Convex Quadratic Reformulation Applied to the Graph Equicut Problem

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Consider the following linearly-constrained zero-one quadratic program :

$$(QP) \ : \ \text{Min} \ \ \left\{q(x) = x^tQx + c^tx : Ax = b, \ A'x \leq b', \ x \in \{0,1\}^n\right\}$$

where c is an n real vector, b an m real vector, b' a p real vector, Q a symmetric $n \times n$ real matrix, A an $m \times n$ matrix and A' a $p \times n$ matrix. A method to solve (QP) is presented in [1]. It consists to reformulate (QP) into an equivalent 0-1 quadratic program with a convex objective function. This reformulation uses equality constraints and requires solutions of a semidefinite relaxation of (QP). In this communication, we choose to present an application of this method to the equicut problem. Given an undirected graph $G = (V, U)$ with n nodes denoted by $v_1, \ldots, v_n$, the problem (which is NP-hard) consists in partitioning the nodes into two components V_1 and V_2 of $\frac{n}{2}$ nodes such that the number of edges which connect this two components is minimal. It can be formulated as follows:

$$(BP) \ : \ \text{Min} \ \ \{g(x) = \sum_{i<j}^{n} c_{ij} \left(x_i\overline{x_j} + \overline{x_i}x_j\right) : \sum_{i=1}^{n} x_i = \frac{n}{2}, \ x \in \{0,1\}^n\}$$

where the real coefficient c_{ij} is the weight of the edge $[v_i, v_j]$ of G. The binary variable x_i is equal to 1 if and only if the node v_i is in V_1.

The convex reformulation of the equicut problem consists in adding to $g(x)$ two functions, null on the feasible set and depending on two parameters $\alpha \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$:

$$g_{\alpha,u}(x) = g(x) + \sum_{i=1}^{n} \alpha_i x_i \left(\sum_{j=1}^{n} x_j - \frac{n}{2}\right) + \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right)$$

α and u are determined by semidefinite programming in order to make $g_{\alpha,u}(x)$ convex and to maximize its value over the relaxed domain $\{\sum_{i=1}^{n} x_i = \frac{n}{2}, x \in [0,1]^n\}$. Experimental results show that, for this problem, the approach outperforms existing methods ([2]).

[1] A. Billionnet, S. Elloumi and M.-C. Plateau "Convex Quadratic Programming for Exact Solution of 0-1 Quadratic Programs", *Technical report CEDRIC 723, http://cedric.cnam.fr/PUBLIS/RC723.pdf*, 2005.

[2] S. E. Karisch, F. Rendl, J. Clausen, "Solving graph bisection problems with semidefinite programming", *INFORMS Journal on Computing* 12(3), 2000, 177-191