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Bayesian analysis for detecting a change in exponential family

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Abstract

We propose a Bayesian analysis of detection of a change of parameter in a sequence of independent random variables from exponential family. The test uses the highest posterior density credible set. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Change-point; Posterior distribution; HPD credible set; p-Value

1. Introduction

We consider a sequence $X_1, \ldots, X_n$ of independent random variables such that there exists $k \in \{1, \ldots, n\}$ so that $X_1, \ldots, X_k$ have a distribution F_{θ_1} and $X_{k+1}, \ldots, X_n$ have a distribution F_{θ_2}.

We suppose that θ_1, θ_2, k are unknown and k is the said change-point. The aim of this work is to estimate k from the observed sequence $(x_1, \ldots, x_n)$. By the Bayes formula, we determine the posterior distribution of the change-point k.

The problem of detection of change in Bayesain context was studied by many authors. We can cite the works of Chernoff and Zacks [1], Kander and Zacks [8] and Sen and Srivatana (1973) where the aim is to detect the change in the mean for normal random variables; Menzefricke [11] proposed a test of

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change in the variance for the same family of observations. Smith [12] proposed a Bayesian procedure for both normal and binomial random variables and Kim [9] in linear regression context. Henderson and Matthews [7] proposed a non-Baysian model for Poisson random variables to detect changes in annual number of cases of haemolytic uraemic syndrome (HUS). They studied not only at most one change but also multi-change. West and Ogden [13] used the HUS data for the estimation problem of change-points in a sequence of Poisson random variables approached by allowing the change to range over a continuous time interval $(0, T)$, and derived a Bayesian-based interval estimator. Lee [10] proposed a Bayesian test for an exponential random family based on the Type II maximum likelihood (ML-II) approach. The objectives of these works are different from our work. We place our work in a similar situation as in Ghorbanzadeh and Lounes [4] who proposed a Bayesian test for exponential random variables. They introduced a parameter nuisance as in [3] and confirmed Smith's result, but their confirmation depended on the value of this parameter.

We apply the results obtained to the data in [2], on the annual volume of discharge from the Nile River at the Aswan dam in the period 1871 to 1970 for the exponential case, and for the Poisson case we use the data in [7].

2. Statistical model and test procedure

Let F_θ denote an exponential family of the form

$$\mathrm{d}F_\theta(x) = a(x)\,(\phi(\theta))^{b(x)}\,\mathrm{e}^{-\psi(\theta)c(x)}\;\mathrm{d}\mu(x)$$

relative to some Lebesgue measure μ.

Our model includes the following cases:

A. *Continuous cases*
1. If $c(x) = x^2/2$, $b(x) = (1/2)a(x) = 1/\sqrt{2\pi}$, $\phi(\theta) = \psi(\theta)$, then $X \sim \mathcal{N}(0, 1/\psi(\theta))$.
2. If $a(x) = x^{\alpha-1}$, $b(x) = \alpha$ (known), $c(x) = x$, $\phi(\theta) = \psi(\theta)$, then $X \sim$ Gamma$(\alpha, \psi(\theta))$.
3. If $a(x) = 1$, $b(x) = 1$, $c(x) = |x|$, $\phi(\theta) = \psi(\theta)$, then $X \sim$ Double-Exponential$(\psi(\theta))$.

B. *Discrete cases*
1. If $a(x) = 1/x!$, $b(x) = x$, $c(x) = 1$, $\phi(\theta) = \psi(\theta)$, then $X \sim$ Poisson $(\psi(\theta))$.
2. If $a(x) = 1$, $b(x) = 1 - x$, $c(x) = x$, $\phi(\theta) = 1 - \theta$, $\psi(\theta) = -\log\theta$, then $X \sim$ Bernoulli (θ).
3. If $a(x) = \binom{m}{x}$, $b(x) = m - x$, $c(x) = x$, $\phi(\theta) = 1 - \theta$, $\psi(\theta) = -\log\theta$, then $X \sim$ Binomial(m, θ).

4. If $a(x) = \binom{x+r+1}{x}$, $b(x) = x$, $c(x) = r$ (known), $\phi(\theta) = 1 - \theta$, $\psi(\theta) = -\log\theta$, then $X \sim$ Negative-Binomial(r, θ).

In this work, we consider a changing model

$$X_i \mid \theta_1, \theta_2, k \sim \begin{cases} F_{\theta_1} & \text{if } i = 1, \ldots, k, \\ F_{\theta_2} & \text{if } i = k+1, \ldots, n. \end{cases}$$

We suppose that θ_1, θ_2 and k are independent.

We build an inference about testing the hypothesis and estimating the change-point k. That is, to test whether or not a change-point occurs, i.e.

$$\mathrm{H}_0\colon \theta_1 = \theta_2 \quad \text{against} \quad \mathrm{H}_1\colon \theta_1 \neq \theta_2.$$

The proposed test uses Bayesian analysis, based on the posterior distribution of the ratio $\lambda = \theta_1/\theta_2$. The hypothesis H_0 meaning "no change" is equivalent to H_0': $k = n$ and H_1 equivalent to H_1': $k \neq n$. The test H_0' against H_1' is based on the posterior distribution of the change-point k [3,10]. The advantage of this procedure is that asymptotics is not required [6], unlike in non-Bayesian classical procedures [4,5].

2.1. Choice of prior

To cover many classical distributions, we suppose the following two hypotheses: we impose
1. $\psi(\theta) = \phi(\theta) = \theta \quad$ if $\theta \in]0, +\infty[$,
2. $\psi(\theta) = -\log\theta, \phi(\theta) = 1 - \theta \quad$ if $\theta \in]0, 1[$.

We suppose that the parameters θ_j, $j = 1, 2$, have the Gamma prior distribution: $\pi_j(\theta_j) \sim \mathit{Gamma}(\mu_j, v_j)$ for the case $\theta_j \in]0, +\infty[$, and the Beta prior distribution: $\pi_j(\theta_j) \sim \mathit{Beta}(v_j, \mu_j)$ for the case $\theta_j \in]0, 1[$ where the parameters, μ_1, μ_2, v_1 and v_2 are assumed to be known.

These choices are motivated by technical calculus and are usually used in Bayesian change-point literature [3,6,12].

For the prior of k we use discrete families defined as $\{1, \ldots, n\}$ built on a uniform distribution of $[0, 1]$ for a parameter θ:

$$\pi(k \mid \theta) = \begin{cases} q_n(\theta) & \text{if } k = n, \\ p_n(k, \theta) & \text{if } k = 1, \ldots, n-1. \end{cases}$$

Let $v(k) = \int_0^1 \pi(k \mid \theta)\, \mathrm{d}\theta$. Then

$$v(k) = \begin{cases} \int_0^1 q_n(\theta)\, \mathrm{d}\theta & \text{if } k = n, \\ \int_0^1 p_n(k, \theta)\, \mathrm{d}\theta & \text{if } k = 1, \ldots, n-1. \end{cases}$$
$$= \begin{cases} q_n & \text{if } k = n, \\ p_n(k) & \text{if } k = 1, \ldots, n-1. \end{cases}$$

2.2. Posterior distribution of k

Let

$$b_k = \sum_{i=1}^{k} b(x_i), \quad b_k^* = \sum_{i=k+1}^{n} b(x_i),$$

$$c_k = \sum_{i=1}^{k} c(x_i), \quad c_k^* = \sum_{i=k+1}^{n} c(x_i), \quad \text{and} \quad \underline{x} = (x_1, \dots, x_n).$$

We have

$$l(\underline{x} \mid \theta_1, \theta_2, k) = \left(\prod_{i=1}^{n} a(x_i) \right) \phi(\theta_1)^{b_k} \phi(\theta_2)^{b_k^*} \mathrm{e}^{-c_k \psi(\theta_1) - c_k^* \psi(\theta_2)}$$

with the convention $b_n^* = c_n^* = 0$.

By the classical Bayes formula, we obtain

$$\pi(\underline{x}, \theta_1, \theta_2, k) = l(\underline{x} \mid \theta_1, \theta_2, k)\pi(\theta_1, \theta_2)v(k),$$

$$\pi(\underline{x}, k) = \int_0^{\infty} \int_0^{\infty} \pi(\underline{x}, \theta_1, \theta_2, k)\pi_1(\theta_1)\pi_2(\theta_2)\ \mathrm{d}\theta_1 \mathrm{d}\theta_2$$

and

$$\pi(\theta_1, \theta_2 | \underline{x}, k) = \begin{cases} \dfrac{(c_k + v_1)^{b_k + \mu_1}}{\Gamma(b_k + \mu_1)} \theta_1^{b_k + \mu_1 - 1} \mathrm{e}^{-\theta_1 (c_k + v_1)} & \\ \quad \times \dfrac{(c_k^* + v_2)^{b_k^* + \mu_2}}{\Gamma(b_k^* + \mu_2)} \theta_2^{b_k^* + \mu_2 - 1} \mathrm{e}^{-\theta_2 (c_k^* + v_2)} & \text{if } \theta \in]0, +\infty[, \\ \dfrac{\theta_1^{c_k + v_1 - 1} (1 - \theta_1)^{b_k + \mu_1 - 1}}{\beta(b_k + \mu_1, c_k + v_1)} \dfrac{\theta_2^{c_k^* + v_2 - 1} (1 - \theta_2)^{b_k^* + \mu_2 - 1}}{\beta(b_k^* + \mu_2, c_k^* + v_2)} & \text{if } \theta \in]0, 1[, \end{cases}$$

where $\Gamma(\cdot)$ and $\beta(\cdot, \cdot)$ are the gamma and beta functions, respectively.

We can state that, given as $(\underline{x}, k)$, θ_1 and θ_2 are independent and follow

$$\theta_1 \mid \underline{x}, k \sim \begin{cases} Gamma(b_k + \mu_1, c_k + v_1) & \text{if } \theta \in]0, +\infty[, \\ Beta(c_k + v_1, b_k + \mu_1) & \text{if } \theta \in]0, 1[. \end{cases}$$

$$\theta_2 \mid \underline{x}, k \sim \begin{cases} Gamma(b_k^* + \mu_2, c_k^* + v_2) & \text{if } \theta \in]0, +\infty[, \\ Beta(c_k^* + v_2, b_k^* + \mu_2) & \text{if } \theta \in]0, 1[. \end{cases}$$

$$\pi(\underline{x}, k) = \begin{cases} v(k) \dfrac{\Gamma(b_k + \mu_1)\Gamma(b_k^* + \mu_2)}{(c_k + v_1)^{b_k + \mu_1} (c_k^* + v_2)^{b_k^* + \mu_2}} & \text{if } \theta \in]0, +\infty[, \\ v(k)\beta(c_k + v_1, b_k + \mu_1)\beta(c_k^* + v_2, b_k^* + \mu_2) & \text{if } \theta \in]0, 1[. \end{cases}$$

Then, the posterior distribution of k is given by $v^*(k \mid \underline{x}) = \pi(\underline{x}, k)/f(\underline{x})$ with $f(\underline{x}) = \sum_{k=1}^{n} \pi(\underline{x}, k)$.

We can remark that v and v^* are conjugate.

2.3. Statistic Test

Let $\lambda = \theta_1/\theta_2$. In this section we suppose that $\theta \in]0, +\infty[$, thus we have

Proposition 1. *Given as* $(\underline{x}, k)$

$$\pi^*(\lambda \mid \underline{x}, k) = \left(\frac{c_k + v_1}{c_k^* + v_2}\right)^{b_k+\mu_1} \frac{1}{\beta(b_k + \mu_1, b_k^* + \mu_2)} \frac{\lambda^{b_k+\mu_1-1}}{\left[1 + \frac{c_k+v_1}{c_k^*+v_2}\lambda\right]^{\mu_1+\mu_2+b_n}}.$$

By making the following change of variables: $(\lambda = \theta_1/\theta_2, u = \theta_1)$ we obtain the result.

Let

$$R_k(\lambda) = (c_k + v_1)/(c_k^* + v_2)\lambda \quad \text{and} \quad D_k(\lambda) = (b_k^* + \mu_2)/(b_k + \mu_1)R_k(\lambda).$$

Then we have

Proposition 2. Given as $(\underline{x}, k)$
1. $R_k(\lambda) \sim \beta_{II}(b_k, b_k^*)$
2. *If for all* k, *both* $b_k + \mu_1$ *and* $b_k^* + \mu_2$ *are integers, then* $D_k(\lambda) \sim F_{2(b_k+\mu_1),2(b_k^*+\mu_2)}$ *where* $\beta_{II}(a, b)$ *denotes the Beta distribution of the second kind with density*

$$\frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}} \quad \forall x > 0, \quad a > 0, \quad b > 0.$$

If $X \sim \beta_{II}(a, b)$ (with $a \in \mathbb{N}^*$, $b \in \mathbb{N}^*$), then $(b/a)X \sim F_{2a,2b,}$ where F_{n_1,n_2} is the Fisher distribution with (n_1, n_2) degrees of freedom.

2.4. Rejection zone

Given as k, a $(1 - \alpha)$ credible set for $D_k(\lambda)$ is defined by

$$S_k(\alpha) = \{D_k(\lambda)\colon \mathscr{F}_{2(b_k+\mu_1),2(b_k^*+\mu_2)}(\alpha/2) \leqslant D_k(\lambda) \leqslant \mathscr{F}_{2(b_k+\mu_1),2(b_k^*+\mu_2)}(1 - \alpha/2)\},$$

where $\mathscr{F}_{n_1,n_2}(\alpha)$ is the α-quantile of F_{n_1,n_2}.

The decision rule for rejecting H_0 at the $(100\alpha)\%$ level is

$$D_k(\lambda = 1) = \frac{b_k^* + \mu_2}{b_k + \mu_1} \frac{c_k + v_1}{c_k^* + v_2} \notin S_k(\alpha).$$

Given as k, the p-value is

$$p_{\lambda=1}(k) = 2 \min\{P(Y < D_k(1) \mid \underline{x}, k),\ P(Y > D_k(1) \mid \underline{x}, k)\},$$

where $Y \sim F_{2(b_k+\mu_1),2(b_k^*+\mu_2)}$. Then we have

$$p_{\lambda=1}(k) = 2\min\{\mathscr{F}_{2(b_k+\mu_1),2(b_k^*+\mu_2)}(D_k(1)),\ 1 - \mathscr{F}_{2(b_k+\mu_1),2(b_k^*+\mu_2)}(D_k(1))\},$$

where $\mathscr{F}_{2(b_k+\mu_1),2(b_k^*+\mu_2)}$ is the cumulative distribution function of $F_{2(b_k+\mu_1),2(b_k^*+\mu_2)}$.

The unconditional p-value is

$$p_{\lambda=1} = \sum_{k=1}^{n} p_{\lambda=1}(k)\ v^*(k|\underline{x}).$$

3. Application

For the parameter k, we propose the three following distributions:

$$\pi_1(k \mid \theta) = \begin{cases} \theta^{n-1} & \text{if} \quad k = n, \\ (1-\theta)\theta^{k-1} & \text{if} \quad 1 \leqslant k \leqslant n-1. \end{cases}$$

$$v_1(k) = \begin{cases} 1/n & \text{if } k = n, \\ 1/k(k+1) & \text{if } 1 \leqslant k \leqslant n-1. \end{cases}$$

$$\pi_2(k \mid \theta) = \begin{cases} \theta & \text{if } k = n, \\ \theta(1-\theta)^k/1-(1-\theta)^{n-1} & \text{if } 1 \leqslant k \leqslant n-1. \end{cases}$$

$$v_2(k) = \begin{cases} 1/2 & \text{if} \quad k = n, \\ \int_0^1 \theta(1-\theta)^{k-1}/1-(1-\theta)^{n-1}\mathrm{d}\theta & \text{if } 1 \leqslant k \leqslant n-1. \end{cases}$$

$$\pi_3(k \mid \theta) = \begin{cases} \theta & \text{if } k = n, \\ 1-\theta/n-1 & \text{if } 1 \leqslant k \leqslant n-1. \end{cases}$$

$$v_3(k) = \begin{cases} 1/2 & \text{if } k = n, \\ 1/2(n-1) & \text{if } 1 \leqslant k \leqslant n-1. \end{cases}$$

3.1. The exponential case

In this section, we apply the previous results to the data in [2], on the annual volume of discharge from the Nile River at the Aswan dam in the period 1871 to 1970. Cobb [2] supposes that the data come from a normal distribution and shows by a different technique that there is a change in the mean at $k = 28$ (year 1898). Since for these data, the change occurs for the mean, and because our model for the normal case is adapted for a change in variance, we transform these data to an exponential one.

Let

$$X = -\sigma \log\left(1 - \Phi\left(\frac{Y-m}{\sigma}\right)\right),$$

where $Y \sim \mathcal{N}(m, \sigma)$ and Φ is the cumulative function of a standard normal random variable; it is easy to show that $X \sim \exp(1/\sigma)$.

In our application, we have replaced m and σ by their empirical estimators.

For the v_1 prior distribution, we obtain the data presented in Table 1 (see also Fig. 1).

Table 1
The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Nile data

Year	k	$v_1^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_1^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1895	25	8.1781×10^{-3}	7.3382×10^{-9}	6.0012×10^{-11}
1896	26	1.0605×10^{-1}	4.8693×10^{-10}	5.1637×10^{-11}
1897	27	1.4780×10^{-1}	3.2077×10^{-10}	4.7410×10^{-11}
1898	28	4.6064×10^{-1}	9.3109×10^{-11}	4.2890×10^{-11}
1899	29	1.5262×10^{-1}	2.6852×10^{-10}	4.0982×10^{-11}
1900	30	6.4868×10^{-2}	6.0223×10^{-10}	3.9066×10^{-11}
1901	31	3.2751×10^{-2}	1.1357×10^{-9}	3.7196×10^{-11}
1902	32	1.0198×10^{-2}	3.5265×10^{-9}	3.5961×10^{-11}
1903	33	7.7135×10^{-3}	4.4168×10^{-9}	3.4069×10^{-11}
1904	34	3.4940×10^{-3}	9.3872×10^{-9}	3.2799×10^{-11}
1905	35	1.1907×10^{-3}	2.6811×10^{-8}	3.1924×10^{-11}

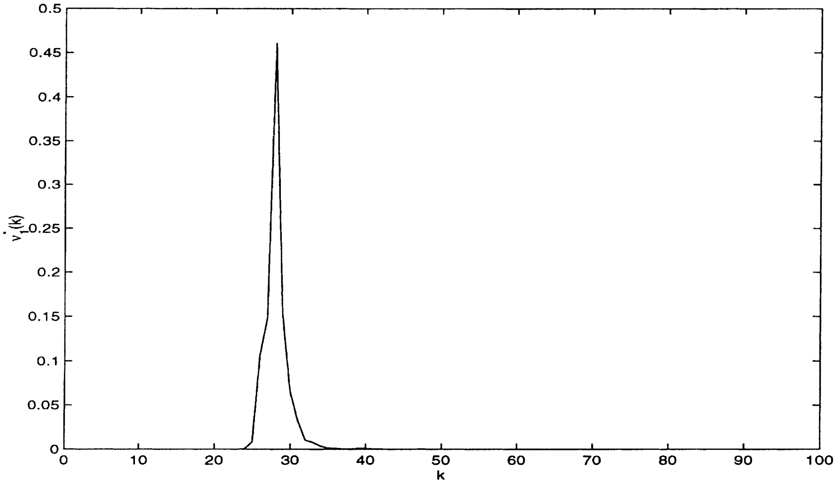

Fig. 1. The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Nile data.

The posterior probability $v_1^*(k \mid \underline{x})$ is maximized at $k = 28$ corresponding to the conditional p-value : $p_{\lambda=1}(28) = 9.3109 \times 10^{-11}$ and to the unconditional p-value $\sum_{k=1}^{100} v_1^*(k \mid \underline{x}) p_{\lambda=1}(k) = 1.2785 \times 10^{-9}$.

The other posterior probability $v_1^*(k \mid \underline{x})$ is below 10^{-4} and then is omitted. The 95% HPD credible set is $\{26, 27, 28, 29, 30, 31\}$. For priors v_2 and v_3, the results are similar and are given in Appendix A (see Tables 4 and 5).

3.2. The Poisson case

In this section, we apply the previous results to the data used by Henderson and Matthews [7] and West and Ogden [13]. The data represent the number of haemolytic uraemic syndrome (HUS) cases in Birmingham and Newcastle from 1970 to 1989.

Henderson and Matthews [7] propose the following Poisson change-point model

$$X_i \mid \theta_1, \theta_2, k \sim \begin{cases} \mathscr{P}(\theta_1) & \text{if } i = 1, \ldots, k, \\ \mathscr{P}(\theta_2) & \text{if } i = k+1, \ldots, n, \end{cases}$$

where X_i denotes the number of HUS cases for the year i and $\mathscr{P}(\theta)$ is the Poisson distribution with mean θ. Henderson and Matthews [7] showed for the at most one change-point case that the change occurs at $k = 11$ for the Birmingham data and $k = 15$ for the Newcastle data. They used the log-likelihood statistic given as the change occurs and the resulting test (χ^2-test) is significant. West and Ogden [13] supposed that data over T unit time periods, $X_1, \ldots, X_T$, are observed where X_i represents the number of events that occurred in the ith time period. Then they used a Poisson process to suppress the hypothesis that the change is an integer. They used the following model:

$$X_i \mid \theta_1, \theta_2, \tau \sim \begin{cases} \mathscr{P}(\theta_1) & \text{if } i = 1, \ldots, [\tau], \\ \mathscr{P}((\tau - [\tau])\theta_1 + (1 - \tau + [\tau])\theta_2) & \text{if } i = [\tau] + 1, \\ \mathscr{P}(\theta_2) & \text{if } i = [\tau] + 2, \ldots, T, \end{cases}$$

where $[x]$ denotes the greatest integer function of x.

They used only the Newcastle data and obtained (maximum-likelihood method), the value $\tau = 14.944$ which closely corresponds with the estimate of 15 given by Henderson and Matthews [7]. The Birmingham data for the v_1 prior distribution are given in Table 2 (see also Fig. 2).

The posterior probability $v_1^*(k \mid \underline{x})$ is maximized at $k = 11$ corresponding to the conditional p-value $p_{\lambda=1}(11) < 10^{-15}$ and to the unconditional p-value $\sum_{k=1}^{20} v_1^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-13}$.

The 98% HPD credible set is $\{11\}$. We obtain the same results as in [7].

Table 2
The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Birmingham data

Year	k	$v_1^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_1^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	1.4876×10^{-11}	4.7390×10^{-4}	7.0498×10^{-15}
1971	2	3.3047×10^{-13}	6.7317×10^{-3}	2.2246×10^{-15}
1972	3	2.9220×10^{-13}	3.1026×10^{-3}	9.0655×10^{-16}
1973	4	9.0299×10^{-13}	4.7893×10^{-4}	4.3246×10^{-16}
1974	5	3.7449×10^{-12}	6.3896×10^{-5}	2.3928×10^{-16}
1975	6	6.2817×10^{-11}	2.2273×10^{-6}	1.3991×10^{-16}
1976	7	6.3035×10^{-9}	1.3591×10^{-8}	8.5672×10^{-17}
1977	8	1.3855×10^{-6}	4.0603×10^{-11}	5.6256×10^{-17}
1978	9	2.4337×10^{-5}	1.6864×10^{-12}	4.1043×10^{-17}
1979	10	3.4866×10^{-3}	8.6597×10^{-15}	3.0193×10^{-17}
1980	11	9.8159×10^{-1}	$< 10^{-15}$	$< 10^{-15}$
1981	12	1.4867×10^{-2}	1.3323×10^{-15}	1.9807×10^{-17}
1982	13	2.6371×10^{-6}	7.5218×10^{-12}	1.9836×10^{-17}
1983	14	1.9022×10^{-5}	8.7619×10^{-13}	1.6667×10^{-17}
1984	15	4.2751×10^{-6}	3.5805×10^{-12}	1.5307×10^{-17}
1985	16	4.1136×10^{-8}	3.7801×10^{-10}	1.5550×10^{-17}
1986	17	1.9867×10^{-12}	1.0500×10^{-5}	2.0859×10^{-17}
1987	18	1.8073×10^{-15}	2.0634×10^{-2}	3.7291×10^{-17}
1988	19	4.8881×10^{-16}	1.0720×10^{-1}	5.2400×10^{-17}
1989	20	1.2913×10^{-13}	9.9505×10^{-3}	1.2849×10^{-15}

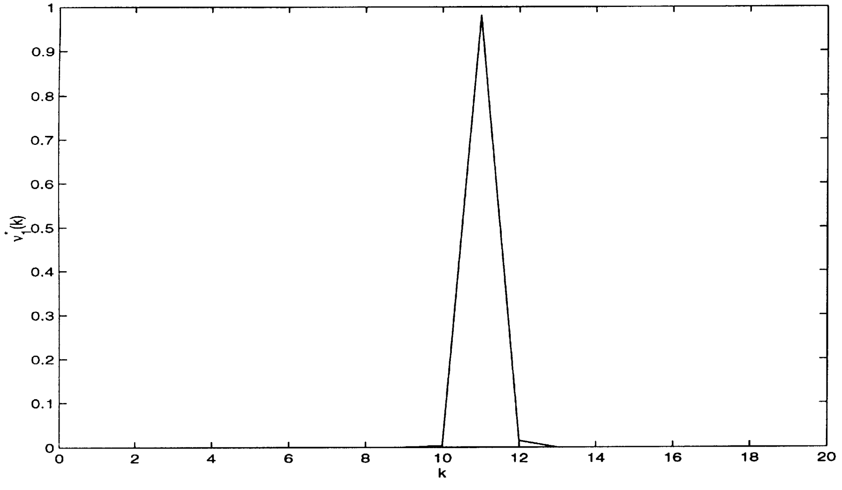

Fig. 2. The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Birmingham data.

For priors v_2 and v_3, the results are similar and are given in Appendix A (see Tables 6 and 7).

The Newcastle data for the v_1 prior distribution are presented in Table 3 (see also Fig. 3).

Table 3
The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Newcastle data

Year	k	$v_1^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_1^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	2.3497×10^{-10}	3.5570×10^{-1}	8.3580×10^{-11}
1971	2	2.8257×10^{-10}	5.4046×10^{-2}	1.5271×10^{-11}
1972	3	2.2793×10^{-9}	2.1065×10^{-3}	4.8015×10^{-12}
1973	4	4.9441×10^{-8}	4.2230×10^{-5}	2.0879×10^{-12}
1974	5	1.3692×10^{-7}	8.7600×10^{-6}	1.1994×10^{-12}
1975	6	7.9344×10^{-6}	8.7104×10^{-8}	6.9112×10^{-13}
1976	7	1.7190×10^{-4}	2.6161×10^{-9}	4.4969×10^{-13}
1977	8	3.3230×10^{-7}	1.1952×10^{-6}	3.9715×10^{-13}
1978	9	2.3912×10^{-7}	1.2942×10^{-6}	3.0947×10^{-13}
1979	10	6.2282×10^{-6}	3.5521×10^{-8}	2.2123×10^{-13}
1980	11	6.8085×10^{-6}	2.6499×10^{-8}	1.8042×10^{-13}
1981	12	1.4901×10^{-3}	8.8467×10^{-11}	1.3183×10^{-13}
1982	13	3.0148×10^{-3}	3.6818×10^{-11}	1.1100×10^{-13}
1983	14	3.8235×10^{-2}	2.4041×10^{-12}	9.1919×10^{-14}
1984	15	9.5703×10^{-1}	8.0380×10^{-14}	7.6926×10^{-14}
1985	16	4.1173×10^{-5}	2.1319×10^{-9}	8.7777×10^{-14}
1986	17	6.3489×10^{-9}	1.8199×10^{-5}	1.1554×10^{-13}
1987	18	1.0839×10^{-9}	1.1593×10^{-4}	1.2566×10^{-13}
1988	19	1.1139×10^{-10}	1.4078×10^{-3}	1.5682×10^{-13}
1989	20	4.0572×10^{-10}	1.8217×10^{-2}	7.3912×10^{-12}

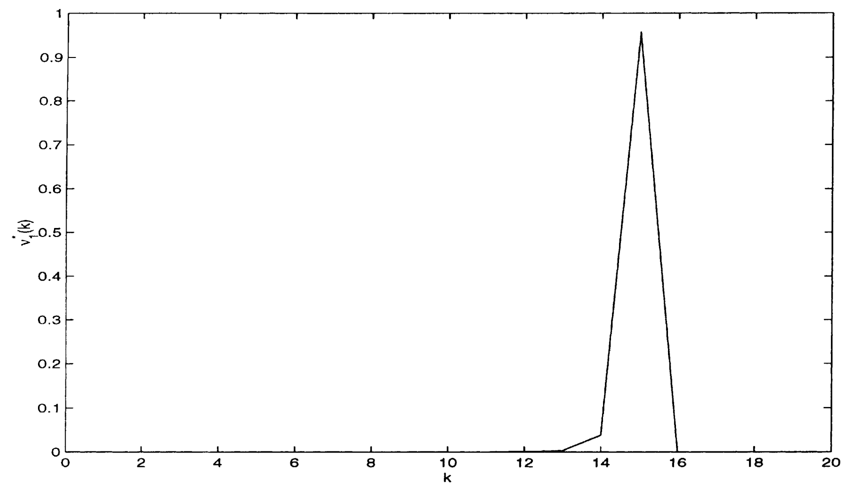

Fig. 3. The posterior probability $v_1^*(k \mid \underline{x})$ of the change-point k in Newcastle data.

The posterior probability $v_1^*(k \mid \underline{x})$ is maximized at $k = 15$ corresponding to the conditional p-value $p_{\lambda=1}(15) < 10^{-13}$ and to the unconditional p-value $\sum_{k=1}^{20} v_1^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-9}$. The 95% HPD credible set is $\{15\}$.

For priors v_2 and v_3, the results are similar and are given in Appendix A (see Tables 8 and 9).

Appendix A

The posterior probability $v_2^*(k \mid \underline{x})$ of the change-point k in Nile data is given in Table 4.

The posterior probability $v_2^*(k \mid \underline{x})$ is maximized at $k = 28$ and corresponding to the conditional p-value $p_{\lambda=1}(28) = 9.3109 \times 10^{-11}$ and to the unconditional p-value $\sum_{k=1}^{100} v_2^*(k \mid \underline{x}) p_{\lambda=1}(k) = 1.4232 \times 10^{-9}$.

The other posterior probability $v_2^*(k \mid \underline{x})$ is below 10^{-4} and is omitted. The 95% HPD credible set is $\{26, 27, 28, 29, 30, 31\}$.

The posterior probability $v_3^*(k \mid \underline{x})$ of the change-point k in Nile data is given in Table 5.

Table 4
The posterior probability $v_2^*(k \mid \underline{x})$ of the change-point k in Nile data

Year	k	$v_2^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_2^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1895	25	7.9807×10^{-3}	7.3382×10^{-9}	5.8563×10^{-11}
1896	26	1.0431×10^{-1}	4.8693×10^{-10}	5.0790×10^{-11}
1897	27	1.4651×10^{-1}	3.2077×10^{-10}	4.6997×10^{-11}
1898	28	4.6013×10^{-1}	9.3109×10^{-11}	4.2843×10^{-11}
1899	29	1.5361×10^{-1}	2.6852×10^{-10}	4.1248×10^{-11}
1900	30	6.5780×10^{-2}	6.0223×10^{-10}	3.9615×10^{-11}
1901	31	3.3458×10^{-2}	1.1357×10^{-9}	3.7999×10^{-11}
1902	32	1.0495×10^{-2}	3.5265×10^{-9}	3.7009×10^{-11}
1903	33	7.9950×10^{-3}	4.4168×10^{-9}	3.5312×10^{-11}
1904	34	3.6484×10^{-3}	9.3872×10^{-9}	3.4249×10^{-11}
1905	35	1.2523×10^{-3}	2.6811×10^{-8}	3.3576×10^{-11}

Table 5
The posterior probability $v_3^*(k \mid \underline{x})$ of the change-point k in Nile data

Year	k	$v_3^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_3^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1895	25	6.4593×10^{-3}	7.3382×10^{-9}	4.7399×10^{-11}
1896	26	9.0458×10^{-2}	4.8693×10^{-10}	4.4047×10^{-11}
1897	27	1.3578×10^{-1}	3.2077×10^{-10}	4.3553×10^{-11}
1898	28	4.5450×10^{-1}	9.3109×10^{-11}	4.2318×10^{-11}
1899	29	1.6134×10^{-1}	2.6852×10^{-10}	4.3324×10^{-11}
1900	30	7.3305×10^{-2}	6.0223×10^{-10}	4.4147×10^{-11}
1901	31	3.9478×10^{-2}	1.1357×10^{-9}	4.4835×10^{-11}
1902	32	1.3085×10^{-2}	3.5265×10^{-9}	4.6144×10^{-11}
1903	33	1.0516×10^{-2}	4.4168×10^{-9}	4.6448×10^{-11}
1904	34	5.0523×10^{-3}	9.3872×10^{-9}	4.7426×10^{-11}
1905	35	1.8230×10^{-3}	2.6811×10^{-8}	4.8877×10^{-11}

The posterior probability $v_3^*(k \mid \underline{x})$ is maximized at $k = 28$ and corresponding to the conditional p-value $p_{\lambda=1}(28) = 9.310910^{-11}$ and to the unconditional p-value $\sum_{k=1}^{100} v_3^*(k \mid \underline{x}) p_{\lambda=1}(k) = 3.2503 \times 10^{-9}$. The other posterior probability $v_3^*(k \mid \underline{x})$ is below 10^{-4} and is omitted. The 95% HPD credible set is $\{26, 27, 28, 29, 30, 31\}$.

The Birmingham data for the v_2 prior distribution are given in Table 6.

The posterior probability $v_2^*(k \mid \underline{x})$ is maximized at $k = 11$ and corresponding to the conditional p-value $p_{\lambda=1}(11) < 10^{-15}$ and to the unconditional p-value $\sum_{k=1}^{20} v_2^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-13}$. The 98% HPD credible set is $\{11\}$.

The Birmingham data for the v_3 prior distribution are given in Table 7.

The posterior probability $v_3^*(k \mid \underline{x})$ is maximized at $k = 11$ and corresponding to the conditional p-value $p_{\lambda=1}(11) < 10^{-15}$ and to the unconditional p-value $\sum_{k=1}^{20} v_3^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-13}$. The 97% HPD credible set is $\{11\}$.

The Newcastle data for the v_2 prior distribution are given in Table 8.

The posterior probability $v_2^*(k \mid \underline{x})$ is maximized at $k = 15$ and corresponding to the conditional p-value $p_{\lambda=1}(15) < 10^{-13}$ and to the unconditional p-value $\sum_{k=1}^{20} v_2^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-9}$. The 95% HPD credible set is $\{15\}$.

The Newcastle data for the v_3 prior distribution are given in Table 9.

Table 6
The posterior probability $v_2^*(k \mid \underline{x})$ of the change-point k in Birmingham data

Year	k	$v_2^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_2^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	4.4244×10^{-12}	4.7390×10^{-4}	2.0967×10^{-15}
1971	2	1.5032×10^{-13}	6.7317×10^{-3}	1.0119×10^{-15}
1972	3	1.6322×10^{-13}	3.1026×10^{-3}	5.0641×10^{-16}
1973	4	5.7517×10^{-13}	4.7893×10^{-4}	2.7546×10^{-16}
1974	5	2.6281×10^{-12}	6.3896×10^{-5}	1.6792×10^{-16}
1975	6	4.7667×10^{-11}	2.2273×10^{-6}	1.0617×10^{-16}
1976	7	5.1129×10^{-9}	1.3591×10^{-8}	6.9491×10^{-17}
1977	8	1.1922×10^{-6}	4.0603×10^{-11}	4.8408×10^{-17}
1978	9	2.2096×10^{-5}	1.6864×10^{-12}	3.7263×10^{-17}
1979	10	3.3266×10^{-3}	8.6597×10^{-15}	2.8807×10^{-17}
1980	11	9.8109×10^{-1}	$< 10^{-15}$	$< 10^{-15}$
1981	12	1.5526×10^{-2}	1.3323×10^{-15}	2.0685×10^{-17}
1982	13	2.8714×10^{-6}	7.5218×10^{-12}	2.1598×10^{-17}
1983	14	2.1554×10^{-5}	8.7619×10^{-13}	1.8885×10^{-17}
1984	15	5.0328×10^{-6}	3.5805×10^{-12}	1.8020×10^{-17}
1985	16	5.0240×10^{-8}	3.7801×10^{-10}	1.8991×10^{-17}
1986	17	2.5138×10^{-12}	1.0500×10^{-5}	2.6395×10^{-17}
1987	18	2.3665×10^{-15}	2.0634×10^{-2}	4.8829×10^{-17}
1988	19	6.6160×10^{-16}	1.0720×10^{-1}	7.0923×10^{-17}
1989	20	1.1263×10^{-12}	9.9505×10^{-3}	1.1208×10^{-14}

Table 7
The posterior probability $v_3^*(k \mid \underline{x})$ of the change-point k in Birmingham data

Year	k	$v_3^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_3^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	2.2492×10^{-13}	4.7390×10^{-4}	1.0659×10^{-16}
1971	2	1.4989×10^{-14}	6.7317×10^{-3}	1.0090×10^{-16}
1972	3	2.6507×10^{-14}	3.1026×10^{-3}	8.2239×10^{-17}
1973	4	1.3653×10^{-13}	4.7893×10^{-4}	6.5386×10^{-17}
1974	5	8.4930×10^{-13}	6.3896×10^{-5}	5.4266×10^{-17}
1975	6	1.9945×10^{-11}	2.2273×10^{-6}	4.4423×10^{-17}
1976	7	2.6685×10^{-9}	1.3591×10^{-8}	3.6268×10^{-17}
1977	8	7.5414×10^{-7}	4.0603×10^{-11}	3.0620×10^{-17}
1978	9	1.6558×10^{-5}	1.6864×10^{-12}	2.7924×10^{-17}
1979	10	2.8993×10^{-3}	8.6597×10^{-15}	2.5107×10^{-17}
1980	11	9.7951×10^{-1}	$< 10^{-15}$	$< 10^{-15}$
1981	12	1.7533×10^{-2}	1.3323×10^{-15}	2.3358×10^{-17}
1982	13	3.6283×10^{-6}	7.5218×10^{-12}	2.7291×10^{-17}
1983	14	3.0198×10^{-5}	8.7619×10^{-13}	2.6459×10^{-17}
1984	15	7.7564×10^{-6}	3.5805×10^{-12}	2.7772×10^{-17}
1985	16	8.4586×10^{-8}	3.7801×10^{-10}	3.1974×10^{-17}
1986	17	4.5957×10^{-12}	1.0500×10^{-5}	4.8253×10^{-17}
1987	18	4.6726×10^{-15}	2.0634×10^{-2}	9.6413×10^{-17}
1988	19	1.4042×10^{-15}	1.0720×10^{-1}	1.5053×10^{-16}
1989	20	3.7094×10^{-13}	9.9505×10^{-3}	3.6910×10^{-15}

Table 8
The posterior probability $v_2^*(k \mid \underline{x})$ of the change-point k in Newcastle data

Year	k	$v_2^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_2^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	2.3497×10^{-10}	3.5570×10^{-1}	8.3580×10^{-11}
1971	2	2.8257×10^{-10}	5.4046×10^{-2}	1.5271×10^{-11}
1972	3	2.2793×10^{-9}	2.1065×10^{-3}	4.8015×10^{-12}
1973	4	4.9441×10^{-8}	4.2230×10^{-5}	2.0879×10^{-12}
1974	5	1.3692×10^{-7}	8.7600×10^{-6}	1.1994×10^{-12}
1975	6	7.9344×10^{-6}	8.7104×10^{-8}	6.9112×10^{-13}
1976	7	1.7190×10^{-4}	2.6161×10^{-9}	4.4969×10^{-13}
1977	8	3.3230×10^{-7}	1.1952×10^{-6}	3.9715×10^{-13}
1978	9	2.3912×10^{-7}	1.2942×10^{-6}	3.0947×10^{-13}
1979	10	6.2282×10^{-6}	3.5521×10^{-8}	2.2123×10^{-13}
1980	11	6.8085×10^{-6}	2.6499×10^{-8}	1.8042×10^{-13}
1981	12	1.4901×10^{-3}	8.8467×10^{-11}	1.3183×10^{-13}
1982	13	3.0148×10^{-3}	3.6818×10^{-11}	1.1100×10^{-13}
1983	14	3.8235×10^{-2}	2.4041×10^{-12}	9.1919×10^{-14}
1984	15	9.5703×10^{-1}	8.0380×10^{-14}	7.6926×10^{-14}
1985	16	4.1173×10^{-5}	2.1319×10^{-9}	8.7777×10^{-14}
1986	17	6.3489×10^{-9}	1.8199×10^{-5}	1.1554×10^{-13}
1987	18	1.0839×10^{-9}	1.1593×10^{-4}	1.2566×10^{-13}
1988	19	1.1139×10^{-10}	1.4078×10^{-3}	1.5682×10^{-13}
1989	20	4.0572×10^{-10}	1.8217×10^{-2}	7.3912×10^{-12}

Table 9
The posterior probability $v_3^*(k \mid \underline{x})$ of the change-point k in Newcastle data

Year	k	$v_3^*(k \mid \underline{x})$	$p_{\lambda=1}(k)$	$v_3^*(k \mid \underline{x})\ p_{\lambda=1}(k)$
1970	1	1.9703×10^{-12}	3.5570×10^{-1}	7.0082×10^{-13}
1971	2	7.1080×10^{-12}	5.4046×10^{-2}	3.8416×10^{-13}
1972	3	1.1467×10^{-10}	2.1065×10^{-3}	2.4156×10^{-13}
1973	4	4.1457×10^{-9}	4.2230×10^{-5}	1.7507×10^{-13}
1974	5	1.7221×10^{-8}	8.7600×10^{-6}	1.5086×10^{-13}
1975	6	1.3971×10^{-6}	8.7104×10^{-8}	1.2170×10^{-13}
1976	7	4.0358×10^{-5}	2.6161×10^{-9}	1.0558×10^{-13}
1977	8	1.0031×10^{-7}	1.1952×10^{-6}	1.1989×10^{-13}
1978	9	9.0227×10^{-8}	1.2942×10^{-6}	1.1677×10^{-13}
1979	10	2.8723×10^{-6}	3.5521×10^{-8}	1.0203×10^{-13}
1980	11	3.7679×10^{-6}	2.6499×10^{-8}	9.9846×10^{-14}
1981	12	9.7458×10^{-4}	8.8467×10^{-11}	8.6218×10^{-14}
1982	13	2.3004×10^{-3}	3.6818×10^{-11}	8.4698×10^{-14}
1983	14	3.3663×10^{-2}	2.4041×10^{-12}	8.0928×10^{-14}
1984	15	9.6297×10^{-1}	8.0380×10^{-14}	7.7403×10^{-14}
1985	16	4.6953×10^{-5}	2.1319×10^{-9}	1.0010×10^{-13}
1986	17	8.1451×10^{-9}	1.8199×10^{-5}	1.4823×10^{-13}
1987	18	1.5541×10^{-9}	1.1593×10^{-4}	1.8017×10^{-13}
1988	19	1.7747×10^{-10}	1.4078×10^{-3}	2.4984×10^{-13}
1989	20	6.4638×10^{-10}	1.8217×10^{-2}	1.1775×10^{-11}

The posterior probability $v_3^*(k \mid \underline{x})$ is maximized at $k = 15$ and corresponding to the conditional p-value $p_{\lambda=1}(15) < 10^{-13}$ and to the unconditional p-value $\sum_{k=1}^{20} v_3^*(k \mid \underline{x}) p_{\lambda=1}(k) < 10^{-10}$. The 96% HPD credible set is $\{15\}$.

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