) :

$$(H \Rightarrow \exists y : \sigma.B) \Rightarrow \exists y : \sigma.(H \Rightarrow B)$$

   is mr-realizable but not provable in HA^ω.

It is known that a strong elimination rule for existential quantification as in P. Martin-Löf's type theory [6, 7, 8] is enough to prove AC. Notice that the distinction between the numerous presentation of the theory (or between intensional and extensional theories) is not relevant in this paper, so we will just call it ML (for further details see [14]).

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On the other hand, IP is not derivable in ML (since ML conservative over HA, see [1] p. 323). Let us try to explain this phenomenon: in ML, proof-terms denote total functions (which are functionals of Gödel's System T, as in HA^ω) but this information is not explicit inside the formal system. In fact, ML differs from HA^ω in its behaviour in that there is a "built-in" realizability interpretation connected with the "formulas-as-types" idea. However, this realizability is more like an abstract Kleene realizability (thanks to A. S. Troelstra for pointing this to me).

To charaterize provability in ML [11, 12], M. D. G. Swaen is led to extend HA^ω with a "conditional application", and consequently (as in ML) there is no way to reflect the totality of realizers into the formal system. Conversely, we will show here that if we add a rule which states that "any realizer is typable" (and thus denotes a total function), provability and mr-realizability collapse. In other words, a formula A is thus provable in the resulting type theory if and only if A is mr-realizable in HA^ω.

This paper is devoted to the definition of the type theory and the proof of the main result (which is partly is given in appendix). In the second section, we give the definition of a realized theory, which allows for a general study of realizibility in many-sorted predicate calculi (and not only finite-type arithmetic). Although the results of the second section are not new, they are necessary to understand the type theory defined in the last section.

Related works

For formal definitions of languages and theories in which r-realizability and mr-realizability have been investigated, the reader is refered to the works of S. C. Kleene [5] (in the theory HA, where a function is represented its Gödel-number), S. Feferman [4] (in the theory APP, where functions are represented in a λ-calculus with a fixed-point operator) and M. J. Beeson [1] p. 148 (in the theory EON, where functions are represented by untyped λ-terms) and the works on G. Kreisel's modified realizability of J. Diller [2] and A. S. Troelstra [13] p. 220 (in the theory HA, where a function is represented by its Gödel-number) and p. 213 (in the theory HA^ω, where functions are represented by typed λ-terms).

1.1 Recursive realisability

We recall informally Kleene's recursive realizability. The reader may assume that the theory is HA (although the same results hold in EON). The notation $(f\ a)\downarrow$ means that the function f is defined on a (this predicate is built-in in EON, but can be defined in HA from Kleene's predicate T). In the following definition, $x\ r\ A$ is a new *formula* where x is a fresh variable (i.e. which does not occur free in A).

- $x\ r\ A \equiv A$, if A is an atomic formula
- $x\ r\ A \wedge B \equiv \pi(x)\ r\ A \wedge \pi'(x)\ r\ B$
- $x\ r\ A \Rightarrow B \equiv \forall y(y\ r\ A \Rightarrow ((x\ y)\downarrow \wedge (x\ y)\ r\ B))$
- $x\ r\ \forall y B(y) \equiv \forall y((x\ y)\downarrow \wedge (x\ y)\ r\ B)$
- $x\ r\ \exists y B(y) \equiv \pi'(x)\ r\ B(\pi(x))$

Remark. The notation $x\ r\ A$ is relevant since the "substitution lemma" holds: $(x\ r\ A)[t/y] = x\ r\ A[t/y]$.

Definition 1.1.1 *We say that a formula A is r-realized (resp. r-realizable) in* HA *iff there is a term t with the same free variables as A such that* $\mathrm{HA} \vdash t\ r\ A$ *(resp.* $\mathrm{HA} \vdash \exists x(x\ r\ A)$*).*

Axiomatization of r-realizability

Definition 1.1.2 Harrop formulas *are defined inductively as follows: atomic formulas are Harrop formulas. If A and B are Harrop formulas and C is* any *formula then $A \wedge B$, $C \Rightarrow A$ and $\forall x A$ are Harrop formulas.*

Definition 1.1.3 *We call* ECT *(for "Extended Church Thesis") the following axiom schema:*

$$\forall x(H \Rightarrow \exists y B(y)) \Rightarrow \exists f \forall x(H \Rightarrow (f\ x) \wedge B(f\ x)) \qquad \textit{where } H \textit{ is a Harrop formula}$$

Lemma 1.1.4 ECT *is r-realized.*

Remark. The restriction of H to Harrop formulas in ECT is needed for the following reason: an intuitionistic proof of $A \Rightarrow B$ is a procedure which turns a proof (realizer) of A into a proof (realizer) of B. In the assumption $H \Rightarrow \exists y.B$, y may depend on the proof of H. This axiom can thus be realized only if there is no "computational content" in H, which is the case for negative formulas such as Harrop formulas (see section 2.5). Consequently, if H is a Harrop formula, a realizer of $\forall x(H \Rightarrow \exists y B(y))$ is a *partial* function which is defined on every x where H is realized, and which map this x to a realizer of $\exists y B(y)$.

Lemma 1.1.5 (soundness) *Any formula provable in* HA *is r-realized in* HA*.*

Proposition 1.1.6 (axiomatization) *For any formula A, there is a term t such that* $\mathrm{HA} \vdash t\ r\ A$ *if and only if* $\mathrm{HA} + \mathrm{ECT} \vdash A$.

The axiom schema ECT is a generalization of CT (take $H = \top$):

$$\forall x \exists y B(y) \Rightarrow \exists f \forall x((f\ x) \downarrow \wedge B(f\ x))$$

Let us now consider the axiom schema IP (where H is a Harrop formula):

$$(H \Rightarrow \exists y B(y)) \Rightarrow \exists y(H \Rightarrow B(y))$$

We have $\mathrm{CT} + \mathrm{IP} \Rightarrow \mathrm{ECT}$:

$$\begin{array}{rcll} \forall x(H \Rightarrow \exists y B(y)) & \Rightarrow & \forall x \exists y(H \Rightarrow B(y)) & (\mathrm{IP}) \\ & \Rightarrow & \exists f \forall x((f\ x) \downarrow \wedge H \Rightarrow B(f\ x)) & (\mathrm{CT}) \\ & \Rightarrow & \exists f \forall x(H \Rightarrow ((f\ x) \downarrow \wedge B(f\ x))) & \end{array}$$

However, $\mathrm{ECT} \not\Rightarrow \mathrm{IP}$ since IP is not r-realized in HA. Indeed, if IP were r-realized in HA, we would get a procedure (the realizer of IP) which could turn a realizer of $\forall x(H \Rightarrow \exists y B(y))$, which is a partial function, into a realizer of $\forall x \exists y(H \Rightarrow B(y))$, which is a total function. Yet, a generic partial recursive function cannot necessarily be extended into a total function. The formal proof requires a diagonalization argument (see [9] or [10] p. 171, for instance).

2 Kreisel's modified realizability

Kreisel's modified realizability is usually defined in HA^ω. It is easy to generalize the definition to many-sorted predicate calculi: the theory HA^ω is a special case of "realized" theories. As usual, formulas are realized by simply typed λ-terms.

2.1 Many-sorted predicate calculi with equality IQC^ω

A many-sorted predicate calculus is a predicate calculus in which the term language contains the simply typed λ-calculus. Equality is used to define the usual conversion rules between λ-terms with product type.

Language. The language of a many-sorted predicate calculus is given by:

- A set Σ of ground types. We denote by Σ^ω the set of types inductively defined from $\Sigma \cup \{I_1\}$ using $\times$ and $\rightarrow$ (where I_1 is the singleton type).

- A set of function and predicate symbols typed by elements of Σ^ω. This set contains an equality symbol $=_\sigma$ for any type σ of Σ^ω.

The set of terms of each type is inductively defined from typed variables, the function symbols, a constant $\mathbf{e} : I_1$ using the following rules:

$$\frac{x : \sigma \quad y : \tau}{\langle x, y\rangle : \sigma \times \tau} \qquad \frac{z : \sigma \times \tau}{\pi(z) : \sigma} \quad \frac{z : \sigma \times \tau}{\pi'(z) : \tau} \qquad \frac{\begin{matrix}[x : \sigma] \\ t : \tau\end{matrix}}{\lambda x : \sigma . t : \sigma \to \tau} \qquad \frac{f : \sigma \to \tau \quad a : \sigma}{f\ a : \tau}$$

Formulas. The set of formulas is inductively defined as follows:

- atomic formulas are formulas;
- if A and B are formulas then $A \wedge B$ and $A \Rightarrow B$ are also formulas;
- if x is a variable of type σ and A is a formula, then $\exists x : \sigma . A$ and $\forall x : \sigma . A$ are also formulas (where the variable x is bound).

Rules for equality

- Reflexivity:

$$t =_\sigma t$$

- Substitution:

$$\frac{u =_\sigma v \quad \phi[v/x]}{\phi[u/x]}$$

- Conversion rules for the λ-calculus with product type:

$$\begin{array}{c}(\lambda x : \sigma . t)\ u = t[u/x] \\ \pi(\langle x, y\rangle) = x \\ \pi'(\langle x, y\rangle) = y\end{array}$$

Rules for connectives and quantifiers

$$\frac{A \qquad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} \qquad \frac{\begin{matrix}[A] \\ B\end{matrix}}{A \Rightarrow B} \qquad \frac{A \Rightarrow B \qquad A}{B}$$

$$\frac{t : \sigma \qquad B[t/x]}{\exists x : \sigma . B} \qquad \frac{\exists x : \sigma . B \qquad \begin{matrix}[B] \\ C\end{matrix}}{C} \qquad \frac{\begin{matrix}[x : \sigma] \\ B\end{matrix}}{\forall x : \sigma . B} \qquad \frac{\forall x : \sigma . B \qquad a : \sigma}{B[a/x]}$$

Notation. We call IQC^ω this calculus (where the entailment symbol will be denoted by $\vdash_I$).

2.2 Example: the theory HA^ω

The language of HA^ω contains the ground type N, a symbol 0 of type N, a symbol S of type $N \to N$ and denumerable set of symbols $\mathbf{rec}_\sigma$ of type $(\sigma \times (N \to \sigma \to \sigma)) \to N \to \sigma$ for each type σ. The theory HA^ω contains the equations which define these constants,

$$\begin{array}{rcl}(\mathbf{rec}_\sigma\ h\ 0) & =_\sigma & \pi(h) \\ (\mathbf{rec}_\sigma\ h\ Sn) & =_\sigma & (\pi'(h)\ n\ (\mathbf{rec}_\sigma\ h\ n))\end{array}$$

and the induction schema $Rec(\phi)$,

$$\phi(0) \wedge \forall n : N.(\phi(n) \Rightarrow \phi(Sn)) \Rightarrow \forall n : N.\phi(n)$$

Remark. If we assume that $\perp$ is defined as $0 = 1$, the axiom $\perp \vdash A$ is derivable by induction for any formula A (see [15] vol. II, p. 592, in the theory ML^i). If the negation $\neg A$ is defined as $A \Rightarrow \perp$ then the fourth axiom of Peano's arithmetic ($\neg 0 =_N 1$) is obvious. Moreover, disjunction can be defined as $A \vee B \equiv \exists n : N((n = 0 \Rightarrow A) \wedge (\neg(n = 0) \Rightarrow B))$

2.3 Kreisel's modified realizability for IQC $^\omega$

Computational content of a formula

We map each formula to a type which represents the function space where we expect to find a realizer for the formula. The presence of the singleton type, product types and arrow types allows for a simple definition of the type of realizers:

- $\mathcal{T}(A) \equiv I_1$, if A is an atomic formula,
- $\mathcal{T}(A \Rightarrow B) \equiv \mathcal{T}(A) \rightarrow \mathcal{T}(B)$
- $\mathcal{T}(A \wedge B) \equiv \mathcal{T}(A) \times \mathcal{T}(B)$
- $\mathcal{T}(\forall x : \sigma.A) \equiv \sigma \rightarrow \mathcal{T}(A)$
- $\mathcal{T}(\exists x : \sigma.A) \equiv \sigma \times \mathcal{T}(A)$

Each formula A is mapped to a formula $f\ mr\ A$ where f is a fresh variable of type $\mathcal{T}(A)$. The definition is the same as in HA^ω (see [13] p. 218, for instance).

- $f\ mr\ A \equiv A$, if A is an atomic formula
- $f\ mr\ A \Rightarrow B \equiv \forall g : \mathcal{T}(A).(g\ mr\ A \Rightarrow (f\ g)\ mr\ B)$
- $f\ mr\ A \wedge B \equiv \pi(f)\ mr\ A \wedge \pi'(f)\ mr\ B$
- $f\ mr\ \forall x : \sigma.B \equiv \forall x : \sigma.(f\ x\ mr\ B)$
- $f\ mr\ \exists x : \sigma.B \equiv \pi'(f)\ mr\ B[\pi(f)/x]$

Proposition 2.3.1 (soundness) *If A is a closed formula derivable in* IQC^ω *from the hypotheses $H_1, \ldots, H_n$, then there is a λ-term $t(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are variables of type $\mathcal{T}(H_1), \ldots, \mathcal{T}(H_n)$, such that $t\ mr\ A$ is derivable from the hypotheses $x_1\ mr\ H_1, \ldots, x_n\ mr\ H_n$.*

2.4 Realized theory and term extraction

Definition 2.4.1 *We say that a formula A is* realized *in a theory Γ if there is a closed term $t : \mathcal{T}(A)$ such that $\Gamma \vdash_I t\ mr\ A$; we say that A is* realizable *if $\Gamma \vdash_I \exists x : \mathcal{T}(A).(x\ mr\ A)$.*

Definition 2.4.2 *We say that a theory Γ is* realized *(resp. realizable) if and only if each axiom of Γ is realized (resp. realizable) in Γ.*

Theorem 2.4.3 (soundness) *If Γ is a realized theory, any formula A which is derivable in Γ is realized in Γ.*

Proof. By proposition 2.3.1, if $H_1, \ldots, H_n$ are the axioms of Γ which occur in the proof of A, there is a term $t(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are variables of type $\mathcal{T}(H_1), \ldots, \mathcal{T}(H_n)$, such that $t\ mr\ A$ is provable from the hypotheses $x_1\ mr\ H_1, \ldots, x_n\ mr\ H_n$. And precisely, since Γ is a realized theory, there are some closed terms $t_1, \ldots, t_n$ with types $\mathcal{T}(H_1), \ldots, \mathcal{T}(H_n)$ such that $\Gamma \vdash_I t_i\ mr\ H_i$. We conclude that $\Gamma \vdash t[t_1/x_1, \ldots t_n/x_n]\ mr\ A$. $\square$

Example: the theory HA^ω

The theory HA^ω is realized. To prove this, it is enough to prove that the induction scheme is realized since all other axioms are equational. Of course, Rec is realized by **rec** (to prove $\mathrm{HA}^\omega \vdash \mathbf{rec}_{A^*}\ mr\ Rec(A)$, consider $Rec((\mathbf{rec}_{A^*}\ h\ n)\ mr\ A(n)))$.

Remark. The notion of realized theory is independant from the computational properties of the term language. For instance, the theory HA^ω is realized, but this result has nothing to do with the properties of the typed λ-calculus with **rec**.

2.5 Aximatization of mr-realizability

The axiom schema AC

Proposition 2.5.1 *The axiom schema AC:*

$$\forall x : \sigma.\exists y : \tau.A(x, y) \Rightarrow \exists f : \sigma \to \tau.\forall x : \sigma.A(x, f\ x)$$

is mr-realized by the λ-term $t_{\mathrm{AC}}^{\sigma,\tau} \equiv \lambda p : \mathcal{T}(\forall x : \sigma.\exists y : \tau.A).\langle \lambda x : \sigma.\pi(p\ x), \lambda x : \sigma.\pi'(p\ x)\rangle$.

The axiom schema IP

To prove that IP is mr-realized, we need the concept of self-realized formula.

Definition 2.5.2 *A formula A is said to be* self-realized *if there is a closed term, denoted by $sr(A)$ of type $\mathcal{T}(A)$ such that for any term t of type $\mathcal{T}(A)$:*

$$\vdash_I t\ mr\ A \Rightarrow sr(A)\ mr\ A$$

Lemma 2.5.3 *Harrop formulas are self-realized.*

Proof. By induction on the Harrop formula, prove that $\vdash_I t\ mr\ A \Rightarrow sr(A)\ mr\ A$ with:

- $sr(A) = \mathbf{e}$ if is an atomic formula.
- $sr(A \wedge B) = \langle sr(A), sr(B)\rangle$ if A and B are Harrop formulas.
- $sr(A \Rightarrow B) = \lambda x : \mathcal{T}(A).sr(B)$ if B is a Harrop formula.
- $sr(\forall x : \sigma.A) = \lambda x : \sigma.sr(A)$ if A is a Harrop formula.

□

Proposition 2.5.4 *The axiom schema IP (where H is a Harrop formula):*

$$(H \Rightarrow \exists y : \sigma.B(y)) \Rightarrow \exists y : \sigma.(H \Rightarrow B(y))$$

is realized by: $t_{\mathrm{IP}}^{\sigma} \equiv \lambda f : \mathcal{T}(H \Rightarrow \exists y : \sigma.B(y)).\langle \pi(f\ sr(H)), \lambda x : \mathcal{T}(H).\pi'(f\ sr(H))\rangle$.

Lemma 2.5.5 (axiomatization) *For any formula A,* $\mathrm{AC} + \mathrm{IP} \vdash_I \exists f : \mathcal{T}(A).f\ mr\ A \Leftrightarrow A$

Proposition 2.5.6 *If Γ is a realized theory then for any formula A, $\Gamma \vdash_I \exists t : \mathcal{T}(A).t\ mr\ A$ if and only if* $\Gamma + \mathrm{AC} + \mathrm{IP} \vdash_I A$.

Corollary 2.5.7 *If Γ is a realized theory then the existential property holds in Γ modulo* $\mathrm{AC}+\mathrm{IP}$. *In other words, for any formula $A(x, y_1, \ldots, y_n)$, if $\Gamma \vdash \exists x : \sigma.A$ then there is a term $t(y_1, \ldots, y_n)$ of type σ such that* $\Gamma + \mathrm{AC} + \mathrm{IP} \vdash A[t/x]$.

3 Internalizing Kreisel's modified realizability

In this section, we define a type theory in which any formula which is mr-realizable is also provable. This system contains a strong elimination rule of $\exists$ as in P. Martin-Löf's type theory, which enables us to derive the axiom of choice.

3.1 The type theory MR

Terms, types and formulas are those of IQC^ω. We have three kinds of judgments:

- Typing judgments, denoted by $t : \sigma$, where σ is a type and t is a term.
- Realizability (or provability) judgments, denoted by $t \in A$, where A is a formula and t is a term of type $\mathcal{T}(A)$.
- Equality judgments, denoted by $u =_\sigma v$, where u and v are both terms of type σ.

A context has the following form (all possible free variables shown):

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n, y_1 \in A_1(x_1, \ldots, x_n), \ldots, y_p \in A_p(x_1, \ldots, x_n)$$

Axioms have the form: $\Gamma, x : \sigma, \Delta \vdash x : \sigma$ and $\Gamma, y \in A, \Delta \vdash y \in A$.
The typing rules are the same as in IQC^ω (see section 2.1). The logical rules are given below.

Rules for connectives

$$\frac{u \in A \qquad v \in B}{\langle u, v\rangle \in A \wedge B} \qquad\qquad \frac{t \in A \wedge B}{\pi(t) \in A} \quad \frac{t \in A \wedge B}{\pi'(t) \in B}$$

$$\frac{\begin{array}{c}[x \in A]\\ t \in B\end{array}}{\lambda x : \mathcal{T}(A).t \in A \Rightarrow B} \qquad\qquad \frac{f \in A \Rightarrow B \qquad a \in A}{(f\ a) \in B}$$

provided that x does not occur free in B in the introduction rule of implication.

Remark. In ML, the introduction rule of $\forall$ is

$$\frac{\begin{array}{c}[x \in A]\\ t \in B\end{array}}{\lambda x \in A.t \in \forall x \in A.B}$$

and $A \Rightarrow B$ is just an abbreviation for $\forall x \in A.B$ when x does not occur free in B.

Rules for quantifiers

$$\frac{t : \sigma \qquad u \in B[t/x]}{\langle t, u\rangle \in \exists x : \sigma.B} \qquad\qquad \frac{t \in \exists x : \sigma.B}{\pi\,(t) : \sigma} \quad \frac{t \in \exists x : \sigma.B}{\pi'(t) \in B[\pi\,(t)/x]}$$

$$\frac{\begin{array}{c}[x : \sigma]\\ t \in B\end{array}}{\lambda x : \sigma.t \in \forall x : \sigma.B} \qquad\qquad \frac{f \in \forall x : \sigma.B \qquad a : \sigma}{(f\ a) \in B[a/x]}$$

provided that x does not occur in any hypothesis in the introduction rule of $\forall$ and in the elimination rules of $\exists$.

Rules for equality. The equations are the same as in IQC$^\omega$, the substitution is now:

$$\frac{u =_\sigma v \qquad t \in \phi[v/x]}{t \in \phi[u/x]}$$

Remark. The elimination rule of $\exists$ given above is strictly stronger than the usual one (given in the definition of IQC$^\omega$), since we are now able to prove the axiom of choice (see section 3.2). The usual rule can be derived as follows (where h does not occur in C and x does not occur in C or in any hypothesis other than B):

$$\dfrac{\dfrac{t \in \exists x:\sigma.B}{\pi'(t) \in B[\pi(t)/x]} \qquad \dfrac{\dfrac{t \in \exists x:\sigma.B}{\pi(t):\sigma} \quad \dfrac{\dfrac{\begin{array}{c}[h \in B]\\ u \in C\end{array}}{\lambda h:\mathcal{T}(B).u \in B \Rightarrow C}}{\lambda x:\sigma.\lambda h:\mathcal{T}(B).u \in \forall x:\sigma(B \Rightarrow C)}}{(\lambda x:\sigma.\lambda h:\mathcal{T}(B).u\ \pi(t)) \in (B \Rightarrow C)[\pi(t)/x]}}{((\lambda x:\sigma.\lambda h:\mathcal{T}(B).u\ \pi(t)\ \pi'(t)) \in C}$$

Proposition 3.1.1 *If $t \in A$ is derivable in* MR *from the hypotheses $x_1 \in \Gamma_1, \ldots, x_n \in \Gamma_n$, then t mr A is derivable in* IQC$^\omega$ *from the hypotheses x_1 mr $\Gamma_1, \ldots, x_n$ mr Γ_n.*

Corollary 3.1.2 *For any closed formula A provable in* IQC$^\omega$ *from the hypotheses $\Gamma_1, \ldots, \Gamma_n$, there is a λ-term $t(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are variables of type $\mathcal{T}(\Gamma_1), \ldots, \mathcal{T}(\Gamma_n)$, such that t mr A is provable from the hypotheses x_1 mr $\Gamma_1, \ldots, x_n$ mr Γ_n.*

Proof. The predicate calculus IQC$^\omega$ can easily be embedded in MR since we have shown that the usual elimination rule of $\exists$ is derivable in MR, and this is the only difference between both systems. $\square$

3.2 Proof of AC

The presence of a strong elimination rule for $\exists$ allows for a proof of AC which is the same as in ML. Notice also that the realizer is exactly $t_{\mathrm{AC}}^{\sigma,\tau}$.

Proposition 3.2.1 *In* MR, $t_{\mathrm{AC}}^{\sigma,\tau} \in AC$ *in derivable.*

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \in \forall x:\sigma.\exists y:\tau.A]^1 \quad [x:\sigma]^2}{(p\ x) \in \exists y:\tau.A}}{\pi(p\ x):\tau}}{\lambda x:\sigma.\pi(p\ x):\sigma \to \tau}(2) \qquad \dfrac{(\lambda x:\sigma.\pi(p\ x)\ x) = \pi(p\ x) \quad \dfrac{\dfrac{\dfrac{[p \in \forall x:\sigma.\exists y:\tau.A]^1 \quad [x:\sigma]^3}{(p\ x) \in \exists y:\tau.A}}{\pi'(p\ x) \in A[\pi(p\ x)/y]}}{\lambda x:\sigma.\pi'(p\ x) \in \forall x:\sigma.A[\pi(p\ x)/y]}(3)}{\lambda x:\sigma.\pi'(p\ x) \in \forall x:\sigma.A[(\lambda x:\sigma.\pi(p\ x)\ x)/y]}}{\langle \lambda x:\sigma.\pi(p\ x), \lambda x:\sigma.\pi'(p\ x)\rangle \in \exists f:\sigma \to \tau.\forall x:\sigma.A[(f\ x)/y]}}{\lambda p:\mathcal{T}(\forall x:\sigma.\exists y:\tau.A).\langle \lambda x:\sigma.\pi(p\ x), \lambda x:\sigma.\pi'(p\ x)\rangle \in \forall x:\sigma.\exists y:\tau.A \Rightarrow \exists f:\sigma \to \tau.\forall x:\sigma.A[(f\ x)/y]}(1)$$

$\square$

3.3 Proof of IP

In order to prove IP, we need a rule which states that we will not consider the computationnal content of atomic formulas, in the same way as in the realizability semantics. For further information on the meaning of this rule, see [3] p. 268. Notice however that atomic formulas are not necessarily decidable.

Rule for atomic formulas $\mathrm{R}_{\mathbf{e}}$

$$\frac{t \in A}{\mathbf{e} \in A} \quad \text{where } A \text{ is atomic}$$

Notation. We call $\mathrm{MR}_{\mathbf{e}}$ the system $\mathrm{MR} + \mathrm{R}_{\mathbf{e}}$.

Proposition 3.3.1 *Proposition* 3.1.1 *still holds in* $\mathrm{MR}_{\mathbf{e}}$.

Proof. By induction hypothesis, $t\ mr\ A$ is derivable in IQC^{ω} from the hypotheses of Γ^{mr}, and precisely, by definition $t\ mr\ A \equiv A \equiv \mathbf{e}\ mr\ A$. Consequently, $\mathbf{e}\ mr\ A$ is derivable. □

Self-realized formulas

The rule for atomic formulas $\mathrm{R}_{\mathbf{e}}$ states exactly that atomic formulas are self-realized in MR. As in the case of realizability, this property extends to Harrop formulas.

Definition 3.3.2 *A formula A is self-realized in MR if there is a closed λ-term $sr(A)$ such that for any t, if $t \in A$ is derivable in* $\mathrm{MR}_{\mathbf{e}}$ *then $sr(A) \in A$ is also derivable from the same hypotheses.*

Proposition 3.3.3 *Harrop formulas are self-realized in* $\mathrm{MR}_{\mathbf{e}}$.

Proof. By induction on the formula A, we prove by induction on A that for any t, if $t \in A$ is derivable from the hypotheses of Γ then $sr(A) \in A$ is also derivable from the same hyphotheses, where $sr(A)$ is defined as in the proof of lemma 2.5.3. We denote by SR_A this derivation (the details are given in appendix). □

Remark. The axiom schema IP is still not derivable in $\mathrm{MR}_{\mathbf{e}}$. Indeed, it is easy (although rather technical) to embedd $\mathrm{MR}_{\mathbf{e}}$ into ML, and IP is not derivable in ML (since ML is conservative over HA, see [1] p. 323). We really need a rule which states that a realizer of a formula A is typable of type $\mathcal{T}(A)$.

The "type extraction" rule $\mathrm{R}^{\mathcal{T}}$

$$\frac{t \in A}{t : \mathcal{T}(A)}$$

Notation. We call $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$ the system $\mathrm{MR} + \mathrm{R}_{\mathbf{e}} + \mathrm{R}^{\mathcal{T}}$.

Proposition 3.3.4 *Proposition* 3.1.1 *still holds in* $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$.

Remark. It is enough to give rule $\mathrm{R}^{\mathcal{T}}$ for hypotheses as follows:

$$f \in A \vdash f : \mathcal{T}(A)$$

The general rule is then easily obtained from this axiom by induction on the term.

Proof of IP

We are now able to give a proof of the axiom schema IP in $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$. The term which annotates the proof is of course t_{IP}^{σ}. Note also the occurrence of the rule $\mathrm{R}^{\mathcal{T}}$ which enable us to derive the type of f and which implies that f is a total function (f is thus defined on $sr(H)$).

Proposition 3.3.5 *In* $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$, $t_{\mathrm{IP}}^{\sigma} \in \mathrm{IP}$ *is derivable.*

Proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ sr(H):\mathcal{T}(H)\end{matrix} \qquad \dfrac{[f \in H \Rightarrow \exists y:\sigma.B]^2}{f:\mathcal{T}(H \Rightarrow \exists y:\sigma.B)}}{(f\ sr(H)):\mathcal{T}(\exists y:\sigma.B)}}{\pi(f\ sr(H)):\sigma} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[x \in H]^1\\ \cdots\\ \mathrm{SR}_H\\ \cdots\\ sr(H) \in H\end{matrix} \qquad [f \in H \Rightarrow \exists y:\sigma.B]^2}{(f\ sr(H)) \in \exists y:\sigma.B}}{\pi'(f\ sr(H)) \in B[\pi(f\ sr(H))/y]}}{\lambda x:\mathcal{T}(H).\pi'(f\ sr(H)) \in H \Rightarrow B[\pi(f\ sr(H))/y]}\,(1)}{\langle \pi(f\ sr(H)), \lambda x:\mathcal{T}(H).\pi'(f\ sr(H))\rangle \in \exists y:\sigma.(H \Rightarrow B)}}{\lambda f:\mathcal{T}(H \Rightarrow \exists y:\sigma.B).\langle \pi(f\ sr(H)), \lambda x:\mathcal{T}(H).\pi'(f\ sr(H))\rangle \in (H \Rightarrow \exists y:\sigma.B) \Rightarrow \exists y:\sigma.(H \Rightarrow B)}\,(2)
$$

$\square$

Let us round off this section with the result we claimed in the introduction: a formula A is provable in $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$ if and only if A is mr-realizable in IQC^{ω}.

Theorem 3.3.6 *For any formula A, there is a term t of type $\mathcal{T}(A)$ such that $t \in A$ is derivable in $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$ if and only if* $\mathrm{IQC}^{\omega} \vdash \exists x : \mathcal{T}(A).(x\ mr\ A)$.

Proof. By proposition 3.3.4, if $t \in A$ derivable in $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$ then $\mathrm{IQC}^{\omega} \vdash t\ mr\ A$ and thus $\mathrm{IQC}^{\omega} \vdash \exists t : \mathcal{T}(A).(t\ mr\ A)$. Conversely, if $\mathrm{IQC}^{\omega} \vdash t\ mr\ A$ then $\mathrm{IQC}^{\omega} + \mathrm{IP} + \mathrm{AC} \vdash A$ and consequently there is a term t of type $\mathcal{T}(A)$ such that $t \in A$ is derivable in $\mathrm{MR}_{\mathbf{e}}^{\mathcal{T}}$. $\square$

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A Proofs

Proposition 3.1.1 *If $t \in A$ is derivable in* MR *from the hypotheses $x_1 \in \Gamma_1, \ldots, x_n \in \Gamma_n$, then $t\ mr\ A$ is derivable in* IQC$^\omega$ *from the hypotheses $x_1\ mr\ \Gamma_1, \ldots, x_n\ mr\ \Gamma_n$.*

Proof. By induction on the derivation of $t : A$. If $\Gamma = x_1 : \Gamma_1, \ldots, x_n : \Gamma_n$ is a set of hypotheses of MR, we denote by Γ^{mr} the set of hypotheses of IQC$^\omega$ composed of $x_1\ mr\ \Gamma_1, \ldots, x_n\ mr\ \Gamma_n$.

- Case of the axiom: if $x_i \in \Gamma_I$ is an hypothesis of Γ, then $x_i\ mr\ \Gamma_i$ is an hypothesis of Γ^{mr}.
- Case of the introduction rule of $\wedge$

$$\frac{u \in A \qquad v \in B}{\langle u, v\rangle \in A \wedge B}$$

  By induction hypothesis, $u\ mr\ A$ and $v\ mr\ B$ are derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}. Consequently, $(u\ mr\ A) \wedge (v\ mr\ B)$ and thus $\langle u, v\rangle\ mr\ A \wedge B$ which is exactly $(\pi(\langle u, v\rangle)\ mr\ A) \wedge (\pi'(\langle u, v\rangle)\ mr\ B)$ are derivable in IQC$^\omega$ from the same hypotheses.
- Case of the elimination rule of $\wedge$ (we treat only the first projection)

$$\frac{t \in A \wedge B}{\pi(t) \in A}$$

  By induction hypothesis, $t\ mr\ A \wedge B$ is derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}. By definition, $t\ mr\ A \wedge B \equiv (\pi(t)\ mr\ A) \wedge (\pi'(t)\ mr\ B)$ and consequently $\pi(t)\ mr\ A$ is also derivable in IQC$^\omega$ from the same hypotheses.
- Case of the introduction rule of $\Rightarrow$

$$\frac{\begin{matrix}[x \in A]\\ t \in B\end{matrix}}{\lambda x : \mathcal{T}(A).t \in A \Rightarrow B}$$

  If $t \in B$ is derivable from the hypotheses $\Gamma, x \in A$, by induction hypothesis, $t\ mr\ B$ is derivable in IQC$^\omega$ from the hypotheses $\Gamma^{mr}, x\ mr\ A$. Consequently, $(x\ mr\ A) \Rightarrow (t\ mr\ B)$ and thus $(x\ mr\ A) \Rightarrow (((\lambda x : \mathcal{T}(A).t)\ x)\ mr\ B)$ are derivables in IQC$^\omega$ from the hypotheses of Γ^{mr}. Finally, since x does not occur in Γ (we recall that in MR, the name of an hypothesis cannot occur in any hypothesis), x does not occur in Γ^{mr}. Then $\forall x : \mathcal{T}(A).(x\ mr\ A \Rightarrow ((\lambda x : \mathcal{T}(A).t)\ x)\ mr\ B)$ which is by definition exactly $\lambda x : \mathcal{T}(A).t\ mr\ (A \Rightarrow B)$ is also derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}.
- Case of the elimination rule of $\Rightarrow$

$$\frac{f \in A \Rightarrow B \qquad a \in A}{(f\ a) \in B}$$

  By induction hypothesis, $f\ mr\ (A \Rightarrow B)$ and $a\ mr\ A$ are derivables in IQC$^\omega$ from the hypotheses of Γ^{mr}. Since $f\ mr\ (A \Rightarrow B) \equiv \forall x : \mathcal{T}(A).(x\ mr\ A \Rightarrow (f\ x)\ mr\ B)$ by definition, $a\ mr\ A \Rightarrow (f\ a)\ mr\ B$ and thus $(f\ a)\ mr\ B$ are also derivable from the same hypothesis.

- Case of the introduction rule of $\exists$

$$\frac{t : \sigma \qquad u \in B[t/x]}{\langle t, u\rangle \in \exists x : \sigma.B}$$

By induction hypothesis, $u\ mr\ B[t/x]$ is derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}. Consequently $\langle t, u\rangle\ mr\ \exists x : \sigma.B \equiv \pi'\langle t, u\rangle\ mr\ B[\pi\langle t, u\rangle/x]$ is also derivable.

- Case of the elimination rule of $\exists$

$$\frac{t \in \exists x : \sigma.B}{\pi'(t) \in B[\pi\,(t)/x]}$$

By induction hypothesis, $t\ mr\ \exists x : \sigma.B$ is derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}. By definition $t\ mr\ \exists x : \sigma.B$ is exactly $\pi'(t)\ mr\ B[\pi\,(t)/x]$.

- Case of the introduction rule of $\forall$

$$\frac{\begin{array}{c}[x : \sigma]\\ t \in B\end{array}}{\lambda x : \sigma.t \in \forall x : \sigma.B}$$

By induction hypothesis, $t\ mr\ B$ is derivable in IQC$^\omega$ from the hypothesis Γ^{mr}. Consequently, $\forall x : \sigma.(t\ mr\ B)$ and thus $\lambda x : \sigma.t\ mr\ \forall x : \sigma.B \equiv \forall x : \sigma.(\lambda x : \sigma.t\ x)\ mr\ B$ are also derivable from the same hypotheses.

- Case of the elimination rule of $\forall$

$$\frac{f \in \forall x : \sigma.B \qquad a : \sigma}{(f\ a) \in B[a/x]}$$

By induction hypothesis, $f\ mr\ \forall x : \sigma.B$ is derivable in IQC$^\omega$ from the hypotheses of Γ^{mr}. By definition, this formula is $\forall x : \sigma.((f\ x)\ mr\ B)$ and consequently $(f\ a)\ mr\ B[a/x]$ is also derivable from the same hypotheses.

- Case of the substitution rule

$$\frac{u =_\sigma v \qquad t \in \phi[v/x]}{t \in \phi[u/x]}$$

By induction hypothesis, $t\ mr\ \phi[v/x]$ is derivable in IQC$^\omega$ from the hypotheses Γ^{mr}. Since x does not occur in t, $t\ mr\ \phi[v/x] = (t\ mr\ \phi)[v/x]$ and consequently $(t\ mr\ \phi)[u/x] = t\ mr\ \phi[u/x]$ is also derivable from the same hypotheses.

- The axioms for equality are the same in IQC$^\omega$ and MR.

$\square$

Proposition 3.3.3 *Harrop formulas are self-realized in* MR$_{\mathbf{e}}$.

Proof.

- If A is an atomic formula, if $t \in A$ is derivable from the hypotheses of Γ then apply the rule for atomic formulas:

$$\frac{t \in A}{\mathbf{e} \in A}$$

- If $t \in A \wedge B$ is derivable from the hypotheses of Γ then take the following derivation:

$$\frac{\begin{array}{c}\dfrac{t \in A \wedge B}{\pi(t) \in A}\\ \cdots\\ \mathrm{SR}_A\\ \cdots\\ sr(A) \in A\end{array} \qquad \begin{array}{c}\dfrac{t \in A \wedge B}{\pi'(t) \in B}\\ \cdots\\ \mathrm{SR}_B\\ \cdots\\ sr(B) \in B\end{array}}{\langle sr(A),\ sr(B)\rangle \in A \wedge B}$$

- If $t \in A \Rightarrow B$ is derivable form the hypotheses of Γ then take the following derivation:

$$\dfrac{\dfrac{t \in A \Rightarrow B \qquad [x \in A]}{(t\ x) \in B} \\ \cdots \\ \mathrm{SR}_B \\ \cdots \\ sr(B) \in B}{\lambda x : \mathcal{T}(A).\ sr(B) \in A \Rightarrow B}$$

- If $t \in \forall x : \sigma.B$ is derivable from the hypotheses of Γ then take the following derivation:

$$\dfrac{\dfrac{t \in \forall x : \sigma.B \qquad [x : \sigma]}{(t\ x) \in B} \\ \cdots \\ \mathrm{SR}_B \\ \cdots \\ sr(B) \in B}{\lambda x : \sigma.\ sr(B) \in \forall x : \sigma.B}$$

□