Quantifiers are not interdefinable in the second-order propositional constant domain logic

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Abstract

We show that the universal quantifier is not definable from the existential quantifier in the second order propositional constant domain logic. To prove this, we exhibit a (full) Kripke model where no closed formula built from $\top, \bot, \vee, \wedge, \rightarrow, \exists$ has the same semantics as the formula $\forall X(X \vee \neg X)$. This is in contrast to second order propositional subtractive logic where the universal quantifier is definable from the existential quantifier and subtraction.

1 Introduction

It is well-known [GLT89] that existential quantifier $\exists$ is definable from the universal quantifier $\forall$ and the implication $\rightarrow$ in the second-order propositional intuitionistic logic by taking $\exists X.F = \forall O(\forall X(F \rightarrow O) \rightarrow O)$. It is also known that the converse is not true in intuitionistic logic, but $\forall$ is definable in the second-order propositional subtractive logic [Cro01] from $\exists$ and $-$. Indeed, by applying the duality, $\forall$ is definable as follows: $\forall X.F = \exists O((O - \exists X(O - F))$.

The second-order propositional subtractive logic is conservative over the second-order propositional constant domain logic (CDL^2) which may be characterized as an extension of intuitionistic logic with the following axiom schema DIS^2 (where X does not occur in ψ):

$$\forall X(\phi \vee \psi) \rightarrow \forall X.\phi \vee \psi$$

A natural question is thus to determine whether $\forall$ is already definable in CDL^2 (without the subtraction). In this paper, we answer this question negatively by exhibiting a (full) Kripke model of CDL^2 where no closed formula built from $\top, \bot, \vee, \wedge, \rightarrow, \exists$ has the same semantics as the formula $\forall X(X \vee \neg X)$.

2 Kripke Models

Let us recall the primary interpretation of second order propositional formulas in Kripke models [Kre97]. This definition includes the semantics of subtraction [Cro01].

Definition 1. *A second-order Kripke structure is a triple $(X, \mathcal{O}, \mathcal{P})$ where $(X, \mathcal{O})$ is a bi-topological space (i.e. the upper-closed sets of a partial order) and $\mathcal{P} \subseteq \mathcal{O}$. If $\mathcal{V}$ is a finite mapping from variables to elements of $\mathcal{P}$. The semantics of a formula is inductively defined as follows:*

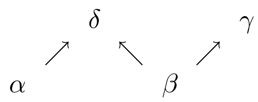
- $[\![A]\!]_{\mathcal{V}} = \mathcal{V}(A)$
- $[\![A \wedge B]\!]_{\mathcal{V}} = [\![A]\!]_{\mathcal{V}} \cap [\![B]\!]_{\mathcal{V}}$

- $[\![A \vee B]\!]_{\mathcal{V}} = [\![A]\!]_{\mathcal{V}} \cup [\![B]\!]_{\mathcal{V}}$
- $[\![A \rightarrow B]\!]_{\mathcal{V}} = int([\![A]\!]^c_{\mathcal{V}} \cup [\![B]\!]_{\mathcal{V}})$
- $[\![A - B]\!]_{\mathcal{V}} = ext([\![A]\!]_{\mathcal{V}} \cap [\![B]\!]^c_{\mathcal{V}})$
- $[\![\forall X A]\!]_{\mathcal{V}} = \bigcap_{O \in \mathcal{P}} [\![A]\!]_{\mathcal{V}, X \leftarrow O}$
- $[\![\exists X A]\!]_{\mathcal{V}} = \bigcup_{O \in \mathcal{P}} [\![A]\!]_{\mathcal{V}, X \leftarrow O}$

Definition 2. *A second-order Kripke model is a second-order Kripke structure $(X, \mathcal{O}, \mathcal{P})$ such that $\mathcal{P}$ is closed under the semantics of formulas, that is, for any A and $\mathcal{V}$, $[\![A]\!]_{\mathcal{V}}$ is in $\mathcal{P}$. In other words, it is a model of the comprehension scheme $\exists X(X \leftrightarrow F)$. A second-order Kripke structure $(X, \mathcal{O}, \mathcal{P})$ is full if $\mathcal{P}$ is exactly $\mathcal{O}$ (and it is thus a model).*

2.1 $\forall^2$ is not definable from $\exists^2$ in CDL^2

Let $\mathcal{M}$ be the following Kripke model:



The collection $\mathcal{O}$ of open sets of $\mathcal{M}$ is:

$$\{\}, \{\gamma\}, \{\delta\}, \{\gamma, \delta\}, \{\alpha, \delta\}, \{\alpha, \gamma, \delta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta, \gamma, \delta\}$$

In $\mathcal{M}$, the semantics of $\forall X(X \vee \neg X)$ is $\{\gamma, \delta\}$. We will show that no closed formula built from $\top, \bot, \vee, \wedge, \rightarrow, \exists$ has this semantics in $\mathcal{M}$. We denote by $\mathcal{E}$ the following subset of $\mathcal{O}$:

$$\mathcal{E} = \{\{\delta\}, \{\gamma, \delta\}, \{\beta, \gamma, \delta\}\}$$

Lemma 3. *For any connector $\square$, if the semantics of $A \square B$ is in $\mathcal{E}$, then at least one of the semantics $[\![A]\!]$ and $[\![B]\!]$ is in $\mathcal{E}$. Moreover, we check in each case that if we replace each semantics $[\![A]\!]$ and $[\![B]\!]$ which is in $\mathcal{E}$ by any open set containing α then the semantics of $A \square B$ also contains α.*

Proof. By systematic checking (see the table given in appendix). □

Proposition 4. *For any formula ϕ, for any assignment $\mathcal{V}$, if the open set $[\![\phi]\!]_{\mathcal{V}}$ is in $\mathcal{E}$ then, by denoting $\mathcal{I}$ the set of the free variables of ϕ whose interpretation by $\mathcal{V}$ is in $\mathcal{E}$, we have:*

1. *$\mathcal{I}$ is nonempty,*
2. *for any assignment $\mathcal{V}'$ such as for any free X variable in ϕ, $\alpha \in \mathcal{V}'(X)$ if $X \in \mathcal{I}$ and $\mathcal{V}'(X) = \mathcal{V}(X)$ otherwise, we have $\alpha \in [\![\phi]\!]_{\mathcal{V}'}$.*

Proof. We prove the proposition by induction on the formula. For the propositional variables, it is obvious. For the connectors, it is clear by the previous lemma and the induction hypothesis.

For the case of $\exists$, let us consider some formula $\phi = \exists Y\psi$ and some assignment $\mathcal{V}$ for the free variables of ϕ. Let us assume that the semantics of ϕ for the assignment $\mathcal{V}$ is in $\mathcal{E}$ and let us denote $\mathcal{I}$ the set of the free variables of ϕ whose interpretation by $\mathcal{V}$ is in $\mathcal{E}$.

By definition $[\![\phi]\!]_{\mathcal{V}} \equiv \bigcup_{O\in\mathcal{O}} [\![\psi]\!]_{\mathcal{V}\cup\{(Y,O)\}}$. For any union of open sets that is in $\mathcal{E}$, since $\{\beta, \delta\}$ is not an open set, at least one of the members of this union must be in $\mathcal{E}$. There is thus some $O\in\mathcal{O}$ such as $[\![\psi]\!]_{\mathcal{V}\cup\{(Y,O)\}}$ is in $\mathcal{E}$. Let us denote $\mathcal{W}$ the assignment $\mathcal{V}\cup\{(Y,O)\}$ and $\mathcal{J}$ the set of the free variables of ψ whose interpretation by $\mathcal{W}$ is in $\mathcal{E}$.

1. Let us show that $\mathcal{I}$ is nonempty. By definition, $\mathcal{J}=\mathcal{I}$ or $\mathcal{J}=\mathcal{I}\cup\{Y\}$. By recurrence hypothesis (1), $\mathcal{J}$ is nonempty. We just have to show that $\mathcal{J}$ is not the singleton $\{Y\}$. Let us assume that it is the case. Let O be some open set containing α and let us denote $\mathcal{W}'$ the assignment $\mathcal{V}\cup\{(Y,O)\}$. This assignment $\mathcal{W}'$ is indeed such as for any free X variable in ψ, $\alpha\in\mathcal{W}'(X)$ if $X\in\mathcal{J}$ and $\mathcal{W}'(X)=\mathcal{W}(X)$ otherwise. We then know by induction hypothesis (2) that $\alpha\in[\![\psi]\!]_{\mathcal{W}'}$. Consequently, $\alpha\in\bigcup_{o\in\mathcal{O}}[\![\psi]\!]_{\mathcal{V}\cup\{(Y,O)\}}\equiv[\![\phi]\!]_{\mathcal{V}}$. Or $[\![\phi]\!]_{\mathcal{V}}$ is in $\mathcal{E}$ by assumption, hence the contradiction.

2. Let us prove that for any assignment $\mathcal{V}'$ such that for any free variable X in ϕ, $\alpha\in\mathcal{V}'(X)$ if $X\in\mathcal{I}$ and $\mathcal{V}'(X)=\mathcal{V}(X)$ otherwise, we have $\alpha\in[\![\phi]\!]_{\mathcal{V}'}$. Let $\mathcal{V}'$ be such an assignment:

    - First case: $Y\in\mathcal{J}$. Let O be an open set containing α, and let us denote $\mathcal{W}'$ the assignment $\mathcal{V}'\cup\{(Y,O)\}$. The assignment $\mathcal{W}'$ is indeed such as for any free X variable in ψ, $\alpha\in\mathcal{W}'(X)$ if $X\in\mathcal{J}$ and $\mathcal{W}'(X)=\mathcal{W}(X)$ otherwise. We then know by recurrence hypothesis (2) that $\alpha\in[\![\psi]\!]_{\mathcal{W}'}$. Consequently $\alpha\in\bigcup_{o\in\mathcal{O}}[\![\psi]\!]_{\mathcal{V}'\cup\{(Y,O)\}}\equiv[\![\phi]\!]_{\mathcal{V}'}$.

    - Second case: $Y\notin\mathcal{J}$. Let us denote $\mathcal{W}'$ the assignment $\mathcal{V}'\cup\{(Y,\mathcal{W}(Y))\}$. The assignment $\mathcal{W}'$ is indeed such as for any free X variable in ψ, $\alpha\in\mathcal{W}'(X)$ if $X\in\mathcal{J}$ and $\mathcal{W}'(X)=\mathcal{W}(X)$ otherwise. Then, we know by induction hypothesis (2) that $\alpha\in[\![\psi]\!]_{\mathcal{W}'}$. Consequently $\alpha\in\bigcup_{o\in\mathcal{O}}[\![\psi]\!]_{\mathcal{V}'\cup\{(Y,O)\}}\equiv[\![\phi]\!]_{\mathcal{V}'}$.

□

Theorem 5. *In the second order propositional CDL, the quantifier $\forall$ is not definable from $\top$, $\bot, \vee, \wedge, \rightarrow, \exists$.*

Proof. In the model $\mathcal{M}$, the semantics of $\forall X(X\vee\neg X)$ is $\{\gamma,\delta\}$, but the semantics of no closed formula built from $\top,\bot,\vee,\wedge,\rightarrow,\exists$ is in $\mathcal{E}$ (since for a closed formula $\mathcal{I}$ si empty). □

Bibliography

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Appendix A Proof of lemma 3

$X \in \mathcal{O}$	$Y \in \mathcal{O}$	$X \cap Y$	$X \cup Y$	$int(X^c \cup Y)$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\delta\}$	$\{\}$	$\{\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\gamma\}$	$\{\}$	$\{\gamma\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\gamma,\delta\}$	$\{\}$	$\{\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\beta,\gamma,\delta\}$	$\{\}$	$\{\beta,\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\alpha,\delta\}$	$\{\}$	$\{\alpha,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\alpha,\gamma,\delta\}$	$\{\}$	$\{\alpha,\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\}$	$\{\alpha,\beta,\gamma,\delta\}$	$\{\}$	$\{\alpha,\beta,\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\delta\}$	$\{\}$	$\{\}$	$\{\delta\}$	$\{\gamma\}$
$\{\delta\}$	$\{\delta\}$	$\{\delta\}$	$\{\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
$\{\delta\}$	$\{\gamma\}$	$\{\}$	$\{\gamma,\delta\}$	$\{\gamma\}$
$\{\delta\}$	$\{\gamma,\delta\}$	$\{\delta\}$	$\{\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
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$\{\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$	$\{\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$	$\{\alpha,\beta,\gamma,\delta\}$
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$\{\gamma\}$	$\{\delta\}$	$\{\}$	$\{\gamma,\delta\}$	$\{\alpha,\delta\}$
$\{\gamma\}$	$\{\gamma\}$	$\{\gamma\}$	$\{\gamma\}$	$\{\alpha,\beta,\gamma,\delta\}$
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