The reconstruction problem of uniform hypergraphs from their degree sequences

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Definitions

Graphs and Hypergraphs

A *graph* is a pair of sets $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.
Generalization: a *hypergraph* is a pair $H = (V, E)$ where V is the set of vertices and $E \subset \mathcal{P}(V)$ is the set of hyperedges.

Remark: in particular a graph is a hypergraph.

The *degree* of a vertex v is the number of edges that intersect v.
The *degree sequence* d of a hypergraph is the sequence of the degrees of vertices arranged in non-increasing order.
A non increasing sequence of positive integers $d = (d_1, \ldots, d_n)$ is (hyper)graphic if there exists a (hyper)graph that realizes d.

Problem:

Can we characterize graphic and hypergraphic sequences?

Definitions

Incidence matrix

The incidence matrix is a binary matrix associated to the hypergraph H and identifies it uniquely.
The element in position (i, j) is equal to 1 if and only if the i-th hyperedge intersects the j-th vertex.

- Columns correspond to vertices,
- rows correspond to hyperedges.

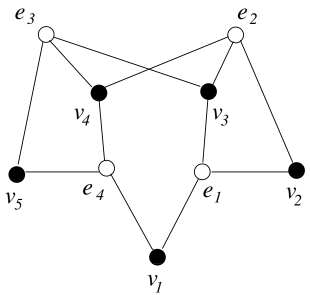
The incidence matrix has the degree sequence d as column sums.
If the cardinality of each edge is k, then the hypergraph is *k-uniform*. Consequently the rows sum of the incidence matrix is $(k, \dots, k)$.

Problem:

Can we reconstruct the incidence matrix (if it exists) of a given integer sequence?

Definitions

Example of 3-uniform hypergraph



$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$d_H = (3, 3, 2, 2, 2)$$

Previous results-Graphs

Theorem

(Erdös, Gallai 1960) *The integer sequence $d = (d_1, d_2, \ldots, d_n)$, with $d_1 \geq d_2 \geq \cdots \geq d_n$, is graphic if and only if $\Sigma_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, 1 \leq k \leq n.$$

Can be checked in time $O(n)$

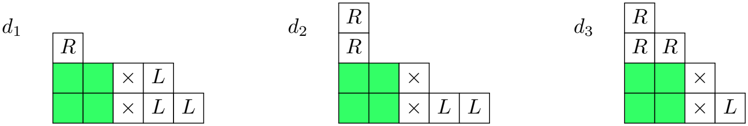
AND

We have a polynomial algorithm for the reconstruction of a graphic sequence (Havel, Hakimi).

Graphical partitions

A geometrical interpretation of Erdös' condition

Through its Ferrers diagram, an integer partition d can be decomposed as follows



$$\begin{array}{ll} d_1 = 3+2+2+2+1 & \text{graphical} \\ d_2 = 4+2+2+1+1 & \text{maximal} \\ d_3 = 4+3+2+1 & \text{not graphical} \end{array}$$

Theorem

An integer partition d is graphical if and only if $R(d) \trianglelefteq L(d)$ w.r.t. dominance order.

Previous results - Hypergraphs

Some sub-classes of uniform hypergraphs have been investigated:

Theorem

(A. F., C. Picouleau) *An homogeneous* 3*-uniform hypergraph can be reconstructed in polynomial time from its degree sequence.*

Theorem

(A. F., C. Picouleau, S.Rinaldi) *A step-one* 3*-uniform hypergraph can be reconstructed in polynomial time from its degree sequence.*

Similar complexity results are obtained for different of 3-uniform hypergraphs' subclasses.

Remark: these results are obtained using notions of combinatorics on words.

Previous results - Hypergraphs

In general, for a 3-uniform hypergraphic sequence the characterization problem is *NP-complete*, and consequently it is *NP-hard* the related reconstruction problem.

Theorem

(Deza, Levin, Meesum, Onn 2018) *For any fixed $k \geq 3$ it is NP-complete to decide if $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a k-uniform hypergraph.*

Dimostrazione: P-time reduction of 3-*partition* to 3-*graphic*.

Research Aim: Detect further classes of uniform hypergraphs that are reconstructable in polynomial time.

The class $\mathcal{D}$

Starting from a non-increasing integer sequence $s = \big(s(1), \ldots, s(n)\big)$, with $n \geq 3$, we define the 3-uniform hypergraph $H = (V, E)$ as
$V = \{v_1, \ldots, v_n\}$,
$E = \{(v_i, v_j, v_k) \text{ s.t. } s(i) + s(j) + s(k) > 0, \text{ for } 1 \leq i < j < k \leq n\}$.

The class $\mathcal{D}$ is thus defined as $\mathcal{D} = \bigcup_{n \geq 3} \mathcal{D}_n$.

Property

If (v_i, v_j, v_k) is an edge of H, then $(v_i, v_j, v_{k'})$ is an edge of H for all $j + 1 \leq k' \leq k$.
The degree sequences in $\mathcal{D}$ are *unique*, i.e., if $\pi \in \mathcal{D}$ then there exists one only 3-hypergraph (up to isomorphism) realizing it.

The class $\mathcal{D}_n$

Example

Let $s = (3, 2, 0, -1, -2)$. The obtained 3-hypergraph is

$$H_s = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and $\pi_s = (5, 4, 4, 3, 2)$.

Incidence matrix structure

Block Graphs

The incidence matrix of a $\mathcal{D}$-hypergraph allows a splitting into blocks, grouping the rows whose first two non-zero entries occupy the same positions.

$$H_\pi = \left(\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 1 & 1 & 1 & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} B_{1,2,6} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} B_{1,3,5} \\
\left.\begin{array}{c} \\ \end{array}\right\} B_{1,4,5} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} B_{2,3,6} \\
\left.\begin{array}{c} \\ \end{array}\right\} B_{2,4,5} \\
\left.\begin{array}{c} \\ \end{array}\right\} B_{3,4,5}
\end{array}$$

with $\pi = (8, 8, 8, 6, 6, 3)$ and $s = (1, 1, 1, 0, 0, -1)$.

Incidence matrix structure

Block Graphs

For each block, we can define an integer sequence considering its columns' sums:

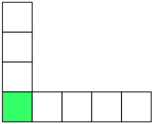
$$H_\pi = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{array}{l} b_{1,2,6} = (4,1,1,1,1) \\ b_{1,3,5} = (0,3,1,1,1) \\ b_{1,4,5} = (0,0,1,1) \end{array}$$

Incidence matrix structure

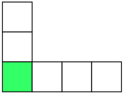
Block Graphs

The integer sequences thus obtained are graphical partitions.

$b_{1,2,6}$



$b_{1,3,5}$



$b_{1,4,5}$

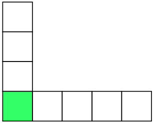
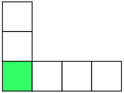
Property

The integer sequence $b_{i,j,k}$ is a *maximal graphical partition* for each $1 \leq i < j < k \leq n$.

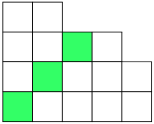
Incidence matrix structure

Block Graphs

For a fixed index i^*, summing $b_{i^*,j,k}$ on varying of j, k is equivalent to stack their Ferrers diagrams

$b_{1,2,6}$  $+$ $b_{1,3,5}$  $+$ $b_{1,4,5}$ 

getting a new integer partition λ^{i^*}

λ^1 

Incidence matrix structure

Block Graphs

Starting from the incidence matrix, we define the *block graphs* of a $\mathcal{D}$-hypergraph:

$$H_\pi = \left(\begin{array}{cccccc}
\cancel{1} & 1 & 1 & 0 & 0 & 0 \\
\cancel{1} & 1 & 0 & 1 & 0 & 0 \\
\cancel{1} & 1 & 0 & 0 & 1 & 0 \\
\cancel{1} & 1 & 0 & 0 & 0 & 1 \\
\cancel{1} & 0 & 1 & 1 & 0 & 0 \\
\cancel{1} & 0 & 1 & 0 & 1 & 0 \\
\cancel{1} & 0 & 1 & 0 & 0 & 1 \\
\cancel{1} & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & \cancel{1} & 1 & 1 & 0 & 0 \\
0 & \cancel{1} & 1 & 0 & 1 & 0 \\
0 & \cancel{1} & 1 & 0 & 0 & 1 \\
0 & \cancel{1} & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & \cancel{1} & 1 & 1 & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \lambda^1 = (4,4,3,3,2) \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \lambda^2 = (3,2,2,1,0) \\
\left.\begin{array}{c} \\ \end{array}\right\} \lambda^3 = (1,1,0,0,0)
\end{array}$$

Property

Block graphs are *maximal graphical partitions*.

Incidence matrix structure

Block Graphs

Starting from the incidence matrix, we define the *block graphs* of a $\mathcal{D}$-hypergraph:

$$H_\pi = \left(\begin{array}{cccccc} \cancel{1} & 1 & 1 & 0 & 0 & 0 \\ \cancel{1} & 1 & 0 & 1 & 0 & 0 \\ \cancel{1} & 1 & 0 & 0 & 1 & 0 \\ \cancel{1} & 1 & 0 & 0 & 0 & 1 \\ \cancel{1} & 0 & 1 & 1 & 0 & 0 \\ \cancel{1} & 0 & 1 & 0 & 1 & 0 \\ \cancel{1} & 0 & 1 & 0 & 0 & 1 \\ \cancel{1} & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & \cancel{1} & 1 & 1 & 0 & 0 \\ 0 & \cancel{1} & 1 & 0 & 1 & 0 \\ 0 & \cancel{1} & 1 & 0 & 0 & 1 \\ 0 & \cancel{1} & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & \cancel{1} & 1 & 1 & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \lambda^1 = (4,4,3,3,2) \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \lambda^2 = (3,2,2,1,0) \\ \left.\vphantom{1}\right\} \lambda^3 = (1,1,0,0,0) \end{array}$$

Property

Block graphs are *maximal graphical partitions*.

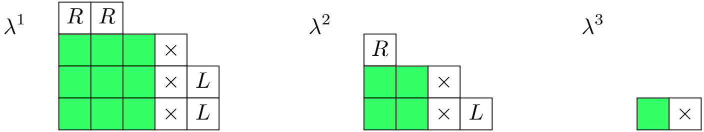
Incidence matrix structure

Block Graphs

Property

Block graphs are *maximal graphical partitions*.

If we look at block graphs as integer partitions, we find strong symmetry properties in their Ferrers diagram:



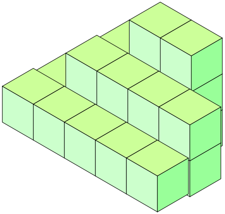
Incidence matrix structure

Plane Partitions

Stacking block graphs, each $\mathcal{D}$-hypergraph can be put in relation with a plane partition. In the previous example,

$$\lambda^1 + \lambda^2 + \lambda^3 = (4,4,3,3,2) + (3,2,2,1) + (1,1)$$

$$P_\pi = \begin{array}{ccccc} 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$



$\mathcal{D}$-plane partitions reflect symmetry properties of maximal graphical partitions.

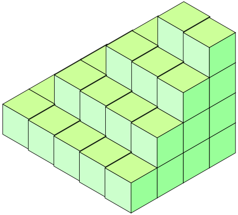
Incidence matrix structure

Plane Partitions

The complete 3-uniform hypergraph on n vertices gives $n-2$ *complete* block graphs. Their Ferrers diagrams are rectangles of decreasing dimension and compose the *complete* plane partition.

For $n=6$, we get

$$\lambda^1+\lambda^2+\lambda^3+\lambda^4=(4,4,4,4,4)+(3,3,3,3)+(2,2,2)+(1,1)$$



$$P_\pi = \begin{matrix} 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

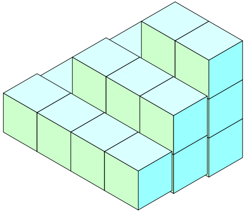
Decomposition of a $\mathcal{D}$-Hypergraph

Property

Given P_π a plane partition related to a generic $\mathcal{D}$-hypergraph with n vertices, there exists P_{π^*}, submatrix of P_π, such that P_{π^*} is the plane partition of a complete $\mathcal{D}$-hypergraph.

So, each $\mathcal{D}$-plane partition contains a complete $\mathcal{D}$-plane partition of lower size.

P_{π^*} and the related $\mathcal{D}$-hypergraph H^* are called *core* of H.



$$P_\pi = \begin{matrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{3} & 1 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & 2 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 \end{matrix}$$

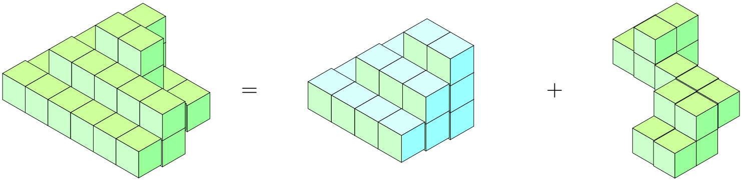
Decomposition of a $\mathcal{D}$-Hypergraph

We can always decompose a $\mathcal{D}$-plane partition as

$$P_\pi = P_{\pi^*} + R_\pi + L_\pi.$$

$R_\pi = L_\pi$ holds, since block graphs are always maximal graphical partitions.

Example



Plane partition of $\pi = (12, 10, 10, 7, 7, 5, 3)$, given by the integer sequence $s = (3, 2, 2, 0, 0, -2, -3)$.

Heuristics from plane partitions

We take advantage from the decomposition of a $\mathcal{D}$-hypergraph as

$$P_\pi = P_{\pi^*} + R_\pi + L_\pi .$$

Strategy: preliminary construction of H^*, the core of the incidence matrix (always exact and in polynomial time).

Aim: avoiding the insertion of extra-rows during the construction of the matrix H.

In general, we do not know the correct order of insertion of rows for the completion of the matrix. Choosing different orders, we can define different subclasses of sequences that admit a polynomial time reconstruction.

Heuristic from plane partitions

Complementarity

Given a $\mathcal{D}$-sequence $\pi = (d_1, \dots, d_n)$, we define its *complementary sequence* as $\pi^c = (d_M - d_1, \dots, d_M - d_n)$, where $d_M = \frac{(n-2)(n-1)}{2}$ is the maximum admitted value for the degree of a vertex.

If $\mathbb{H}$ is the complete 3-uniform hypergraph, π^c is the degree sequence of $H_\pi^c = \mathbb{H} - H_\pi$, the *complementary hypergraph* of the $\mathcal{D}$-hypergraph H_π.

We underline that complementary hypergraphs have the same properties of $\mathcal{D}$-hypergraphs. In particular, they are *unique* and their incidence matrix allows a splitting into blocks.

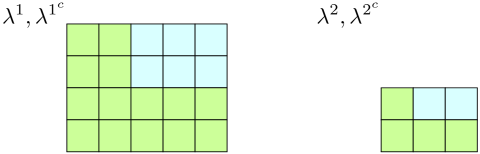
Heuristic from plane partitions

Complementarity

Given a $\mathcal{D}$-sequence $\pi \in \mathcal{D}_n$ and its block graphs λ^i, we define the *complementary block graph* of λ^i as

$$\lambda^{i^c} = \underbrace{(n-i-1,\ldots,n-i-1)}_{(n-i)} - \lambda^i$$

for $i = 1,\ldots,n-2$. It is obtained as the complementary respect to the complete block graph.



Block and complementary block graphs $\pi = (7,6,6,3,3,2)$.

Heuristic from plane partitions

Reconstruction strategy

We propose a new reconstruction strategy, based on the interpretation of hypergraphs as plane partitions and complementarity.

Key point: we reconstruct the plane partition of the hypergraph stacking block and complementary block graphs at the same time.

RecP-core

0. Input: π of length n;
1. Reconstruct P_{π^*} the core of the plane partition;
2. Construct the planes from P^1 to P^{n-2}, avoiding ambiguous insertions;
3. Complete the construction of planes, from P^{n-2} to P^1;
4. Output: P_π.

Heuristic from plane partitions

Example of reconstruction

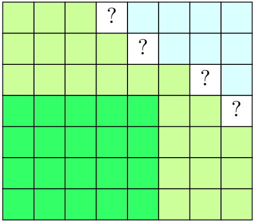
- Input: $\pi_s = (23, 21, 20, 19, 17, 12, 11, 10, 5)$;
- Construction of P_{π^*}, the plane partition of the complete $\mathcal{D}$-hypergraph with 6 vertices;
- Update: $\pi = (13, 11, 10, 9, 7, 2, 11, 10, 5)$ and $\pi^c = (5, 7, 8, 9, 11, 16, 17, 18, 23)$;
- Partial construction of planes, from P^1 to P^4, avoiding ambiguous insertions;

We avoid the insertion of extra-rows during the reconstruction process!

Heuristic from plane partitions

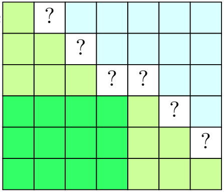
Example of reconstruction

P^1



one choice between $(1, 5, 9), (1, 6, 8)$

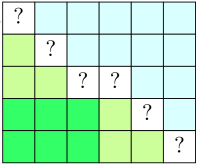
P^2



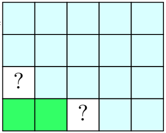
two choices among $(2, 4, 9), (2, 5, 8), (2, 6, 7)$

Heuristics from plane partitions

Example of reconstruction

P^3


one choice between $(3,4,9),(3,5,8),(3,6,7)$

P^4


? $(4,5,7)$

- Update: $\pi = (1,2,1,2,3,1,1,3,1)$;
- Completion of planes, from P^4 to P^1, according to implications required from $\mathcal{D}$-hypergraphs' structure.

Heuristic from plane partitions

Example of reconstruction

The update sequence is $\pi = (1, 2, 1, 2, 3, 1, 1, 3, 1)$, we have to insert five hyperedges to complete the reconstruction.

1. By cardinality reasons, $(4, 5, 7) \in H_\pi$, *inserted*.
2. $(3, 4, 9) \in H_\pi$ implies $(2, 4, 9) \in H_\pi$, in conflict with $\pi_9 = 1$, *excluded*.
3. $(3, 6, 7) \in H_\pi$ implies $(2, 6, 7) \in H_\pi$, in conflict with $\pi_6 = 1$ and $\pi_7 = 1$, *excluded*.
4. It follows that $(3, 5, 8) \in H_\pi$ and $(2, 5, 8) \in H_\pi$, *inserted*.
5. $(2, 4, 9)$ is the last hyperedges involving v_4, and we have $\pi_4 \neq 0$, *inserted*.
6. $(1, 6, 8)$, *inserted*.

Heuristic from plane partitions

- There is not any insertion of extra-rows in H_π and H_{π^c};
- It is necessary to find an algorithmic way to implement the partial reconstruction of planes, based on *geometrical properties* of maximal graphical (complementary) partitions;
- Avoiding ambiguous insertions, we have a partial and *correct* reconstruction of π in polynomial time;
- Implementation (in polynomial time) of the implications required for the completion of the hypergraph;
- *Uniqueness* of $\mathcal{D}$-sequences should ensure the correct completion of the incidence matrix, but we need a rigorous proof.

$\mathcal{D}$-sequences and Order Theory

Triplets' Poset

We consider Ω_n the set of all triplets (a_1, a_2, a_3) whose elements are in $\{1, \dots, n\}$ and s.t. $1 \leq a_1 < a_2 < a_3 \leq n$. We define a partial order relation on it:

$$(a_1, a_2, a_3) \leq (b_1, b_2, b_3) \text{ if and only if } a_i \leq b_i \text{ for } i = 1, 2, 3.$$

We get the partially ordered set $\mathcal{T}_n = (\Omega_n, \leq)$.

Each triplet $(i, j, k) \in \Omega_n$ identifies the hyperedge of the complete 3-hypergraph on n vertices that intersects v_i,v_j and v_k.

Property

Given $h_1, h_2 \in \Omega_n$ s.t. $h_1 \leq h_2$, if $h_2 \in H_\pi$ then $h_1 \in H_\pi$, with H_π the $\mathcal{D}$-hypergraph with degree sequence π. Similarly, if $h_1 \leq h_2$ and $h_1 \notin H_\pi$, then $h_2 \notin H_\pi$.

$\mathcal{D}$-sequences and Order Theory

Theorem

Each hypergraph related to a $\mathcal{D}$-sequence π can be uniquely associated to an ideal I_π of the poset $\mathcal{T}_n$.

Remark: the correspondence is not bijective. There exist ideals in $\mathcal{T}_n$ that identify 3-hypergraphs that are *not* $\mathcal{D}$-hypergraphs.

Hypergraphs identified by ideals keep the properties of $\mathcal{D}$-hypergraphs: their incidence matrix allows a splitting into blocks but they are *not unique* (in general).

We define a new class of 3-graphic sequences, $\mathcal{D}_n^{ext}$, as the class of sequences identified by ideals of $\mathcal{T}_n$.

An extension of $\mathcal{D}$-sequences

Example

$$
\begin{array}{c}
\begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\
\left|\begin{array}{ccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & & & & \vdots & & & & \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right|
\end{array}
\qquad
\begin{array}{c}
\begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\
\left|\begin{array}{ccccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
 & & & & \vdots & & & & \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}\right|
\end{array}
$$

3-uniform hypergraph not in $\mathcal{D}$ and related to the ideal $I_\pi = \downarrow \{(1,6,9),(3,5,8)\}$ (degree sequence $d_H = (25,19,19,16,16,12,10,10,5)$).

Conclusions and Future Developments

- Extend the defined heuristics to get the polynomial time reconstruction of (subclasses of) $\mathcal{D}_n^{ext}$;
- As an alternative, prove that the reconstruction problem is NP-hard also for $\mathcal{D}$-sequences;
- Find bijections between classes $\mathcal{D}_n$, $\mathcal{D}_n^{ext}$ and known combinatorial structures, with the aim of establishing their cardinalities. Investigate $\mathcal{T}_n$ and its ideals;
- Generalize our results to k-uniform degree sequences and hypergraphs, detecting new classes of *unique* sequences that admit a polynomial time solution for the consistency and reconstruction problems.

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