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Optimal interval observers for discrete-time linear switched systems

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ABSTRACT
The main goal of this paper is to present a methodology to design interval observers for discrete-time linear switched systems affected by bounded, but unknown disturbances. Two design techniques are presented. The first one requires that the observation error dynamics are nonnegative while the second one relaxes this restrictive requirement by a change of coordinates. Furthermore, ideas of using H_∞ formalism to compute optimal gains are proposed. Finally, illustrative examples highlight the performance of our methodology.

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Interval observer; discrete time; linear switched systems; LMI; H_∞ design

1. Introduction

One solution to estimate the state variables of a system when some of them are not available for measurements is to use a real-time estimation algorithm. For monitoring, fault detection and feedback control purposes, all of state variables must be available and observers have demonstrated the ability to reproduce efficiently this expected information. Therefore state estimation has become a fundamental problem in control theory and has been developed in many directions. A traditional estimator is the Luenberger observer (Luenberger, 1971) which computes point estimates of the state from input–output data. However, in some cases, this technique may not provide component-wise information of the state vector due to uncertainties. In the last two decades, a new technique of state estimation has been proposed to meet the practical demand. It is based on the notions of framers and interval observers (Gouzé, Rapaport, & Hadj-Sadok, 2000). Framers and interval observers belong to a specific class of estimators called guaranteed state estimation methods whose strength is to provide a region of the state space where the unknown variables are sure to belong. Framers and interval observers are composed of a dynamic extension with two outputs giving upper and lower bounds for the solutions of the considered system at each instant. More precisely, an upper and a lower bound are provided for each component of the state and, in the absence of disturbances, the norm of the error between the bounds converges to zero. There are two other reasons why framers and interval observers become more and more popular. First, they make it possible to cope with large uncertainties, which is very important for example when we consider biological models. Second, they have been successfully applied to many real-life problems (see e.g. Alcaraz-Gonzalez & Gonzalez-Alvarez, 2007; Bernard & Gouzé, 2004; Goffaux, Vande Wouwer, & Bernard, 2009 and references therein). Framers and interval observers have been proposed in many contributions for both continuous-time and discrete-time systems. Some works are devoted to various classes of finite or infinite-dimensional linear systems (Efimov, Perruquetti, Raïssi, & Zolghadri, 2013; Loukkas, Martinez, & Meslem, 2017; Mazenc & Bernard, 2010, 2011; Mazenc & Dinh, 2014; Mazenc, Dinh, & Niculescu, 2014), bilinear systems (Dinh & Ito, 2016) and others concern some classes of nonlinear systems (Ito & Dinh, 2018; Mazenc, Dinh, & Niculescu, 2013; Moisan, Bernard, & Gouzé, 2009; Raïssi, Efimov, & Zolghadri, 2012; Raïssi, Ramdani, & Candau, 2005).

Recently, in the context of systems that exhibit changes along the time among a finite number of possible dynamical behaviours (i.e. switched systems), framers and interval observers have started to be designed (Briat & Khammas, 2017; Ethabet, Rabehi, Efimov, & Raïssi, 2018; Ethabet, Raïssi, Amairi, & Aoun, 2017; Guo & Zhu, 2017; He & Xie, 2016; Ifqir, Oufroukh, Ichalal, & Mammar, 2017). The study of switched systems has received growing attention which can be explained by the fact of the different application domains treated as switched systems, such as the control of mechanical systems, the automotive industry and the automatic process control (Branicky, 1998; Lin & Antsaklis, 2009). Actually, most of interval observer designs in this context are for continuous-time switched systems. To the best of the authors' knowledge, the case of discrete-time switched systems, as e.g. in Guo and Zhu (2017), has not been fully considered in the literature. In this paper, the main objective is to design interval observers for the family of discrete-time linear switched systems affected by bounded but unknown disturbances. Two design techniques are mentioned. The first one requires that the observation error dynamics are nonnegative while the second one relaxes this restrictive requirement by a change of coordinates. It is worth pointing out that the designs we propose are not derived directly from the interval observers constructed for continuous-

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time systems in Dinh, Mazenc, and Niculescu (2014), Ethabet et al. (2017), Ethabet et al. (2018), although some of the key ideas of these works are used along our construction. In fact, changing the system from non-switched case to switched case and from continuous time to discrete time raises the radical changes of stability properties. Consequently, new criteria of stability need to be stated and proved. Additionally, our designs differ from Guo and Zhu (2017): the interval observer we propose is simpler in its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system. This fact is the crucial difference compared with the designs of interval observers introduced in Guo and Zhu (2017) which are carried out for cooperative systems after a coordinate transformation. In fact, we will use the notion of nonnegative and cooperative system as well, but only indirectly to select for the interval observer appropriate initial conditions and upper and lower bounds for the solutions of the studied system. The simplicity of introduced interval observer designs in this paper not only makes the stability analysis easier but also allows one to avoid the hybrid behaviour due to changes of coordinates. In addition to their simplicity, the interval observers we present offer the possibility to construct a bundle of interval observers, as done for instance in Bernard and Gouzé (2004), without having to introduce extra dynamics, simply by proposing several choices of initial conditions and bounding outputs. Furthermore, this paper extends the preliminary work introduced in Marouani, Dinh, Raïssi, and Messaoud (2018). The most important improvement with respect to Marouani et al. (2018) is the idea of using H_∞ formalism to compute optimal gains. In fact until now, the problem of optimising the accuracy of the interval between upper and lower bounds is not yet fully investigated, e.g. in Wang, Lim, and Shen (2018), H_∞ technique has been considered in interval observer design for discrete-time linear systems. We bring in this paper a solution by employing bounded-real lemma to design H_∞ interval observer to obtain a tighter interval width for a class of complex systems. Comparative simulations are given to illustrate these ideas.

The paper is organised as follows. The preliminaries with the introduction of a definition of interval observers are given in Section 2. The main results are stated and proved in Section 3. Comparative simulations are given in Section 4. Finally, a conclusion is drawn in Section 5.

2. Preliminaries

2.1 Notation, definitions, basic result

The set of natural numbers, integers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of nonnegative real numbers and nonnegative integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, respectively. For a vector $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. For a signal $x(k) : N \to \mathbb{R}^n$, the $\mathcal{L}_2$ norm is denoted by $\|x(k)\|_2$. For a measurable and locally essentially bounded input $u : \mathbb{Z} \to \mathbb{R}$, the symbol $\|u\|_{[t_0,t_1]}$ denotes its $\mathcal{L}_\infty$ norm:

$$\|u\|_{[t_0,t_1]} = \sup\{|u|, t \in [t_0, t_1]\}.$$

If $t_1 = \infty$, then we will simply write $\|u\|$. We denote $\mathcal{L}_\infty$ as the set of all inputs u with the property $\|u\| < \infty$. We denote the sequence of integers $1, \dots, N$ as $\overline{1, N}$. Inequalities must be understood *component-wise*, i.e. for $x_a = [x_{a,1}, \dots, x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, \dots, x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \leq x_b$ if and only if, for all $i \in \overline{1, n}$, $x_{a,i} \leq x_{b,i}$. For a matrix $Q \in \mathbb{R}^{m\times n}$, define Q^+, $Q^- \in \mathbb{R}^{m\times n}$ such as $Q^+ = \max(Q, 0)$ and $Q^- = Q^+ - Q$ and the matrix of absolute values of all elements be defined by $|Q| = Q^+ + Q^-$, the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be nonnegative if all its entries are nonnegative. I_n is the identity matrix of $n \times n$ dimension. Any $n \times m$ (resp. $p \times 1$) matrix, whose entries are all 1 is denoted $E_{n\times m}$ (resp. E_p) and whose entries are all 0 is denoted $0_{n\times m}$ (resp. 0_p). The vector of eigenvalues of each matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\lambda(A)$. $P \in \mathbb{R}^{n\times n}$ is positive (resp. negative) (semi-)definite is denoted as $P \succ (\succcurlyeq)\ 0$ (resp. $P \prec (\preccurlyeq)\ 0$).

Lemma 1 (Efimov, Fridman, Raïssi, Zolghadri, & Seydou, 2012): *Consider a vector $x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, and a constant matrix $A \in \mathbb{R}^{m\times n}$, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

Lemma 2 (Schur Complement Boyd, El Ghaoui, Feron, & Balakrishnan, 1994): *Given the matrices $A = A^{\mathrm{T}}$, $C = C^{\mathrm{T}}$ and B with appropriate dimensions. The following LMIs are equivalent:*

(1)

$$\begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} \succ 0.$$

(2)

$$C \succ 0;\ A - BC^{-1}B^{\mathrm{T}} \succ 0.$$

(3)

$$A \succ 0;\ C - B^{\mathrm{T}}A^{-1}B \succ 0.$$

A discrete-time system described by $x(k+1) = f(x(k))$ is nonnegative if for any integer k_0 and any initial condition $x(k_0) \geq 0$, the solution x satisfies $x(k) \geq 0$ for all integers $k \geq k_0$.

As a consequence of the definition of nonnegative systems right above, a linear system described by $x(k+1) = Ax(k) + u(k)$, with $x(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is nonnegative if and only if the matrix A is elementwise nonnegative, $u(k) \geq 0$ and $x(k_0) \geq 0$. In this case, the system is also called cooperative. This property is essential in the design of interval observers since the estimation errors should follow nonnegative dynamics.

2.2 Interval observer for switched systems

Definition 1: Consider a switched system:

$$\begin{cases} x(k+1) = f_q(x(k), d(k)), \\ y(k) = g_q(x(k)), \end{cases} \tag{2}$$

with the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$, the index of the active subsystem $q \in \overline{1, N}$, the number of subsystems is $N \in \mathbb{N}$

and f_q, g_q are functions. The uncertainties $d(k) \in \mathbb{R}^\ell$ are such that there exists a sequence $\overline{d}(k) \in \mathbb{R}^\ell$ such that, for all $k \geq 0$, $-\overline{d}(k) \leq d(k) \leq \overline{d}(k)$. The initial condition $x(0)$ is assumed to be bounded by two known bounds:

$$\underline{x}(0) \leq x(0) \leq \overline{x}(0). \tag{3}$$

Then, the dynamical system

$$z(k+1) = h_q\left(z(k), y(k), \overline{d}(k)\right), \quad q \in \overline{1,N},\ N \in \mathbb{N}, \tag{4}$$

associated with the initial condition $z(0) = r_q(\overline{x}(0), \underline{x}(0)) \in \mathbb{R}^{n_z}$ and bounds for the solution $x(k)$: $\overline{x}(k) = \overline{h}_q(z(k), y(k))$, $\underline{x}(k) = \underline{h}_q(z(k), y(k))$, where $q \in \overline{1,N}$, $N \in \mathbb{N}$, h_q, r_q, $\overline{h}_q$ and $\underline{h}_q$ are functions, is called(i) a framer for (2) if for any vectors $x(0), \underline{x}(0)$ and $\overline{x}(0)$ in $\mathbb{R}^n$ satisfying (3), the solutions denoted respectively x and z of (2) – (4) with respectively $x(0)$, $z(0) = r_q(\overline{x}(0), \underline{x}(0))$ as initial condition at 0, satisfy for all $k \geq 0$, the inequalities

$$\underline{x}(k) = \underline{h}_q(z(k), y(k)) \leq x(k) \leq \overline{h}_q(z(k), y(k)) = \overline{x}(k), \tag{5}$$

(ii) an interval observer for (2) if in addition $|\overline{h}_q(z(k), y(k)) - \underline{h}_q(z(k), y(k))|$ is input-to-state stable (ISS) with respect to $d(k) \in \mathbb{R}^\ell$ for all $q \in \overline{1,N}$, $N \in \mathbb{N}$.

3. Main results

Consider the following discrete-time linear switched system:

$$\begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k) \\ y(k) = C_q x(k) \end{cases}, \quad q \in \overline{1,N},\ N \in \mathbb{N}, \tag{6}$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $d \in \mathbb{R}^n$ is the disturbances. q is the index of the active subsystem and N is the number of subsystems. A_q, B_q and C_q are time-invariant matrices of the corresponding dimensions.

In this section, the goal is to design framers and interval observers for discrete-time linear switched systems. First, two approaches to construct framers are considered: the first one is based on the nonnegativity (cooperativity) of the estimation errors in the original coordinates, while the second is more general and follows a transformation of coordinates. Using the available information and considering that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$ for some known $\underline{x}(0), \overline{x}(0) \in \mathbb{R}^n$, the objective is to calculate two estimates $\underline{x}, \overline{x} \in \mathcal{L}_\infty^n$, such that

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k),\ k \in \mathbb{Z}_+. \tag{7}$$

Later, the gains can be computed by H_∞ formalism which turn framers into interval observers and allow to cope with uncertainties. In addition, thank to this design, we can improve accuracy of the interval observers because the width of enclosure depends on selecting the observer gains.

Some assumptions are introduced for the rest of the paper.

Assumption 1: The state vector $x \in \mathbb{R}^n$ is bounded, i.e. $x \in \mathcal{L}_\infty^n$.

Assumption 2: Let a function $\overline{d} \in \mathcal{L}_\infty^n$ such that for all $k \in \mathbb{Z}_+$

$$-\overline{d}(k) \leq d(k) \leq \overline{d}(k).$$

Assumption 1 states that the state x is bounded. This assumption is common in the theory of observers design since the control design (stabilisation) is not considered at this stage. Assumption 2 is basic in the literature of interval observers where the disturbances are assumed bounded with known bounds.

3.1 Framer design in the original coordinates

In this part, we introduce the following assumption in order to design the framer without a transformation of coordinates:

Assumption 3: There exist $L_q \in \mathbb{R}^{n\times p}$ such that $A_q - L_q C_q$ are nonnegative, for all $q \in \overline{1,N}$.

Assumption 3 is an important condition to design an interval observer, it is rather restrictive and it will be relaxed in Section 3.2.

As a solution to (7), the following framer candidate is considered:

$$\begin{cases} \overline{x}(k+1) = (A_q - L_q C_q)\overline{x}(k) + B_q u(k) + \overline{d}(k) + L_q y, \\ \underline{x}(k+1) = (A_q - L_q C_q)\underline{x}(k) + B_q u(k) - \overline{d}(k) + L_q y, \\ \underline{x}(0) = \underline{X}_0,\ \overline{x}(0) = \overline{X}_0, \end{cases} \tag{8}$$

where $L_q \in \mathbb{R}^{n\times p}$ is an appropriate observer gain associated to the q-subsystem with $q \in \overline{1,N}$, which satisfies Assumption 3 and later in Section 3.3 ensures that (8) becomes an interval observer.

Theorem 1 (First framer design): *Let Assumptions 1–3 hold, the lower and upper bounds $\underline{x}(k)$, $\overline{x}(k)$ for the state $x(k)$ given by (8) satisfy (7), provided that $\underline{X}_0 \leq x(0) \leq \overline{X}_0$.*

Proof: Let $\overline{e}(k) = \overline{x}(k) - x(k)$ and $\underline{e}(k) = x(k) - \underline{x}(k)$ be the upper observation and lower observation errors, respectively. The aim is to prove that $\overline{e}(k)$ and $\underline{e}(k)$ are nonnegative and bounded. The dynamics of the upper error follow:

$$\overline{e}(k+1) = (A_q - L_q Cq)\overline{e}(k) + \overline{d}(k) - d(k). \tag{9}$$

Similarly, the dynamics of the lower error are described by

$$\underline{e}(k+1) = (A_q - L_q C_q)\underline{e}(k) + \overline{d}(k) + d(k). \tag{10}$$

According to Assumption 2, we have $\overline{d}(k) + d(k) \geq 0$ and $\overline{d}(k) - d(k) \geq 0$. Bearing in mind Assumption 3 and from the fact that $\overline{e}(0) = \overline{x}(0) - x(0) \geq 0$ and $\underline{e}(0) = x(0) - \underline{x}(0) \geq 0$, it follows that, for all $k \in \mathbb{Z}_+$, $\overline{e}(k) \geq 0$ and $\underline{e}(k) \geq 0$. Thus, for all $k \in \mathbb{Z}_+$, $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$. This allows us to conclude that (8) is a framer for (6). ∎

3.2 Framer design in the new coordinates

Even though we have proposed in Theorem 1 the different steps to design a framer, in some cases, it is not possible to find gains L_q such that the matrices $A_q - L_qC_q$ are nonnegative, i.e. Assumption 3 is restrictive. Naturally, one can think about finding a nonsingular transformation $z = Rx$ such that the matrices $R(A_q - L_qC_q)R^{-1}$ are nonnegative. Subsequently, framer can be constructed in these new coordinates. However, the existence of a common transformation R for all $q \in \overline{1,N}$ is not obvious, even impossible.

A new methodology is proposed. It is based on the design, in the original base, of two conventional observers. The structure is inspired by the one proposed in Dinh et al. (2014) for non-switched systems. We introduce the assumption needed in the following.

Assumption 4: There exist changes of coordinates R_q, $q = \overline{1,N}$ such that the matrices

$$R_q(A_q - L_qC_q)R_q^{-1} \tag{11}$$

are nonnegative for all $q = \overline{1,N}$.

Theorem 2 (Second framer design): *Let Assumptions 1, 2 and 4 hold. Consider the discrete-time linear switched system (6) and the dynamic extension as follows:*

$$\begin{cases} \hat{x}^+(k+1) = (A_q - L_qC_q)\hat{x}^+(k) + B_qu(k) \\ \quad +R_q^{-1}|R_q|\overline{d}(k) + L_qy(k), \\ \hat{x}^-(k+1) = (A_q - L_qC_q)\hat{x}^-(k) + B_qu(k) \\ \quad -R_q^{-1}|R_q|\overline{d}(k) + L_qy(k), \end{cases} \tag{12}$$

associated with the suitably selected initial conditions:

$$\begin{cases} \hat{x}^+(0) = S_q(R_q^+\overline{x}(0) - R_q^-\underline{x}(0)), \\ \hat{x}^-(0) = S_q(R_q^+\underline{x}(0) - R_q^-\overline{x}(0)), \end{cases} \tag{13}$$

where

$$S_q = R_q^{-1}, \quad \forall q \in \overline{1,N},\ N \in \mathbb{N}. \tag{14}$$

Then,

$$\begin{cases} \underline{x}(k) = S_q^+R_q\hat{x}^-(k) - S_q^-R_q\hat{x}^+(k), \\ \overline{x}(k) = S_q^+R_q\hat{x}^+(k) - S_q^-R_q\hat{x}^-(k), \end{cases} \tag{15}$$

is a framer for (6) satisfying (7).

Remark 1: Assumptions 3 and 4, which are related to the notion of nonnegative and cooperative system, are fundamental and frequently used in designing interval observers. Assumption 4 allows one to relax the restrictive Assumption 3. In fact thank to changes of coordinates, L_q in (12) will be selected only for the purpose of stability instead of having to ensure both stability and nonnegativity constraints as in (8). Section 3.3 introduces the main theorem as guidelines for selecting L_q for (8) and (12).

Proof: Let us prove that $\overline{x}(k) - x(k) \geq 0$ and $x(k) - \underline{x}(k) \geq 0$. First, let us define errors $E_q^+(k)$ and $E_q^-(k)$ as

$$E_q^+(k) = R_q\hat{x}^+(k) - R_qx(k), \tag{16}$$

$$E_q^-(k) = R_qx(k) - R_q\hat{x}^-(k). \tag{17}$$

Thus

$$E_q^+(k+1) = R_q\hat{x}^+(k+1) - R_qx(k+1), \tag{18}$$
$$E_q^-(k+1) = R_qx(k+1) - R_q\hat{x}^-(k+1). \tag{19}$$

From (6) and (12), the dynamics of E_q^+ are given by

$$E_q^+(k+1) = F_qE_q^+(k) + \gamma_q^+, \tag{20}$$

where

$$F_q = R_q(A_q - L_qC_q)R_q^{-1} \tag{21}$$

and

$$\gamma_q^+(k) = |R_q|\overline{d}(k) - R_qd(k). \tag{22}$$

Similarly, the dynamics of E_q^- are given by

$$\begin{aligned} E_q^-(k+1) &= R_qx(k+1) - R_q\hat{x}^-(k+1) \\ &= F_qE_q^-(k) + \gamma_q^-, \end{aligned} \tag{23}$$

with

$$\gamma_q^-(k) = |R_q|\overline{d}(k) + R_qd(k). \tag{24}$$

Bearing in mind Lemma 1, then

$$-R_q^+\overline{d}(k) - R_q^-\overline{d}(k) \leq R_qd(k) \leq R_q^+\overline{d}(k) + R_q^-\overline{d}(k). \tag{25}$$

By using (25), we have $R_qd(k) + |R_q|\overline{d}(k) \geq 0$, $|R_q|\overline{d}(k) - R_qd(k) \geq 0$, then we deduce that $\gamma_q^-(k) \geq 0$ and $\gamma_q^+(k) \geq 0, \forall k \geq 0$.

Moreover, we have $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, then $E_q^-(0) = R_qx(0) - (R_q^+\underline{x}(0) - R_q^-\overline{x}(0))$ and $E_q^+(0) = R_q^+\overline{x}(0) - R_q^-\underline{x}(0) - R_qx(0)$ are nonnegative. As $F_q = R_q(A_q - L_qC_q)R_q^{-1}$ is nonnegative due to Assumption 4, we deduce that $E_q^-(k) \geq 0$ and $E_q^+(k) \geq 0, \forall k \geq 0$.

Consequently, we obtain

$$R_q\hat{x}^-(k) \leq R_qx(k) \leq R_q\hat{x}^+(k). \tag{26}$$

From (26) and (15), it can be verified that

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k).$$

■

Remark 2: The second approach based on the changes of coordinates is general since it is always possible to transform any real square matrix into a nonnegative form. The existence of such a transformation is not a strong assumption. For instance, it has been shown in Efimov et al. (2013) that there always exists an invertible matrix P such that in the coordinates $z(k) = Px(k)$, the matrix $E = P(A - LC)P^{-1}$ is nonnegative. In addition, it has been shown in Mazenc et al. (2014) that based on the Jordan canonical form, it is always possible to transform any square constant matrix into a nonnegative form with a constant or a time-varying transformation.

Remark 3: The cooperativity property has motivated the need for state transformation. The interest of the second structure proposed above is that even by using changes of coordinates $z(k) = R_q x(k)$, the framer (12)–(13) is designed in the original coordinates 'x' (i.e. $\hat{x}^+(k+1) = (A_q - L_q C_q)\hat{x}^+(k) + \cdots$) instead of in the basis 'z' (i.e. $\hat{x}^+(k+1) = R_q(A_q - L_q C_q)R_q^{-1}\hat{x}^+(k) + \cdots$) as in Guo and Zhu (2017). This makes the stability analysis simpler and allows one to avoid jumping of the framer state and a hybrid behaviour in the basis 'z'.

3.3 Optimal gain computation

We devote this section to the computation of gains L_q to ensure stability properties which turn framers (8) and (15) into interval observers. Notice that these gains decide also the tightness of the interval width. Hence, the goal is not only to ensure the stability of the ultimate bound but also to improve the accuracy of the proposed framers. The idea is based on H_∞ design. In other words, we are interested in computing observer gains L_q which minimise the following cost function:

$$\begin{aligned} &\underset{L_q \in \mathbb{R}^{n\times p}}{\text{minimise}} \quad \gamma^2, \quad q = 1, \ldots, N \\ &\text{subject to} \quad \frac{\|e\|_2^2}{\|\delta\|_2^2} \le \gamma^2. \end{aligned} \tag{27}$$

With e is the estimation error depending on the interval width $\overline{x}(k) - \underline{x}(k)$ and δ is an input which takes into account the bound of the disturbances d, where γ is a positive real number. The effect of the known bound of the uncertainties δ on the estimation error e is reduced by the observer gain matrices L_q. According to first or second framer design ((8) or (15)), we define e and δ, respectively, as follows.

(1) *In the case of first framer design*. Given the framer (8) for the system (6). We define the estimation error $e(k) = \overline{x}(k) - \underline{x}(k)$ and the input $\delta(k) = \overline{d}(k)$. Then, from (9) and (10), we obtain immediately

$$e(k+1) = (A_q - L_q C_q)e(k) + T_n\delta(k),$$

with $T_n = 2I_n$.

(2) *In the case of second framer design*. Given the framer (12)–(15) for the system (6). We define the estimation error $e(k) = R_q^{-1}|S_q|^{-1}(\overline{x}(k) - \underline{x}(k))$ and the input $\delta(k) = R_q^{-1}|R_q|\overline{d}(k)$, with R_q, S_q are constant changes of coordinates in (14). Then, from (15) we have

$$\begin{aligned} e(k+1) &= R_q^{-1}|S_q|^{-1}(S_q^+ R_q + S_q^- R_q) \\ &\quad \times \left(\hat{x}^+(k+1) - \hat{x}^-(k+1)\right) \\ &= R_q^{-1}|S_q|^{-1}|S_q|R_q\left(\hat{x}^+(k+1) - \hat{x}^-(k+1)\right) \\ &= \hat{x}^+(k+1) - \hat{x}^-(k+1). \end{aligned}$$

Due to (18)–(20) and (23), one can prove through simple calculations that

$$\begin{aligned} e(k+1) &= R_q^{-1}\left(E_q^+(k+1) + E_q^-(k+1)\right) \\ &= (A_q - L_q C_q)R_q^{-1}\left(E_q^+(k) + E_q^-(k)\right) \\ &\quad + 2R_q^{-1}(R_q^+\overline{d}(k) + R_q^-\overline{d}(k)) \\ &= (A_q - L_q C_q)R_q^{-1}R_q e(k) + 2R_q^{-1}|R_q|\overline{d}(k) \\ &= (A_q - L_q C_q)e(k) + T_n\delta(k), \end{aligned}$$

with $T_n = 2I_n$.

Remark 4: In both two cases, with the suitable choices of e and δ, we always have

$$e(k+1) = (A_q - L_q C_q)e(k) + T_n\delta(k), \tag{28}$$

with $T_n = 2I_n$.

To simplify exposition, from now on we replace $e(k)$ and $\delta(k)$ by the subscripts e_k and δ_k, respectively. Note that the optimisation problem (27) can be reformulated under LMI form through the bounded-real lemma (Boyd et al., 1994) to the linear switched dynamics (28). This leads to the following inequality:

$$e_{k+1}{}^{\mathrm{T}} P e_{k+1} - e_k{}^{\mathrm{T}} P e_k \le -e_k{}^{\mathrm{T}} e_k + \gamma^2 \delta_k{}^{\mathrm{T}} \delta_k, \tag{29}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

We are ready to propose the theorem as guidelines for selecting gains L_q for (8) and (15).

Theorem 3 (Gain computation for (8) and (15)): *If Assumptions 1 and 2 are satisfied and if there exists a symmetric positive definite matrix P such that*

$$\begin{bmatrix} -P + I_n & 0_{n\times n} & A_q{}^{\mathrm{T}}P - C_q{}^{\mathrm{T}}U_q^{\mathrm{T}} \\ 0_{n\times n} & -\gamma^2 I_n & 2P \\ PA_q - U_q C_q & 2P & -P \end{bmatrix} \preccurlyeq 0, \tag{30}$$

with $U_q = PL_q$, the following statements hold true:

(i) *If Assumption 3 is satisfied, the framers proposed in (8) become interval observers for (6).*
(ii) *If Assumption 4 is satisfied, the framers proposed in (15) become interval observers for (6).*

Moreover, the optimal observer gain matrix,

$$L_q = P^{-1}U_q \tag{31}$$

is computed via the solution of the following constrained minimisation problem:

$$\begin{aligned} &\underset{P,U_q}{\text{minimise}} \quad \gamma^2,\ q = 1, \ldots, N \\ &\text{subject to} \quad (30). \end{aligned} \tag{32}$$

Proof: From (28) and (29), one can obtain

$$((A_q - L_qC_q)e_k + T_n\delta_k)^{\mathrm{T}}P((A_q - L_qC_q)e_k + T_n\delta_k) - {e_k}^{\mathrm{T}}Pe_k + {e_k}^{\mathrm{T}}e_k - \gamma^2{\delta_k}^{\mathrm{T}}\delta_k \leq 0. \quad (33)$$

Inequality (33) is satisfied if and only if there exist P and γ such that

$$\begin{bmatrix} (A_q - L_qC_q)^{\mathrm{T}}P(A_q - L_qC_q) - P + I_n & (A_q - L_qC_q)^{\mathrm{T}}PT_n \\ {T_n}^{\mathrm{T}}P(A_q - L_qC_q) & {T_n}^{\mathrm{T}}PT_n - \gamma^2 I_n \end{bmatrix} \preccurlyeq 0. \quad (34)$$

To get LMI version for (34), some intermediate steps are necessary. We can rewrite (34) as

$$\begin{bmatrix} -P + I_n & 0_{n\times n} \\ 0_{n\times n} & -\gamma^2 I_n \end{bmatrix} + \begin{bmatrix} (A_q - L_qC_q)^TP \\ T_n^TP \end{bmatrix} P^{-1}\begin{bmatrix} P(A_q - L_qC_q) & PT_n \end{bmatrix} \preccurlyeq 0. \quad (35)$$

Then, based on the Schur complement introduced in Lemma 2, we can verify that

$$\begin{bmatrix} -P + I_n & 0_{n\times n} & {A_q}^{\mathrm{T}}P - {C_q}^{\mathrm{T}}L_q^{\mathrm{T}}P \\ 0_{n\times n} & -\gamma^2 I_n & 2P \\ PA_q - PL_qC_q & 2P & -P \end{bmatrix} \preccurlyeq 0. \quad (36)$$

Hence,

$$\begin{bmatrix} -P + I_n & 0_{n\times n} & {A_q}^{\mathrm{T}}P - {C_q}^{\mathrm{T}}U_q^{\mathrm{T}} \\ 0_{n\times n} & -\gamma^2 I_n & 2P \\ PA_q - U_qC_q & 2P & -P \end{bmatrix} \preccurlyeq 0, \quad (37)$$

with $U_q = PL_q$. ■

What is important to note here is that this methodology can be applied for the two different proposed framer designs. The same LMI can be used to compute optimal observer gains based on H_∞ formalism under suitable choices of e and δ.

4. Comparative simulations

In this section, we compare our results with the ones introduced in Marouani et al. (2018) to highlight the contributions of this note. Consider the discrete-time linear switched system subject to disturbances of the form (6):

$$\begin{cases} x(k+1) = A_qx(k) + B_qu(k) + d(k) \\ y(k) = C_qx(k) \end{cases}, \quad q \in \overline{1,N},\ N \in \mathbb{N},$$

where $N = 3$,

$$A_1 = \begin{pmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.3 & -2 \\ 0 & 0.6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.5 & -1.1 \\ 0 & 0.16 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0.1 & 1 \end{pmatrix},$$

$$d(k) = 0.1\begin{pmatrix} \sin(0.5k) \\ \cos(0.5k) \end{pmatrix}$$

is the bounded disturbances such that $-\overline{d} \leq d(k) \leq \overline{d}$ with $\overline{d} = \binom{0.1}{0.1}$.

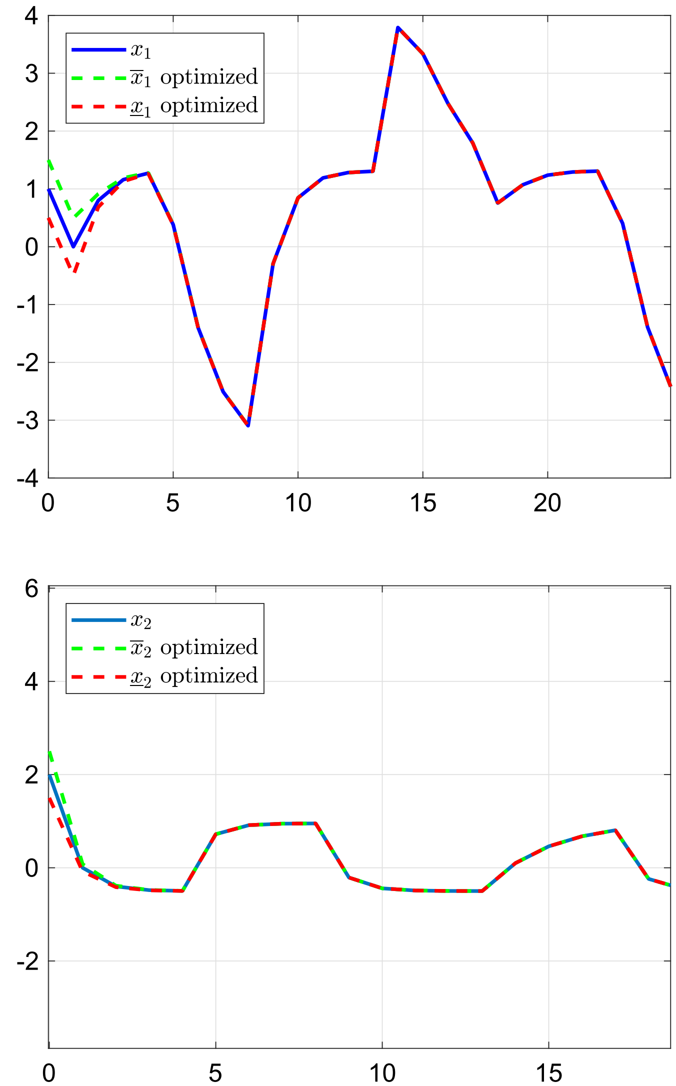

Figure 1. State and estimated bounds without disturbances.

4.1 Optimised gains

The solution which minimises $\alpha = \gamma^2$, obtained using the package CVX (Grant & Boyd, 2014) is given by

$$L_1 = \begin{pmatrix} -0.026 \\ 0.0914 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.5416 \\ -0.0758 \end{pmatrix}, L_3 = \begin{pmatrix} -0.857 \\ 0.1028 \end{pmatrix},$$

$$P = \begin{pmatrix} 2.3947 & 1.8427 \\ 1.8427 & 14.8912 \end{pmatrix}, \quad \alpha = 99.6689.$$

We verify that the matrices $A_q - L_qC_q$ are not nonnegative for all $q \in \overline{1,3}$ so changes of coordinates are required. As discussed above, it is difficult to find a common R such that $R(A_q - L_qC_q)R^{-1}$ are all nonnegative for all $q \in \overline{1,3}$. Consequently, we propose for the next changes of coordinates R_q satisfying that the matrices $R_q(A_q - L_qC_q)R_q^{-1}$ are nonnegative for all $q \in \overline{1,3}$

$$R_1 = \begin{pmatrix} 0.0901 & 0.6930 \\ -0.0901 & 0.3070 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -0.2372 & -0.8173 \\ 0.2372 & 1.8173 \end{pmatrix},$$

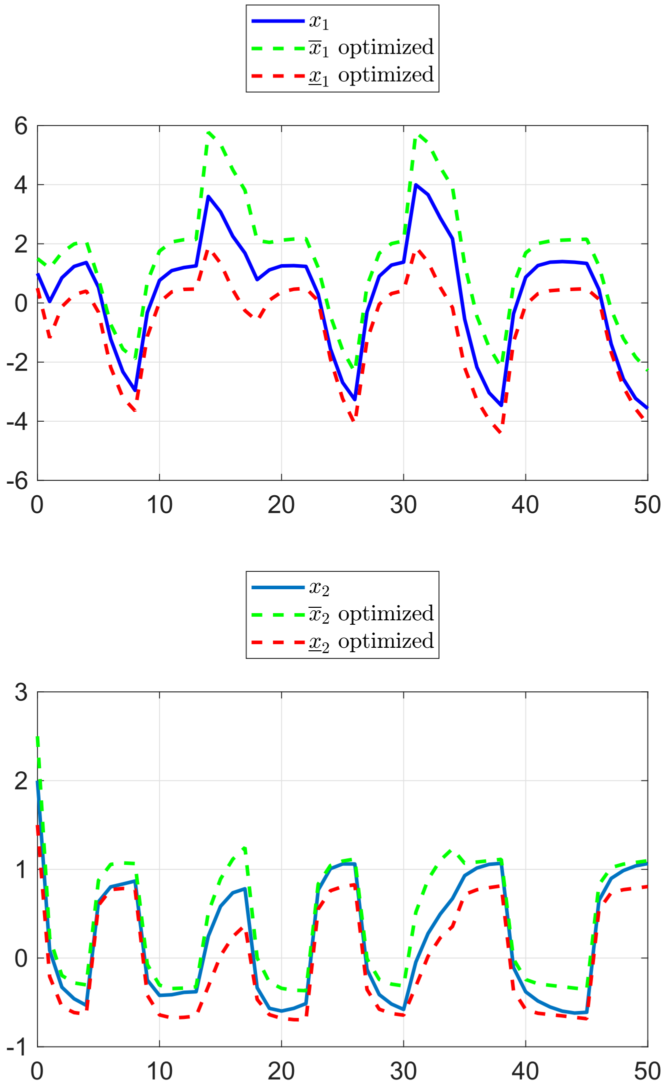

Figure 2. State and estimated bounds with the disturbances.

$$R_3 = \begin{pmatrix} 0.0191 & 0.9913 \\ -0.0191 & 0.0087 \end{pmatrix}, \quad R_1^{-1} = \begin{pmatrix} 3.4083 & -7.6972 \\ 1 & 1 \end{pmatrix},$$
$$R_2^{-1} = \begin{pmatrix} -7.661 & -3.4453 \\ 1 & 1 \end{pmatrix}, \quad R_3^{-1} = \begin{pmatrix} 0.4558 & -51.8598 \\ 1 & 1 \end{pmatrix}.$$

For simulations, we use the initial conditions $x_0 = [1, 2]^{\mathrm{T}}$, $\overline{x}_0 = (1.5, 2.5)^{\mathrm{T}}$, $\underline{x}_0 = (0.5, 1.5)^{\mathrm{T}}$. The results of simulation of the optimised observer are depicted in Figures 1 and 2 for both coordinates where solid lines present the state and dashed lines present the estimated bounds. Figure 1 illustrates the result in the case where there is no disturbances. We see clearly that the interval length converges to zero. Figure 2 shows that in the case where the system is affected by additive disturbances, the interval observer still provides the solutions with bounds. The switching signal between the three subsystems is plotted in Figure 3.

4.2 Comparisons

In Marouani et al. (2018), the following LMI, which turns framers (8) and (15) into interval observers without considering the tightness problem of the interval widths, is proposed

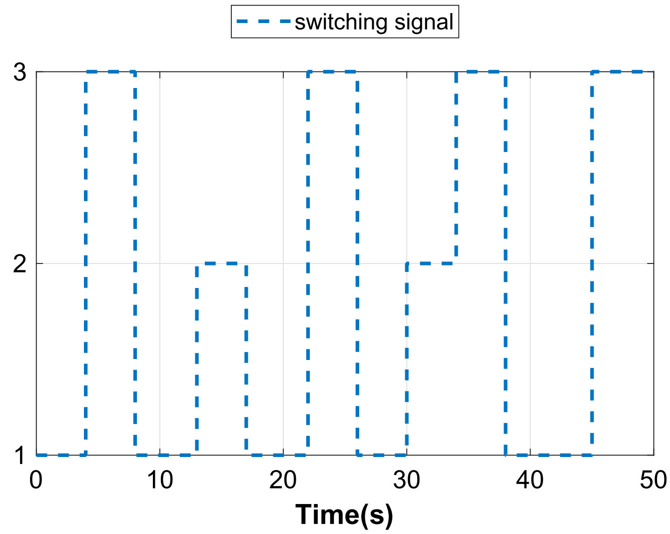

Figure 3. Switching signal.

$$\begin{bmatrix} P & U_q C_q \\ C_q^{\mathrm{T}} U_q^{\mathrm{T}} & \frac{\delta_q}{1+\delta_q} P - A_q^{\mathrm{T}} P A_q + A_q^{\mathrm{T}} U_q C_q + (A_q^{\mathrm{T}} U_q C_q)^{\mathrm{T}} \end{bmatrix} \succ 0,$$
$$q \in \overline{1, N}, \; N \in \mathbb{N}, \tag{38}$$

with P, U_q defined in Theorem 3 and $\delta_q > 0$. Using the Yalmip toolbox (Löfberg, 2004), the gains obtained in Marouani et al. (2018) are given by

$$L_1 = \begin{pmatrix} -0.0954 \\ 0.0388 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.7253 \\ -0.1286 \end{pmatrix}, \quad L_3 = \begin{pmatrix} -0.8067 \\ 0.166 \end{pmatrix},$$
$$P = \begin{pmatrix} 0.1818 & 0.23 \\ 0.23 & 1.0198 \end{pmatrix}, \quad \delta_q = 1.588 \quad \forall q \in \overline{1, 3}.$$

We verify that the matrices $A_q - L_q C_q$ are not nonnegative for all $q \in \overline{1, 3}$ so changes of coordinates are required. Therefore for the solution of the LMI (38), we propose changes of coordinates R_q such that the matrices $R_q(A_q - L_q C_q)R_q^{-1}$ are nonnegative for all $q \in \overline{1, 3}$ as follows:

$$R_1 = \begin{pmatrix} 0.062 & 0.7002 \\ -0.062 & 0.2998 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -0.8664 & -2.9526 \\ 0.8664 & 3.9526 \end{pmatrix},$$
$$R_3 = \begin{pmatrix} 0.0275 & 0.9864 \\ -0.0275 & 0.0136 \end{pmatrix}, \quad R_1^{-1} = \begin{pmatrix} 4.8366 & -11.2966 \\ 1 & 1 \end{pmatrix},$$
$$R_2^{-1} = \begin{pmatrix} -4.562 & -3.4078 \\ 1 & 1 \end{pmatrix}, \quad R_3^{-1} = \begin{pmatrix} 0.4932 & -35.8415 \\ 1 & 1 \end{pmatrix}.$$

With the same initial conditions introduced in Section 4.1, the simulation of the optimised bounds and of the bounds given by Marouani et al. (2018) are depicted in Figure 4 for the purpose of comparison. Both coordinates are drawn where solid lines present the state and dashed lines present the estimated bounds.

The simulation results show that the interval width obtained by our H_∞ design is tighter than the approach proposed in Marouani et al. (2018). The accuracy of framer is clearly improved.

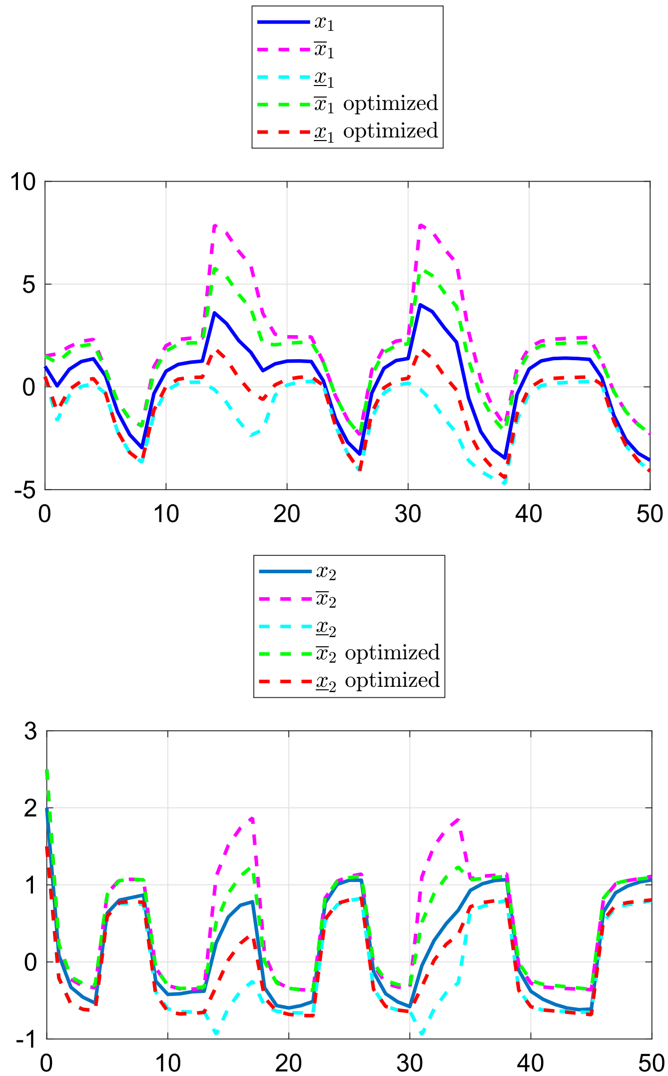

Figure 4. State and estimated bounds.

5. Conclusion

In this paper, two techniques to design interval observers for a class of discrete-time linear switched systems in the presence of additive disturbances are proposed. The assumptions given in the first one are not always feasible. Therefore, a second approach based on changes of coordinates is proposed to relax the condition of nonnegativeness of $A_q - L_q C_q$, $q = \overline{1, N}$. In this context, two copies of classical observers associated with suitably selected initial conditions are reformulated in the base 'x'. For improvement of performance for estimation, the present note has also introduced H_∞ method to estimate optimally the state. The effectiveness of the method is confirmed through comparative simulations. Observer gains can be computed in term of LMIs. Extensions to switched systems with unknown switching instants are expected for further studies.

Disclosure statement

No potential conflict of interest was reported by the authors.

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