Polyhedral results for a constrained quadratic 0-1 problem

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Abstract.

In this paper we consider the problem of optimizing a quadratic pseudo-Boolean function subject to the cardinality constraint $\sum_{1\le i\le n} x_i = k$ with a polyhedral method. More precisely we propose a study of the convex hull of feasible points included in the Boolean quadric polytope proposed by M.Padberg [16]. Finally, we give numerical results for some instances of the problem like bipartition and edge-weighted minimal clique in a graph.

Key words: polytope, face, facet, cutting plane algorithm.

1. Introduction and notations.

The unconstrained quadratic 0-1 optimization problem consists of determining the optimum over $B^n = \{0,1\}^n$ of a quadratic pseudo-Boolean function (qpBf)

$$f(x_{i=1,\dots,n}) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} x_i x_j$$

where all q_i and q_{ij} are real numbers. Optimizing an arbitrary qpBf belongs to the class of the NP-hard problems for which no polynomial algorithm is known. This problem can be extended to the constrained case. In the sequel, all quadratic pairs i, j are assumed to be present even if $q_{ij}=0$.

Given $X \subset B^n$, $X \ne \emptyset$, let us consider the following problem:

$$P_f \equiv \underset{x\in X}{\mathrm{Min}}\left(f(x_{i=1,\dots,n}) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} x_i x_j \right)$$

Let $L_X = \left\{ (x,y) \in B^{n+\frac{1}{2}n(n-1)} : x \in X,\ y_{ij} = x_i x_j,\ 1 \le i < j \le n \right\}$ be the linearization of X. The two problems P_f and $P_{L_f} \equiv \underset{(x,y)\in L_X}{\mathrm{Min}}\left(L_f(x,y) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} y_{ij} \right)$ are easily seen to be equivalent, and if C is the convex hull of L_X, P_{L_f} is also equivalent to the continuous linear program $\underset{(x,y)\in C}{\mathrm{Min}}\ L_f(x,y)$ because the set of the extreme points of C is exactly L_X.

To solve this last problem with some linear programming techniques, a full description of C is necessary. This can be given by a minimal set of equalities and inequalities. This method has been used first in the case of $X = B^n$, whose convex hull, called Boolean quadric polytope, was initially studied by M.Padberg [16], followed by E.Boros, P.L.Hammer [6] and N.Boissin [4]. However, the case with X strictly included in B^n has been less explored. E.Johnson et al. [13] have studied the quadratic 0-1 knapsack polytope where X is defined through a knapsack constraint. We can also give the example of K.Park et al. [17], A.Mehrotra [14] who worked on the set of points satisfying the criterion $\sum_{1\le i\le n} x_i \le k$. Here, we are interested in the more constrained problem with the constraint $\sum_{1\le i\le n} x_i = k$.

Some optimization problems in edge-weighted graphs (bipartition or subclique of fixed cardinality) or some facility location problems can be modelized this way. Note for $k = \left\lfloor \frac{n}{2} \right\rfloor$ or $k = \left\lceil \frac{n}{2} \right\rceil$, the problem is generally NP-hard. The minimization of a qpBf under the constraint $\sum_{1\le i\le n} x_i = k$ has been already studied in [3] where the height of a qpBf notion is generalized to the constrained case. A lower bound is proposed that can be computed by a linear programming method involving the original x_i variables plus, as in the linearization technique above, the y_{ij} variables. The aim of this work is to emphasize some properties of the set of points belonging to the Boolean quadric polytope and satisfying the constraint $\sum_{1\le i\le n} x_i = k$ that can be useful for a cutting plane method. Though the quadratic problem with the strict equality constraint studied here seems to have some similarity to the one with the large inequality constraint studied by K.Park et al.[17], A.Mehrotra[14], it has appeared that the respective structures of the induced polytopes are quite different.

We will first define the following notations.

- $\Re$ the real numbers.
- N the set of integers $\{1,..,n\}$.
- BP_n the set of binary points $\{(x,y): x_i \in B,\ y_{ij} = x_i x_j \quad i,j \in N,\ i < j\}$.
- $BP_{n,k}$ the set of constrained binary points $\left\{(x,y) \in BP_n : \sum_{i \in N} x_i = k\right\}$.
- Let P be a set of points in $\Re^d$ then $Aff(P)$ is the affine hull and $Conv(P)$ is the convex hull of P in $\Re^d$.
- $QP_n = Conv(BP_n)$ the Boolean quadric polytope with n linear variables x_i, see [16].
- $QP_{n,k}$ the polytope defined by $QP_{n,k} = Conv(BP_{n,k})$.

In the whole paper, we consider the polytope $QP_{n,k}$ for $2 \le k \le n-2$. Also the pair (x,y) can be represented by a single letter like "z".

This paper is organized as follows. In section 2, we recall some specific methods and results that will be applied on $QP_{n,k}$ through the polyhedral approach. In section 3, we present some properties of $QP_{n,k}$ and some operations on its facets. In section 4, we give some usual families of facets of $QP_{n,k}$. In section 5, we investigate the relationship of $QP_{n,k}$ with some polytopes of the literature [9, 14, 17]. Finally, in section 6, we report some computational tests.

2. General results of polyhedral theory.

Here, we give the basic results that will be used in our treatment. See [7, 14] for more details.
All the following notions take place in $\Re^d$. First we begin by some basic concepts.

a) Affine, convex hulls [7, 14].
A hyperplane is a set of points z s.t. f(z)=0 where f is a non-constant, affine function. An affine space is an intersection of a finite number of hyperplanes (and conversely). The affine hull of a set S of $\Re^d$, denoted by Aff(S), is the intersection of all the affine spaces containing S and hence of all the hyperplanes containing S. The dimension of a convex set is defined through the dimension of its affine hull. A convex set of dimension e has e+1 affinely independent points.
The convex hull of a set S of $\Re^d$, denoted by Conv(S), is the intersection of all the convex sets containing S and for any S we have Aff(Conv(S))=Aff(S). Conv(S) is equal to the set of all convex combination of points from S; for any affine mapping φ, we have $\varphi(\mathrm{Conv}(S)) = \mathrm{Conv}(\varphi(S))$.

A polytope is the convex hull of a finite number of points. A simplex is a polytope which can be expressed as the convex hull of affinely independent points. We now give some results on polytopes.

b) Facial structure of polytopes [7].
First recall the definition of a face of a closed convex set. A face of a closed convex set C of $\Re^d$ is a convex subset F of C s.t. for any two points $y,z \in C$ s.t. the open segment between y and z intersects F, we actually have $y,z \in F$. The extreme points of C are the faces of C reduced to one point. It follows immediately from the above definition that if F is a face of a closed convex set C and if G is a subset of F, then G is a face of C iff G is a face of F.
A consequence on the facial structure of polytopes is the following. Let F be a face of a polytope P, then $\mathrm{ext}(F) = F \cap \mathrm{ext}(P)$ and $F = \mathrm{Conv}(\mathrm{ext}(F))$, where ext(.) designates the extreme points of the specified set. Remark that a face of a polytope is again a polytope since ext(F) is finite. An extreme point of a polytope is usually called a vertex.
A facet of a polytope P is a face whose dimension is the dimension of P less one. By a proper face of a polytope P we mean a face which is not equal to P. A facet of a polytope P is a maximal (in the inclusion sense) proper face of P. Let F be a proper face of a polytope P, then F is the intersection of the facets of P containing F. A simplex of dimension e has e+1 facets.

A closed halfspace is a set of points z s.t. f(z)≤0 where f is a non-constant, affine function. A polyhedron is the intersection of a finite number of closed halfspaces. A polytope is a bounded polyhedron. So let us give some results on polyhedra.

c) Representation of polyhedral sets [14].
The inequality f(z)≤0, where f is an affine function, is said to be valid for a polyhedron P if $P \subseteq \{z: f(z) \le 0\}$ and then $P \cap \{z: f(z) = 0\}$ is a face of P called the face induced on P by the inequality f(z)≤0.
A minimal representation of a polyhedron is given by a set of valid inequalities inducing its facets and a set of independent equations describing its affine hull.
The following result is a useful tool for checking facets.
Let P be a polyhedron s.t. $\mathrm{Aff}(P) = \{z: A(z) = 0\}$ where A is a vector of affine functions and f(z)≤0 be a valid inequality inducing a proper face F of P, then the two statements are equivalent:

i) F is a facet of P

ii) If g(z)≤0 is a valid inequality which induces a face of P containing F then there exists a scalar α and a vector λ s.t. $g = \alpha f + \lambda^t A$

The previous results have the following consequences on polytopes which are faces of polytope.

Consequence.2.1. Let F be a face of a polytope P and let G be a facet of F. Then there exists a facet H of P s.t. $G = H \cap F$.

Consequence.2.2. Let F be a face of a polytope P s.t. $\mathrm{Aff}(F) = \{z: A(z) = 0\}$ and let g(z)≤0 be a valid inequality for F inducing a facet of F. Then there exists an inequality h(z)≤0 valid for P and inducing a facet of P, a scalar α>0 and a vector λ s.t. $h = \alpha g + \lambda^t A$.

3. Some characteristics and properties of $QP_{n,k}$

3.1. Affine hull of $QP_{n,k}$.

In this part we define $B_{n,k} = \left\{ x \in B^n : \sum_{i \in N} x_i = k \right\}$. Before arriving at the description of the affine hull of $QP_{n,k}$, we recall a characterization of the qpBf's vanishing on $B_{n,k}$.

Theorem.3.1 [3]. For $2 \le k \le n-2$, the qpBf $c(x) = c_0 + \sum_{i \in N} c_i x_i + \sum_{i,j \in N, i<j} c_{ij} x_i x_j$ is equal to zero on $B_{n,k}$ if and only if

$$\exists \lambda_0, \lambda_{i \in N} \text{ such that } c(x) = \lambda_0 \left(\sum_{i \in N} x_i - k \right) + \sum_{i \in N} \lambda_i \left(\sum_{j \in N, j<i} x_j x_i + \sum_{j \in N, i<j} x_i x_j - (k-1) x_i \right)$$

Corollary.3.1. For $2 \le k \le n-2$, the function $f(x,y) = q_0 + \sum_{i \in N} q_i x_i + \sum_{i,j \in N, i<j} q_{ij} y_{ij}$ is equal to zero on $BP_{n,k}$ if and only if

$$\exists \lambda_0, \lambda_{i \in N} \text{ such that } f(x,y) = \lambda_0 \left(\sum_{i \in N} x_i - k \right) + \sum_{i \in N} \lambda_i \left(\sum_{j \in N, j<i} y_{ji} + \sum_{j \in N, i<j} y_{ij} - (k-1) x_i \right)$$

Proof. It follows from the Theorem.3.1 and the definition of $BP_{n,k}$.

We can now easily get a representation of the affine hull of $QP_{n,k}$.

Proposition.3.1. Let $A_n^k(x,y)$ be the vector of functions defined as follows

$$A_n^k(x,y) \equiv \begin{cases} \sum_{i \in N} x_i - k \\ \sum_{j \in N, j<i} y_{ji} + \sum_{j \in N, i<j} y_{ij} - (k-1) x_i & i \in N \end{cases}$$

then, for $2 \le k \le n-2$, $Aff(QP_{n,k}) = \left\{ (x,y) \in \Re^{n(n+1)/2} : A_n^k(x,y) = 0 \right\}$, and the dimension of $QP_{n,k}$ is $\frac{1}{2} n(n-1) - 1$.

Proof. To prove the first part let us recall that $Aff(BP_{n,k}) = \left\{ (x,y) \in \Re^{n(n+1)/2} : E(x,y) = 0 \right\}$, where $E(x,y) = 0$ is the set of all the affine equalities verified by the points of $BP_{n,k}$. From Corollary.3.1, $A_n^k(x,y) = 0$ on $BP_{n,k}$ and any f such that $f(x,y) = 0$ on $BP_{n,k}$ is a linear combination of functions of A_n^k, which are finally sufficient to describe $Aff(BP_{n,k})$ and then $Aff(QP_{n,k})$.

If in the system of equations $A_n^k(x,y) = 0$ we substitute the equation $\sum_{i \in N} x_i - k = 0$ by the linear combination $\left(\sum_{i \in N} x_i - k \right) + \frac{1}{k-1} \sum_{i \in N} \left(\sum_{j \in N, j<i} y_{ji} + \sum_{j \in N, i<j} y_{ij} - (k-1) x_i \right) = \frac{2}{k-1} \sum_{i \in N} \sum_{j \in N, i<j} y_{ij} - k = 0$, we obtain an equivalent system from which we can extract the following lower triangular matrix of order n+1 whose columns correspond to variables $y_{12}, x_1, x_2, \ldots, x_n$

$$\begin{pmatrix} \frac{2}{k-1} & 0 & \cdots & & & 0 \\ 1 & -(k-1) & & & & \vdots \\ 1 & 0 & -(k-1) & & & \\ 0 & & & -(k-1) & & \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & & & 0 & -(k-1) \end{pmatrix}$$

which is full rank for k≥2. Then the dimension of $Aff(QP_{n,k})$ is $\frac{1}{2} n(n+1) - (n+1) = \frac{1}{2} n(n-1) - 1$ and the dimension of $QP_{n,k}$ follows.

Consequence.3.1. If $f(x,y) \le 0$ induces a facet of $QP_{n,k}$ and $g(x,y) \le 0$ is a valid inequality for $QP_{n,k}$ whose face induced contains the facet induced by $f(x,y) \le 0$, there exists a scalar α and a vector $\lambda \in \Re^{n+1}$ such that $g = \alpha f + \lambda^t A_n^k$.

Remark.3.1. For $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_{i \in N} \end{pmatrix} \in \Re^{n+1}$, $\lambda^t A_n^k = -\lambda_0 k + \sum_{i \in N} \left(\lambda_0 - (k-1) \lambda_i \right) x_i + \sum_{i,j \in N, i<j} \left(\lambda_i + \lambda_j \right) y_{ij}$.

In the rest of this paper we note A_n^k the vector of functions defined above and consider polytopes $QP_{n,k}$ for $2 \le k \le n-2$.

3.2. Operations on facets of $QP_{n,k}$

We describe in the following lemmas some useful operations to build a new inequality inducing facet from a given inequality inducing facet.

Indices permutation operation.

Lemma.3.1. Let us consider this notation $y_{\sigma(i)\sigma(j)} = y_{\mathrm{Min}(\sigma(i),\sigma(j))\mathrm{Max}(\sigma(i),\sigma(j))}$, where σ is a permutation function on N. Thus if the valid inequality $a_0 + \sum_{i \in N} a_i x_i + \sum_{i,j \in N, i<j} a_{ij} y_{ij} \le 0$ induces a facet of $QP_{n,k}$, then $a_0 + \sum_{i \in N} a_i x_{\sigma(i)} + \sum_{i,j \in N, i<j} a_{ij} y_{\sigma(i)\sigma(j)} \le 0$ stays valid and induces a facet on $QP_{n,k}$.

Complement operation.

Lemma.3.2. Let $f = a_0 + \sum_{i \in N} a_i x_i + \sum_{i,j \in N, i<j} a_{ij} y_{ij}$ be the function corresponding to a valid inequality that induces a facet of $QP_{n,k}$. Then the function $g = q_0 + \sum_{i \in N} q_i x_i + \sum_{i,j \in N, i<j} q_{ij} y_{ij}$ defined by the transformation

$$\Psi : \begin{cases} q_0 = a_0 + \sum_{i \in N} a_i + \sum_{i,j \in N, i<j} a_{ij} \\ q_i = -a_i - \left(\sum_{j \in N, j<i} a_{ji} + \sum_{j \in N, i<j} a_{ij} \right) & i \in N \\ q_{ij} = a_{ij} \quad i,j \in N, i<j \end{cases}$$

corresponds to a valid inequality inducing a facet of $QP_{n,n-k}$.

Proof. Let $\varphi : (x,y) \in \Re^{n(n+1)/2} \to (x',y') \in \Re^{n(n+1)/2}$ be the affine mapping defined by

$$\begin{cases} x'_i = 1 - x_i \quad i = 1,\dots,n \\ y'_{ij} = 1 - x_i - x_j + y_{ij} \quad 1 \le i < j \le n \end{cases}$$

φ is clearly bijective and $\varphi^{-1} = \varphi$. Each point of $BP_{n,k}$ has through φ an unique image in $BP_{n,n-k}$ and, as φ is affine, the image of $QP_{n,k}$ through φ is $QP_{n,n-k}$ and vice versa. Given f an affine function, suppose g an affine function found such that

$$[C.0]: g(\varphi(z)) = f(z) \quad z \in QP_{n,k}$$

Then

- $f(z) \le 0$ is valid on $QP_{n,k} \Leftrightarrow g(z) \le 0$ is valid on $QP_{n,n-k}$.
- The affine independency is conserved through the invertible affine mapping φ, so $f(z) \le 0$ induces a facet of $QP_{n,k} \Leftrightarrow g(z) \le 0$ induces a facet of $QP_{n,n-k}$.

As $\varphi^{-1} = \varphi$ [C.0] is equivalent to $g(z') = f(\varphi(z')) \quad z' \in QP_{n,n-k}$ and then it suffices to choose $g = f \circ \varphi = \Psi(f)$ for making $[C.0]$ satisfied.

Remark.3.2. As $\Psi \circ \Psi$ is the identity, the converse of Lemma.3.2 holds that is to say if $\Psi(f)(x,y) \le 0$ induces a facet of $QP_{n,n-k}$ then $\Psi(\Psi(f))(x,y) = f(x,y) \le 0$ induces a facet of $QP_{n,k}$.

3.3. Property relative to QP_n

Let $clq^{N,k}(x,y) = k \sum_{i \in N} x_i - \sum_{i,j \in N, i<j} y_{ij} - \frac{k(k+1)}{2} \le 0$ be the clique inequality that is valid for QP_n and inducing a facet of QP_n for all $1 \le k \le n-2$, see [16]. A more complete description of the clique inequality will be given in the next section, just observe now that for any binary point (x,y) of BP_n, $clq^{N,k}(x,y) = 0$ iff $\sum_{i \in N} x_i = k$ or $k+1$.

Proposition.3.2. $QP_{n,k}$ is a face of QP_n.

Proof. Let F be the face of QP_n induced by the valid inequality $f(x,y) = clq^{N,k-1}(x,y) + clq^{N,k}(x,y) \leq 0$. The vertices of F are the vertices of QP_n (i.e. BP_n) for which f(x,y) is null, that is to say $clq^{N,k-1}(x,y)$ and $clq^{N,k}(x,y)$ vanish. Then the vertices of F are the vertices of QP_n (i.e. BP_n) which satisfy $\sum_{i \in N} x_i = k$ that is to say $BP_{n,k}$. As F is the convex hull of its vertices, we have $F = Conv(BP_{n,k}) = QP_{n,k}$.

Consequence.3.2. Because of Consequence.2.2 and Proposition.3.1, if a valid inequality $f(z) \leq 0$ for $QP_{n,k}$ induces a facet of $QP_{n,k}$ then there exists a valid inequality $g(z) \leq 0$ for QP_n inducing a facet of QP_n such that $\exists \alpha > 0 \in \Re, \lambda \in \Re^{n+1}$ and $g(z) = \alpha f(z) + \lambda^t A_n^k(z)$.

4. Some facets families of $QP_{n,k}$.

In order to get the facial description of $QP_{n,k}$, Consequence.3.2 allows us to search among the inequalities inducing facets of QP_n. Therefore, inequalities considered in this part are valid on $QP_{n,k}$ because they are on QP_n .

In this section, we need more notations.

• Let z be a point of $QP_{n,k}$

- x_i^z is the component x_i of z.
- $x(z) = \{i \in N: x_i^z = 1\}$.

• Let S and T, be disjoint (i.e $S \cap T = \varnothing$) subsets of N, then $[S:T] = \{(\mathrm{Min}(i,j), \mathrm{Max}(i,j)): i \in S, j \in T\}$

• Let e=(i,j) be an element of [S:T], then T(e) designates the element of the pair {i,j} which belongs to T.

• If no precision is given, we consider $f = a_0 + \sum_{1\le i\le n} a_i x_i + \sum_{1\le i<j\le n} a_{ij} y_{ij}$ as the general expression of an affine function.

• The coefficient a_{ij} , i<j, will also be denoted by a_{ji} (i.e $a_{ij}=a_{ji}$). For any element i of N and any subset U of N-{i} we define $a_{iU} = \sum_{j\in U} a_{ij}$.

We recall that the linearization of a qpBf (see section 1) consists in substituting each product $x_i x_j$ (i<j) by the corresponding variable y_{ij} and each square x_i^2 by x_i (keep in mind that $x_i \in \{0,1\} \Leftrightarrow x_i^2 = x_i$). For each valid inequality considered, we will briefly recall the way it is constructed and the reason of its validity.

4.1. The clique inequality family

Let S be a non-empty subset of N and α be an integer. Then consider the following quadratic inequality $-\frac{1}{2}\left(\sum_{i\in S} x_i - \alpha\right)\left(\sum_{i\in S} x_i - \alpha - 1\right) \le 0$. Such an inequality is valid for any 0-1 vector x (a member of the product is never strictly positive when the other is strictly negative) and then provides through the linearization of the qpBf involved, the following inequality, called clique inequality,

$clq^{S,\alpha}(x,y) = \alpha \sum_{i\in S} x_i - \sum_{i,j\in S, i<j} y_{ij} - \frac{\alpha(\alpha+1)}{2} \le 0$ which is valid for BP_n and hence for QP_n.

The vertices of QP_n (i.e. BP_n) which saturate this inequality can be easily deduced from the above quadratic formulation and are characterized by letting $z \in BP_n$; then

$clq^{S,\alpha}(z) = 0$ iff $\alpha \le |x(z) \cap S| \le \alpha + 1$.

It has been proved by M. Padberg [16] that the clique inequality induces a facet of QP_n for $|S| = 2$ if $\alpha \in \{0,1\}$ and for $|S| \ge 3$ if $1 \le \alpha \le |S| - 2$.

Now we focus on the investigation of conditions for which this inequality induces a facet of $QP_{n,k}$ too. As mentioned before, we consider 2≤k≤n-2. Some additional restrictions are imposed on k by the magnitude of the integer α and the size of the set N-S.

Proposition.4.1.1.

Let S be a subset of N such that $2 \le |S| \le n-3$ and α be an integer satisfying

$$\begin{cases} |S| = 2 \Rightarrow \alpha \in \{0,1\} & [C4.1.1] \\ |S| \ge 3 \Rightarrow 1 \le \alpha \le |S| - 2 & [C4.1.2] \end{cases}$$

Then, for 2≤k≤n-2, the clique inequality $clq^{S,\alpha}(x,y) = \alpha \sum_{i\in S} x_i - \sum_{i,j\in S, i<j} y_{ij} - \frac{\alpha(\alpha+1)}{2} \le 0$ induces a facet of $QP_{n,k}$ if and only if $\alpha + 2 \le k \le |N - S| + \alpha - 1$ $[C4.1.3]$.

Proof: sufficiency. Let f(x,y)≤0 be a valid inequality for $QP_{n,k}$ such that the face induced on $QP_{n,k}$ contains the face induced by the clique inequality.

[C4.1.1] and [C4.1.2] imply the existence of a subset U of S such that $\alpha = |U| \le |S| - 1$. Let $j \in S - U$.

[C4.1.3] implies the existence of a subset R of N-S such that $|R| = k - \alpha - 1$ and $1 \le |R| \le |N - S| - 2$.

Then let $h' \in R$ and $h, h'' \in N - S - R$ $(h \ne h'')$.

Now consider $z_1, z_2 \in BP_{n,k}$ such that $\begin{cases} x(z_1) = U + R + h \\ x(z_2) = U + R + j \end{cases}$.

As $clq^{S,\alpha}(z_1) = clq^{S,\alpha}(z_2) = 0$ then $f(z_1) - f(z_2) = 0$ which gives

$$a_h + a_{hU} + a_{hR} = a_j + a_{jU} + a_{jR} \ (1).$$

Now let $R' = R - h' + h''$. Substituting R by R' in (1) leads to $a_h + a_{hU} + a_{hR'} = a_j + a_{jU} + a_{jR'}$ (1') and (1) - (1') gives

$$a_{hh'} - a_{hh''} = a_{jh'} - a_{jh''} \ (2)$$

which is valid $\forall j \in S, \forall h, h', h'' \in N - S$, since the sets U, R are arbitrarily chosen.

If $|S| \geq 3$ then [C4.1.2] implies that $1 \leq |U| \leq |S| - 2$. Let $i \in U$, $i' \in S - U - j$ and $U' = U - i + i'$.

Substituting U by U' in (1) leads to $a_h + a_{hU'} + a_{hR} = a_j + a_{jU'} + a_{jR}$ (1") and (1)-(1") gives

$$a_{hi} - a_{hi'} = a_{ji} - a_{ji'} \ (3)$$

which is valid $\forall j, i, i' \in S$, $\forall h \in N - S$, since the sets U, R are arbitrarily chosen.

Without lost of generality (see Lemma.3.1) let s, t be two elements of S (recall $|S| \geq 2$) and 1, 2, 3 be three elements of N-S (recall $|N - S| \geq 3$).

Let λ_1, λ_2, λ_3 be solution of the non-singular system $\begin{cases} a_{12} = \lambda_1 + \lambda_2 \\ a_{13} = \lambda_1 + \lambda_3 \\ a_{23} = \lambda_2 + \lambda_3 \end{cases}$

and then let λ_h $h \in (N - S) - \{1,2,3\}$ defined through $a_{1h} = \lambda_1 + \lambda_h$
and λ_i $i \in S$ defined through $a_{1i} = \lambda_1 + \lambda_i$
and finally μ defined through $a_{st} = \lambda_s + \lambda_t - \mu$

Then

(2) with $h'' = 1, h' = 2, h = 3$ gives $a_{j2} = a_{j1} + a_{23} - a_{13} = \lambda_j + \lambda_2 \quad \forall j \in S$

(2) with $h'' = 2$, $h = 1$ gives $a_{jh'} = a_{j2} + a_{1h'} - a_{12} = \lambda_j + \lambda_{h'} \quad \forall j \in S, \forall h' \in (N - S) - \{1,2\}$

(2) with $h'' = 1$ gives $a_{hh'} = a_{jh'} - a_{j1} + a_{1h} = \lambda_h + \lambda_{h'} \quad \forall h, h' \in (N - S) - \{1\}$

If $|S| \geq 3$ then

(3) with $j = s, i' = t$ gives $a_{si} = a_{hi} - a_{ht} + a_{st} = \lambda_i + \lambda_s - \mu \quad \forall i \in S - \{s,t\}$

(3) with $i' = s$ gives $a_{ij} = a_{sj} + a_{hi} - a_{hs} = \lambda_j + \lambda_i - \mu \quad \forall i, j \in S - \{s\}$

Now, since $|U| + |R| = k - 1$ and $|U| = \alpha$, (1) becomes $a_h + (k-1)\lambda_h = a_j + (k-1)\lambda_j - \alpha\mu$. As the sets R, U are arbitrarily chosen, this relation is available $\forall h \in N - S$ and $\forall j \in S$.

We therefore have a constant λ_0 such that $\begin{cases} a_h = \lambda_0 - (k-1)\lambda_h & \forall h \in N - S \\ a_j = \lambda_0 - (k-1)\lambda_j + \alpha\mu & \forall j \in S \end{cases}$

The last coefficient a_0 can be found by solving the equation f(z)=0, for any z of $BP_{n,k}$ such that $clq^{S,\alpha}(z) = 0$, whose one and only solution necessarily is $a_0 = -k\lambda_0 - \frac{1}{2}\alpha(\alpha+1)\mu$.

Now let us verify that the face induced by the clique inequality is strictly contained in $QP_{n,k}$. As [C4.1.3] implies $k - \alpha + 1 \leq |N - S|$ and $k-2 \geq 0$, there exists subsets U of S and R of N-S such that

if $\alpha \geq 1$ $|U| = \alpha - 1$ and $|R| = k - \alpha + 1$
if $\alpha = 0$ $|U| = 2$ and $|R| = k - 2$

Let z of $BP_{n,k}$ such that x(z)=U+R then $clq^{S,\alpha}(z) \neq 0$.

Finally we have exhibited $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_{i \in N} \end{pmatrix}$ and μ such that $f = \mu clq^{S,\alpha} + \lambda^t A_n^k$. As $clq^{S,\alpha}(z) \neq 0$ for some z of $QP_{n,k}$, the clique inequality $clq^{S,\alpha}(x,y) \leq 0$ induces a facet of $QP_{n,k}$.

Proof: necessity. For the necessary part, observe the following inclusions.

If $2 \leq k \leq \alpha + 1$, the condition $|S| \leq n - 3$ (i.e. $|N - S| \geq 3$) implies there exists $h, r \in N - S$ and $N - S \neq \{h, r\}$. Let z be a vertex of $QP_{n,k}$ (i.e. point of $BP_{n,k}$) which saturates the clique inequality $clq^{S,\alpha}(x,y) \leq 0$. Then $|x(z) \cap S| \geq \alpha$ and thus $|x(z) \cap (N - S)| \leq k - \alpha \leq 1$, that is to say z saturates $clq^{\{h,r\},0}(x,y) \leq 0$ as well. As $N - S \neq \{h, r\}$ there exist a vertex z such that $clq^{\{h,r\},0}(z) = 0$ and $clq^{S,\alpha}(z) \neq 0$ (take z s.t. $0 \leq |x(z) \cap S| \leq \alpha - 1$ and $x_r^z = 0$). Hence the face induced by the clique inequality is strictly included in the face induced by $clq^{\{h,r\},0}$ which is itself strictly contained in $QP_{n,k}$ for k≥2.

For the case $n - 2 \geq k \geq |N - S| + \alpha$, applying the complement operation of Lemma.3.2, if

$\mathrm{clq}^{S,\alpha}(x,y) \le 0$ induces a facet of $QP_{n,k}$, then $\Psi(\mathrm{clq}^{S,\alpha})(x,y) = \mathrm{clq}^{S,|S|-\alpha-1}(x,y) \le 0$ induces a facet of $QP_{n,n-k}$. Letting $k' = n-k, \alpha' = |S|-\alpha-1$, we obtain $\mathrm{clq}^{S,\alpha'}(x,y) \le 0$ inducing a facet of $QP_{n,k'}$ with $2 \le k' \le \alpha'+1$, that contradicts the result above.

Remark.4.1.1. Here we give additional information on the necessity of the conditions involved in the previous proposition. In this paragraph $F(\mathrm{clq}^{S,\alpha})$ denotes the face induced by the clique inequality $\mathrm{clq}^{S,\alpha}(z) \le 0$ on $QP_{n,k}$. Recall it is sufficient to verify the validity of the following inclusions on the vertices of $QP_{n,k}$.

1. We have $F(\mathrm{clq}^{S,\alpha}) = F(\mathrm{clq}^{T,\beta})$ with $T = N-S, \beta = k-\alpha-1$. We shall say that $\mathrm{clq}^{S,\alpha}$ is symmetric of $\mathrm{clq}^{T,\beta}$.
2. For $|N-S| = 2$, we must have $\alpha \in \{k-3, k-2, k-1, k\}$ in order to get $F(\mathrm{clq}^{S,\alpha}) \ne \emptyset$. Let $N-S = \{h,l\}$ then:
    a. For $\alpha = k$, $F(\mathrm{clq}^{S,\alpha})$ is strictly included in the face induced by $-y_{hl} \le 0$.
    b. For $\alpha = k-1$, $\mathrm{clq}^{S,\alpha}$ is the clique inequality symmetric of $\mathrm{clq}^{T,\beta}$ with $\beta = 0$ and $T = \{h,l\}$; that is $-y_{hl} \le 0$.
    c. For $\alpha = k-2$, likewise $\mathrm{clq}^{S,\alpha}$ is the clique inequality symmetric of $\mathrm{clq}^{\{h,l\},1}(x,y) \le 0$.
    d. For $\alpha = k-3$, $F(\mathrm{clq}^{S,\alpha})$ is strictly included in the face induced by $\mathrm{clq}^{\{h,l\},1}(x,y) \le 0$.
3. For $|N-S| = 1$, we must have $\alpha \in \{k-2, k-1, k\}$ in order to get $F(\mathrm{clq}^{S,\alpha}) \ne \emptyset$. Let $N-S = \{t\}, s \in S$, then:
    a. For $\alpha = k-2$, $F(\mathrm{clq}^{S,\alpha})$ is equal to the face induced by $x_t - 1 \le 0$ which is strictly included in the one induced by $\mathrm{clq}^{\{s,t\},1}(x,y) \le 0$.
    b. For $\alpha = k-1$, $F(\mathrm{clq}^{S,\alpha})$ is $QP_{n,k}$ itself.
    c. For $\alpha = k$, $F(\mathrm{clq}^{S,\alpha})$ is equal to the face induced by $-x_t \le 0$,which is strictly included in the one induced by $-y_{st} \le 0$.
4. For $|S| = |N|$ we must have $\alpha = k$ or $\alpha = k-1$ in order to get $F(\mathrm{clq}^{S,\alpha}) \ne \emptyset$, and $F(\mathrm{clq}^{S,\alpha})$ is $QP_{n,k}$ itself.
5. The cases $|S| \in \{0,1,2\}$ are the symmetric ones, in the sense given in 1, of the previous cases $|S| \in \{|N|, |N|-1, |N|-2\}$.
6. More generaly, if $\alpha = |S|-1 \ge 1$ then $F(\mathrm{clq}^{S,\alpha})$ is included in the face induced by $\mathrm{clq}^{\{i,j\},1}(x,y) \le 0,\ i,j \in S,\ i<j$.
7. At last if $\alpha = |S| \ge 1, |S| \le |N|-1$, then $F(\mathrm{clq}^{S,\alpha})$ is strictly included in $F(\mathrm{clq}^{U,\alpha})$ where $S \subset U, |U| = |S|+1$.

Remark.4.1.2. We have seen (points 3-5 of the previous remark) that the face induced by the clique inequality can be equal to $QP_{n,k}$ when $|N-S| \le 1$ and $\alpha \in \{k-1,k\}$ or symmetrically when $|S| \le 1$ and $\alpha \in \{-1,0\}$. It is easy to verify that these are the only cases. Indeed, let us suppose that $|S| \ge 2$ and $\mathrm{clq}^{S,\alpha} = \lambda^t A_n^k$ for some $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_{i\in N} \end{pmatrix}$. Then we have the following system

$$\begin{cases} \frac{\alpha(\alpha+1)}{2} = \lambda_0 k \\ \left.\begin{array}{l} \alpha = \lambda_0 - (k-1)\lambda_i \\ \alpha = \lambda_0 - (k-1)\lambda_j \\ -1 = \lambda_i + \lambda_j \end{array}\right] i,j \in S \end{cases}$$

which implies the equation $\alpha^2 + \alpha(1-2k) + k(k-1) = 0$ whose solutions are $\alpha = k$ and $\alpha = k-1$. On the other hand, $\mathrm{clq}^{S,\alpha}$ is symmetric (in the sense given in point 1 of the previous remark) of $\mathrm{clq}^{N-S,\beta}$ with $\beta = k-\alpha-1$. $\alpha \in \{k-1,k\}$ implies $\beta \in \{-1,0\}$ and then $\beta \notin \{k-1,k\}$ (k≥2). As $\mathrm{clq}^{S,\alpha}(z) \le 0$ and $\mathrm{clq}^{N-S,\beta}(z) \le 0$ induce the same face (that is to say $QP_{n,k}$) necessarily $|N-S| \le 1$.

For $k = 2$ and $k = n - 2$, the elementary clique inequalities (those where $|S| = 2$) give a minimal description of $QP_{n,k}$, as we are going to see.

Proposition.4.1.2. The following equalities hold true:

$$QP_{n,2} = \left\{(x,y): A_n^2(x,y) = 0,\ clq^{\{i,j\},0}(x,y) \le 0,\ \forall(i,j) \in N \times N,\ i < j\right\}$$

$$QP_{n,n-2} = \left\{(x,y): A_n^{n-2}(x,y) = 0,\ clq^{\{i,j\},1}(x,y) \le 0,\ \forall(i,j) \in N \times N,\ i < j\right\}$$

Moreover, these polytopes are simplices and their facets correspond, in a one-to-one manner, to the inequalities given in their above respective descriptions.

Proof. First, let us focus on $QP_{n,2}$. Let f be an affine function which is negative or null on $QP_{n,2}$. In particular, f is negative or null on $BP_{n,2}$; that is $\forall i, j, 1 \le i < j \le n,\ a_0 + a_i + a_j + a_{ij} \le 0$.
As a consequence $\forall i, j, 1 \le i < j \le n, \exists \alpha_{ij} \ge 0$ s.t. $a_{ij} = -\alpha_{ij} - a_0 - a_i - a_j$.

Letting $\lambda_0 = -\frac{a_0}{2}$, $\lambda_i = -\frac{a_0}{2} - a_i$, $i = 1,\dots,n$, f can be written

$$f = -\sum_{1\le i<j\le n}\alpha_{ij}y_{ij} - 2\lambda_0 + \sum_{1\le i\le n}(\lambda_0 - \lambda_i)x_i + \sum_{1\le i<j\le n}(\lambda_i + \lambda_j)y_{ij} = \sum_{1\le i<j\le n}\alpha_{ij}clq^{\{i,j\},0} + \lambda^t A_n^2$$

Then f is a positive linear combination of $clq^{\{i,j\},0}$ plus a linear combination of A_n^2. Thus the $clq^{\{i,j\},0}$ for $1 \le i < j \le n$ are sufficient to describe the proper faces of $QP_{n,2}$.

$QP_{n,2}$ is a simplex. Indeed it has $\frac{1}{2}n(n-1)$ vertices and contains the same number of affinely independent points (Proposition.3.1) then necessarily its vertices are affinely independent.

Now, since $QP_{n,2}$ is a simplex of dimension $\frac{1}{2}n(n-1)$ -1, it has $\frac{1}{2}n(n-1)$ facets. As the number of inequalities $clq^{\{i,j\},0}(x,y) \le 0$ is $\frac{1}{2}n(n-1)$ too, each of these inequalities induces necessarily a facet of $QP_{n,2}$ and each facet of $QP_{n,2}$ is represented by one (and only one) such inequality.

Now, let $f(x,y)\le 0$ be a valid inequality inducing a facet of $QP_{n,n-2}$. Then, $\Psi(f)(x,y) \le 0$ induces a facet of $QP_{n,2}$ (Lemma.3.2) that is to say $\Psi(f) = \alpha clq^{\{i,j\},0} + \lambda^t A_n^2$ for some $i, j \in N$ with a scalar $\alpha > 0$, $\lambda = \begin{pmatrix}\lambda_0 \\ \lambda_{i\in N}\end{pmatrix} \in \Re^{n+1}$. Since $\Psi \circ \Psi$ is the identity, we have

$f = \Psi(\Psi(f)) = \alpha\Psi\left(clq^{\{i,j\},0}\right) + \Psi\left(\lambda^t A_n^2\right) = \alpha clq^{\{i,j\},1} + \mu^t A_n^{n-2}$ with $\mu = \begin{pmatrix} -\sum_{i\in N\cup\{0\}}\lambda_i \\ \lambda_{i\in N}\end{pmatrix}$. Hence the facets of $QP_{n,n-2}$ can be described by the clique inequalities $clq^{\{i,j\},1}(x,y) \le 0$. As $QP_{n,n-2}$ is a simplex of dimension $\frac{1}{2}n(n-1)$-1 too, it has $\frac{1}{2}n(n-1)$ facets described, in a one-to-one correspondence, by the inequalities $clq^{\{i,j\},1}(x,y) \le 0$ for $i, j \in N$, $i<j$.

4.2. The cut inequality family

Let S and T be non-empty, disjoint subsets of N. Then consider the following quadratic inequality $-\frac{1}{2}\left(\sum_{i\in T}x_i - \sum_{i\in S}x_i\right)\left(\sum_{i\in T}x_i - \sum_{i\in S}x_i - 1\right) \le 0$. Such an inequality is valid for any 0-1 vector x and then provides through the linearization of the qpBf involved, the following inequality, called cut inequality, $cut^{S,T}(x,y) = -\sum_{i\in S}x_i - \sum_{i,j\in S,i<j}y_{ij} + \sum_{e\in[S:T]}y_e - \sum_{i,j\in T,i<j}y_{ij} \le 0$ which is valid for BP_n and hence for QP_n.

The vertices of QP_n (i.e. BP_n) which saturate this inequality can be easily deduced from the above quadratic formulation and are characterized in the following manner: let $z \in BP_n$ then $cut^{S,T}(z) = 0$ iff $0 \le |x(z) \cap T| - |x(z) \cap S| \le 1$.

It has been proved by M. Padberg [16] that the cut inequality induces a facet of QP_n if $|T| = |S| = 1$ or if $|T| \ge 2$.

In the investigation of conditions for which this inequality also induces a facet of $QP_{n,k}$, we focus on the case 3≤k≤n-3 since the full description of $QP_{n,k}$ for k=2 and k=n-2 is provided by a subset of the clique inequality family (Proposition.4.1.3). These conditions are based on the minimal size of the sets S, T and on the magnitude of the integer k. When k dominates the size of the set $N - (S \cup T)$, minimal sizes of S and T are required which are computed by considering an equal distribution in S and T of the elements of x(z), z belonging to $BP_{n,k}$, which cannot enter $N-(S\cup T)$.

We recall the following usual notation. For any fractional number a≥0, $\lfloor a \rfloor$ designates the greatest integer smaller or equal to a and $\lceil a \rceil$ the smallest integer greater or equal to a.

Proposition.4.2.1. Let S and T be non-empty, disjoint subsets of N, let $\overline{S\cup T}=N-(S\cup T)$ and let k such that 3≤k≤n-3.

Then the cut inequality $cut^{S,T}(x,y)=-\sum_{i\in S}x_i-\sum_{i,j\in S,i<j}y_{ij}+\sum_{e\in[S:T]}y_e-\sum_{i,j\in T,i<j}y_{ij}\le 0$ induces a facet of $QP_{n,k}$ if and only if the following conditions hold

$$\begin{cases} |T|=1\Rightarrow |S|=1 & [C4.2.1]\\ 2\le |S|\Rightarrow 4\le k & [C4.2.2]\\ \left|\overline{S\cup T}\right|<k\Rightarrow 2+\left\lfloor \dfrac{k-\left|\overline{S\cup T}\right|}{2}\right\rfloor\le |S| & [C4.2.3]\\ \left|\overline{S\cup T}\right|\le k\Rightarrow 2+\left\lceil \dfrac{k-\left|\overline{S\cup T}\right|}{2}\right\rceil\le |T| & [C4.2.4]\\ 4\le\left|\overline{S\cup T}\right| & [C4.2.5]\end{cases}$$

Proof: sufficiency. Let f(x,y)≤0 be a valid inequality for $QP_{n,k}$ such that the face induced on $QP_{n,k}$ contains the face induced by the cut inequality.
We shall consider points z_1, z_2 of $BP_{n,k}$ constructed with subsets $U\underset{\neq}{\subset}S$, $V\underset{\neq}{\subset}T$ s.t. $|U|=|V|$ and $R'\underset{\neq}{\subset}R\underset{\neq}{\subset}\overline{S\cup T}$ s.t. $|U|+|V|+|R|=k-1$ and $|R'|=|R|-2$. In the following, we will design i an element of S-U, j an element of T-V and h an element of $\overline{S\cup T}-R$.
Let z_1, z_2 s.t. $\begin{cases}x(z_1)=U+V+j+R\\ x(z_2)=U+V+h+R\end{cases}$. As $cut^{S,T}(z_1)=cut^{S,T}(z_2)=0$ then $f(z_1)-f(z_2)=0$ which gives

$$a_j+a_{jU}+a_{jV}+a_{jR}=a_h+a_{hU}+a_{hV}+a_{hR}\ (1)$$

Now if $1\le|R|\le\left|\overline{S\cup T}\right|-2$ then let $h'\in R$, $h''\in\overline{S\cup T}-R-h$ and let $R''=R-h'+h''$ obtained from R by permuting h' and h". Substituting R by R" in (1) leads to a new relation (1') and (1)-(1') gives

$$a_{jh'}-a_{jh''}=a_{hh'}-a_{hh''}\ (a)$$

Now suppose that $|T-V|\ge 2$ and let $j'\in T-V-j$. Let z_1, z_2 s.t. $\begin{cases}x(z_1)=U+i+V+j'+j+R'\\ x(z_2)=U+i+V+j'+h+R'\end{cases}$. As $cut^{S,T}(z_1)=cut^{S,T}(z_2)=0$ then $f(z_1)-f(z_2)=0$ which gives
$a_j+a_{jU}+a_{jV}+a_{jR'}+a_{ij}+a_{jj'}=a_h+a_{hU}+a_{hV}+a_{hR'}+a_{ih}+a_{hj'}$ (2). Let {h', h"}=R-R' then (1)-(2) gives

$$a_{ij}+a_{jj'}=a_{jh'}+a_{jh''}-a_{hh'}-a_{hh''}+a_{ih}+a_{hj'}\ (b)$$

Now if $1\le|R'|$ then let $h'\in R'$, $h''\in\overline{S\cup T}-R'-h$ and let $R''=R'-h'+h''$ obtained from R' by permuting h' and h". Substituting R' by R" in (2) leads to a new relation (2') and (2)-(2') as above gives (a).

Now let us prove that the conditions [C4.2] allow us to construct z_1, z_2 with subsets U, V, R, R' and R" this last one obtained from R or R'.
If $3\le k<\left|\overline{S\cup T}\right|$ let us choose $U=V=\varnothing$. Then $|R|=k-1$ and $1\le|R|\le\left|\overline{S\cup T}\right|-2$ this last showing that R" can be constructed from R. Now suppose that $|T|\ge 2$ then the points z_1, z_2 can be constructed with R' and $|R'|=k-3\ge 0$.

If $k\ge\left|\overline{S\cup T}\right|$ let us choose $|U|=|V|=\eta$ where $\eta=\left\lfloor\dfrac{k-\left|\overline{S\cup T}\right|}{2}\right\rfloor$. Then [C.4.2.3] and [C4.2.4] imply that $|S-U|\ge 1$ and $|T-V|\ge 2$ showing that the sets S-U and T-V have sufficient cardinalities to construct z_1 and z_2 in every case.
It follows from the definition of η that $\left|\overline{S\cup T}\right|-1\le k-2\eta\le\left|\overline{S\cup T}\right|$. Then

$|R| = k - 2\eta - 1 \le \left|\overline{S \cup T}\right| - 1$ and $|R'| = k - 2\eta - 3 \ge \left|\overline{S \cup T}\right| - 4 \ge 0$ where non-negativity is ensured by [C4.2.5]. Moreover, if $k - 2\eta = \left|\overline{S \cup T}\right|$ then [C4.2.5] implies $1 \le \left|\overline{S \cup T}\right| - 3 = |R'|$ showing that R" can be constructed from R' and if $k - 2\eta = \left|\overline{S \cup T}\right| - 1$ then [C4.2.5] implies $1 \le |R| = \left|\overline{S \cup T}\right| - 2$ showing that R" can be constructed from R.

At last, as the sets U, V, R, R' are arbitrarily chosen, we have proved that (a) is valid $\forall j \in T, \forall h, h', h'' \in \overline{S \cup T}$ and that if $|T| \ge 2$ (b) is valid $\forall i \in S, \forall j, j' \in T, \forall h, h', h'' \in \overline{S \cup T}$.

Next we shall consider points z_1, z_2 of $BP_{n,k}$ constructed with subsets U, V, R, R' defined as in the previous part except that now $|U| + |V| + |R| = k - 2$. As before, we will design i an element of S-U, j an element of T-V and h an element of $\overline{S \cup T} - R$.

Let z_1, z_2 s.t. $\begin{cases} x(z_1) = U + i + V + j + R \\ x(z_2) = U + h + V + j + R \end{cases}$. As $cut^{S,T}(z_1) = cut^{S,T}(z_2) = 0$ then $f(z_1)$-$f(z_2)$=0 which gives

$$a_i + a_{iU} + a_{iV} + a_{ij} + a_{iR} = a_h + a_{hU} + a_{hV} + a_{hj} + a_{hR} \quad (3)$$

Now if $1 \le |R| \le \left|\overline{S \cup T}\right| - 2$ then let $h' \in R, h'' \in \overline{S \cup T} - R - h$ and let $R'' = R - h' + h''$ obtained from R by permuting h' and h". Substituting R by R" in (3) leads to a new relation (3') and (3)-(3') gives

$$a_{ih'} - a_{ih''} = a_{hh'} - a_{hh''} \quad (c)$$

Now suppose that $|S - U| \ge 2$ and $|T - V| \ge 2$ and let $i' \in S - U - i$ and $j' \in T - V - j$. Let z_1, z_2 s.t. $\begin{cases} x(z_1) = U + i' + i + V + j' + j + R' \\ x(z_2) = U + i' + h + V + j' + j + R' \end{cases}$. As $cut^{S,T}(z_1) = cut^{S,T}(z_2) = 0$ then $f(z_1)$-$f(z_2)$=0 which gives $a_i + a_{iU} + a_{iV} + a_{ij} + a_{iR'} + a_{ii'} + a_{ij'} = a_h + a_{hU} + a_{hV} + a_{hj} + a_{hR'} + a_{i'h} + a_{hj'}$ (4). Let {h', h"}=R-R' then (3)-(4) gives

$$a_{ii'} + a_{ij'} = a_{ih'} + a_{ih''} - a_{hh'} - a_{hh''} + a_{i'h} + a_{hj'} \quad (d)$$

Now if $1 \le |R'|$ then let $h' \in R', h'' \in \overline{S \cup T} - R' - h$ and let $R'' = R' - h' + h''$ obtained from R' by permuting h' and h". Substituting R' by R" in (4) leads to a new relation (4') and (4)-(4') as above gives (c).

Now let us prove that the conditions [C4.2] allow us to construct z_1, z_2 with subsets U, V, R, R' and R" this last one obtained from R or R'.

If $3 \le k \le \left|\overline{S \cup T}\right|$ let us choose $U = V = \emptyset$. Then $|R| = k - 2$ and $1 \le |R| \le \left|\overline{S \cup T}\right| - 2$ this last showing that R" can be constructed from R. Now suppose that $|S| \ge 2$ then [C4.2.1] implies $|T| \ge 2$ and the points z_1, z_2 can be constructed with R' and $|R'| = k - 4 \ge 0$ where non-negativity is ensured by [C4.2.2].

If $k > \left|\overline{S \cup T}\right|$ let us choose $|U| = |V| = \mu$ where $\mu = \left\lfloor \frac{k - \left|\overline{S \cup T}\right|}{2} \right\rfloor$. Then [C4.2.3] and [C4.2.4] imply that $|S - U| \ge 2$ and $|T - V| \ge 2$ showing that the sets S-U and T-V have sufficient cardinalities to construct z_1 and z_2 in every case.

It follows from the definition of μ that $\left|\overline{S \cup T}\right| \le k - 2\mu \le \left|\overline{S \cup T}\right| + 1$. Then $|R| = k - 2\mu - 2 \le \left|\overline{S \cup T}\right| - 1$ and $|R'| = k - 2\mu - 4 \ge \left|\overline{S \cup T}\right| - 4 \ge 0$ where non-negativity is ensured by [C4.2.5]. Moreover, if $k - 2\mu = \left|\overline{S \cup T}\right| + 1$ then [C4.2.5] implies $1 \le \left|\overline{S \cup T}\right| - 3 = |R'|$ showing that R" can be constructed from R' and if $k - 2\mu = \left|\overline{S \cup T}\right|$ then [C4.2.5] implies $1 \le |R| = \left|\overline{S \cup T}\right| - 2$ showing that R" can be constructed from R.

At last, as the sets U, V, R, R' are arbitrarily chosen, we have proved that (c) is valid $\forall i \in S, \forall h, h', h'' \in \overline{S \cup T}$ and that if $|S| \ge 2$ (d) is valid $\forall i, i' \in S, \forall j' \in T, \forall h, h', h'' \in \overline{S \cup T}$.

Without loss of generality (see Lemma.3.1) let s be an element of S, let t be an element of T and 1, 2, 3 be three elements of $\overline{S \cup T}$ (recall $\left|\overline{S \cup T}\right| \ge 3$).

Let $\lambda_1, \lambda_2, \lambda_3$ be solution of the non-singular system $\begin{cases} a_{12} = \lambda_1 + \lambda_2 \\ a_{13} = \lambda_1 + \lambda_3 \\ a_{23} = \lambda_2 + \lambda_3 \end{cases}$

and then let	$\lambda_h\ h \in \overline{S \cup T} - \{1,2,3\}$	defined through	$a_{1h} = \lambda_1 + \lambda_h$
and	$\lambda_i\ i \in S$	defined through	$a_{1i} = \lambda_1 + \lambda_i$
and	$\lambda_j\ j \in T$	defined through	$a_{1j} = \lambda_1 + \lambda_j$
and finally	μ	defined through	$a_{st} = \lambda_s + \lambda_t + \mu$

Then

(a) with	$h'' = 1, h' = 2, h = 3$	gives	$a_{j2} = a_{j1} + a_{23} - a_{13} = \lambda_j + \lambda_2$	$\forall j \in T$
(a) with	$h'' = 2,\ h = 1$	gives	$a_{jh'} = a_{j2} + a_{1h'} - a_{12} = \lambda_j + \lambda_{h'}$	$\forall j \in T, \forall h' \in \overline{S \cup T} - \{1,2\}$
(a) with	$h'' = 1$	gives	$a_{hh'} = a_{jh'} - a_{j1} + a_{1h} = \lambda_h + \lambda_{h'}$	$\forall h, h' \in \overline{S \cup T} - \{1\}$
(c) with	$h'' = 1$	gives	$a_{ih'} = a_{i1} + a_{hh'} - a_{h1} = \lambda_i + \lambda_{h'}$	$\forall h' \in \overline{S \cup T} - \{1\}$

If $|T| \geq 2$ then

(b) now becomes $a_{ij} + a_{jj'} = 2\lambda_j + \lambda_i + \lambda_{j'}$ $\forall i \in S, \forall j, j' \in T, \forall h, h', h'' \in \overline{S \cup T}$

(b) with	$j = t, i = s$	gives	$a_{tj'} = -a_{st} + 2\lambda_t + \lambda_s + \lambda_{j'} = -\mu + \lambda_t + \lambda_{j'}$	$\forall j' \in T - \{t\}$
(b) with	$j' = t, i = s$	gives	$a_{sj} = -a_{tj} + 2\lambda_j + \lambda_s + \lambda_t = \mu + \lambda_j + \lambda_s$	$\forall j \in T - \{t\}$
(b) with	$i = s$	gives	$a_{jj'} = -a_{sj} + 2\lambda_j + \lambda_s + \lambda_{j'} = -\mu + \lambda_j + \lambda_{j'}$	$\forall j, j' \in T$
(b)		gives	$a_{ij} = -a_{jj'} + 2\lambda_j + \lambda_i + \lambda_{j'} = \mu + \lambda_j + \lambda_i$	$\forall i \in S, \forall j \in T$

If $|S| \geq 2$ then

(d) now becomes $a_{ii'} = -a_{ij'} + 2\lambda_i + \lambda_{i'} + \lambda_{j'} = -\mu + \lambda_i + \lambda_{i'}$ $\forall i, i' \in S$

Now (1) becomes $a_j + (k-1)\lambda_j = a_h + (k-1)\lambda_h$ with $j \in T - V,\ h \in \overline{S \cup T} - R$

Now (3) becomes $a_i + (k-1)\lambda_i + \mu = a_h + (k-1)\lambda_h$ with $i \in S - U,\ h \in \overline{S \cup T} - R$

As the sets U, V, R are arbitrarily chosen we have

$a_j + (k-1)\lambda_j = a_h + (k-1)\lambda_h = a_i + (k-1)\lambda_i + \mu$ $\forall i \in S, \forall j \in T, \forall h \in \overline{S \cup T}$

and there exists a constant λ_0 such that $\begin{cases} a_j = \lambda_0 - (k-1)\lambda_j & \forall j \in T \\ a_h = \lambda_0 - (k-1)\lambda_h & \forall h \in \overline{S \cup T} \\ a_i = \lambda_0 - (k-1)\lambda_i - \mu & \forall i \in S \end{cases}$

The last coefficient a_0 can be found by solving the equation f(z)=0, for any z of $BP_{n,k}$ such that $cut^{S,T}(z) = 0$, whose one and only solution necessarily is $a_0 = -k\lambda_0$.

Now let us verify that the face induced by the cut inequality is strictly contained in $QP_{n,k}$.
There exists subsets $U \subset S, V \subset T, R \subset \overline{S \cup T}$ such that

if $k \leq \left|\overline{S \cup T}\right|$ $|U| = 1$, $V = \varnothing$, $|R| = k - 1$

if $k > \left|\overline{S \cup T}\right|$ $|U| = \left\lfloor \frac{k - \left|\overline{S \cup T}\right|}{2} \right\rfloor + 1$, $|V| = \left\lceil \frac{k - \left|\overline{S \cup T}\right|}{2} \right\rceil - 1$, $R = \overline{S \cup T}$

Let z of $BP_{n,k}$ such that x(z)=U+V+R then $cut^{S,T}(z) \neq 0$.

Finally we have exhibited $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_{i \in N} \end{pmatrix}$ and μ such that $f = \mu cut^{S,T} + \lambda^t A_n^k$. As $cut^{S,T}(z) \neq 0$ for some z of $QP_{n,k}$, the cut inequality $cut^{S,T}(x,y) \leq 0$ induces a facet of $QP_{n,k}$.

Proof: necessity. We now observe the different points of the necessary part.
If $|T| = 1, 2 \leq |S|$, then $cut^{S,T}$ is a positive linear combination of the elementary cliques and cuts $clq^{\{i,i'\},0}$ and $cut^{\{i\},\{j\}}$ where $i, i' \in S, T = \{j\}$.
For the other cases, we show that the cut inequality induces a face strictly contained in some proper face of $QP_{n,k}$. We denote by $F\left(cut^{S,T}\right)$ the face induced by the cut inequality $cut^{S,T}(x,y) \leq 0$ on $QP_{n,k}$.
Recall it is sufficient to check the following inclusion properties on the vertices of $QP_{n,k}$.
If $2 \leq |S|, k = 3$, then let $h, l \in S, h \neq l$, $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the elementary clique inequality $-y_{hl} \leq 0$.
Let us now denote $R = \overline{S \cup T}$.

If $|R| < k$ then:

1. If $|S| < \left\lfloor \frac{k-|R|}{2} \right\rfloor$, then $F\left(cut^{S,T}\right)$ is empty.
2. If $|S| = \left\lfloor \frac{k-|R|}{2} \right\rfloor \geq 1$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the valid inequality $x_i - 1 \leq 0$ where $i \in S$.
3. If $|S| = \left\lfloor \frac{k-|R|}{2} \right\rfloor + 1$ then
   a. For $|S| = 1$, let $S = \{i\}$. As $|R| = k-1 > 0$ the set R is non empty and let $h \in R$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the elementary clique inequality $clq^{\{i,h\},1}(z) \leq 0$.
   b. For $|S| \geq 2$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the elementary clique inequality $clq^{\{i,j\},1}(z) \leq 0$ where $i, j \in S$.

If $|R| \leq k$ then:

1. If $|T| < \left\lceil \frac{k-|R|}{2} \right\rceil$, then $F\left(cut^{S,T}\right)$ is empty.
2. If $|T| = \left\lceil \frac{k-|R|}{2} \right\rceil \geq 1$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the valid inequality $x_i - 1 \leq 0$ where $i \in T$.
3. If $|T| = 1 + \left\lceil \frac{k-|R|}{2} \right\rceil$ then
   a. For $|T| = 1$, let $T = \{i\}$. As $|R| = k > 0$ the set R is non empty and let $h \in R$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the elementary clique inequality $clq^{\{i,h\},1}(z) \leq 0$.
   b. For $|T| \geq 2$, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the elementary clique inequality $clq^{\{i,j\},1}(z) \leq 0$ where $i, j \in T$.

If $|R| \leq 3$ then:

1. If $|R| = k$ then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the clique inequality $clq^{T+\{h\},1}(z) \leq 0$ where $h \in R$. As $|T + \{h\}| \geq 2$ and $1 \notin \{k-1, k\}$ (k≥3), this clique inequality induces a proper face of $QP_{n,k}$ (see Remark.4.1.2).

For $k > |R|$, we can suppose $|S| \geq 2$ and $|T| \geq 2$ since the necessity of these conditions has been established in the previous points. Then we have $|N - S| \geq |T| \geq 2$ and $|N - T| \geq |S| \geq 2$.

2. If $k - |R| > 0$ is even, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the clique inequality $clq^{S,\alpha}(z) \leq 0$ with $\alpha = \left\lfloor \frac{k-|R|}{2} \right\rfloor$. As $|S| \geq 2$ and $|N - S| \geq 2$, this clique inequality induces a proper face of $QP_{n,k}$ (see Remark.4.1.2).
3. If $k - |R|$ is odd, then $F\left(cut^{S,T}\right)$ is strictly included in the face induced by the clique inequality $clq^{T,\alpha}(z) \leq 0$ with $\alpha = \left\lceil \frac{k-|R|}{2} \right\rceil$. As $|T| \geq 2$ and $|N - T| \geq 2$, this clique inequality induces a proper face of $QP_{n,k}$ (see Remark.4.1.2).

4.3. The generalized cut inequality family

A new family of valid inequalities is given when the complement operation of Lemma.3.2 is applied to the cut inequality family. The new inequality obtained this way is called the generalized cut inequality [16].

Proposition.4.3.1. Let S and T be non-empty, disjoint subsets of N, let $\overline{S \cup T} = N - (S \cup T)$ and let k such that 3≤k≤n-3. Let us define $s = |S|$, $t = |T|$, then the generalized cut inequality

$$gct^{S,T}(x,y) = (s-t)\sum_{i \in S} x_i + (t-s-1)\sum_{i \in T} x_i - \sum_{i,j \in S, i<j} y_{ij} + \sum_{e \in [S:T]} y_e - \sum_{i,j \in T, i<j} y_{ij} - \frac{(t-s)(t-s-1)}{2} \leq 0$$

induces a facet of $QP_{n,n-k}$ if and only if the conditions [C4.2] hold.

Proof. From Proposition.4.2.1, Lemma.3.2 and Remark.3.2 $\Psi\left(cut^{S,T}\right)(z) \le 0$ induces a facet of $QP_{n,n-k}$ if and only if [C4.2] hold, and $\Psi\left(cut^{S,T}\right) = gct^{S,T}$.

4.4. The hypermetric inequality family

The hypermetric inequality is based on the following quadratic inequality $-\frac{1}{2}\left(\sum_{i\in N} l_i x_i\right)\left(\sum_{i\in N} l_i x_i - 1\right) \le 0$ where l_i is an integer for any i of N. This inequality is valid for any 0-1 vector x since then $\sum_{i\in N} l_i x_i$ is integer and the two members of the product cannot have opposite signs. The linearization of the qpBf involved provides some valid inequalities for BP_n and then for QP_n. These inequalities belong to a more general class of valid inequalities of QP_n introduced by E. Boros and P.L. Hammer in [6]. The case $l_i \in \{-1,0,1\}$ corresponds to the cut inequality of M. Padberg [16]. Here we consider the case studied by N. Boissin in [4] where $l_i = 1\ \forall i \in S, l_i = -\alpha_i < 0\ \forall i \in T, l_i = 0$ otherwise for some non-empty, disjoint subsets S, T of N. The inequality becomes $-\frac{1}{2}\left(\sum_{i\in S} x_i - \sum_{i\in T}\alpha_i x_i\right)\left(\sum_{i\in S} x_i - \sum_{i\in T}\alpha_i x_i - 1\right) \le 0$ and its linearization then leads to the following inequality which will be called hypermetric inequality,

$hyp^{S,T,\alpha_{i\in T}}(z) = -\sum_{i\in T}\tfrac{1}{2}\alpha_i(\alpha_i+1)x_i - \sum_{i,j\in T, i<j}\alpha_i\alpha_j y_{ij} + \sum_{e\in[S:T]}\alpha_{T(e)}y_e - \sum_{i,j\in S, i<j} y_{ij} \le 0$.

The vertices of QP_n (i.e. BP_n) which saturate this inequality can be easily deduced from the above quadratic formulation and are characterized in the following manner: let $z \in BP_n$ then $hyp^{S,T,\alpha_{i\in T}}(z) = 0$ iff $\sum_{i\in x(z)\cap T}\alpha_i \le |x(z)\cap S| \le \sum_{i\in x(z)\cap T}\alpha_i + 1$ where the sum over the empty set is set to 0.

It has been proved by N. Boissin [4] that the hypermetric inequality induces a facet of QP_n if $1 \le \alpha_i \le |S|-2\ \ \forall i \in T$ and $|S| \ge \underset{i,j\in T,\, i\ne j}{Max}\left(\alpha_i+\alpha_j\right)$ for $|T| \ge 2$. Additionnal conditions will provide hypermetric inequalities inducing facets of $QP_{n,k}$. Consider points z of $BP_{n,k}$ such that the set $x(z)\cap T$ contains exactly one element, say i. Such points are, through a suitable projection, points of $BP_{n',k'}$ with $n' = n-|T|$ and $k' = k-1$. The restriction of the hypermetric inequality to this set of points z leads to the clique inequality $clq^{S,\alpha_i}(z) \le 0$. Thus in the following conditions, we will find some constraints on the magnitude of k which are similar to those found in the clique inequality for k'=k-1. However, the upper bound on k is somewhat more restrictive but appears necessary, at least in the case where T contains only one element.

Proposition.4.4.1. Let S and T be non-empty, disjoint subsets of N and $\alpha_{i\in T}$ integers satisfying

$$\begin{cases} 1 \le \alpha_i \le |S|-2\ \forall i \in T & [C4.4.1] \\ \alpha_i + 3 \le k \le |N-(S\cup T)|\ \ \forall i \in T & [C4.4.2] \\ |T| \ge 2 \Rightarrow \underset{i,j\in T,\, i\ne j}{Max}\left(\alpha_i+\alpha_j\right) \le Min\left(|S|, k-2\right) & [C4.4.3] \end{cases}$$

Then the hypermetric inequality

$hyp^{S,T,\alpha_{i\in T}}(z) = -\sum_{i\in T}\tfrac{1}{2}\alpha_i(\alpha_i+1)x_i - \sum_{i,j\in T, i<j}\alpha_i\alpha_j y_{ij} + \sum_{e\in[S:T]}\alpha_{T(e)}y_e - \sum_{i,j\in S, i<j} y_{ij} \le 0$ induces a facet of $QP_{n,k}$.

Proof. Let f(x,y)≤0 be a valid inequality for $QP_{n,k}$ such that the face induced on $QP_{n,k}$ contains the face induced by the hypermetric inequality.

Let $i \in T$.

[C4.4.1] implies the existence of subset U of S such that $1 \le \alpha_i = |U| \le |S|-2$. Then let $j' \in U$ and $j, j'' \in S-U\ (j \ne j'')$.

[C4.4.2] implies the existence of subset R of $N-(S\cup T)$ such that $|R| = k-\alpha_i-2$ and $1 \le |R| \le |N-S\cup T|-2$. Then let $h' \in R$ and $h, h'' \in (N-S\cup T)-R\ \ (h \ne h'')$.

Now consider $z_1, z_2 \in BP_{n,k}$ such that $\begin{cases} x(z_1) = i+U+R+h \\ x(z_2) = i+U+R+j \end{cases}$.

As $hyp^{S,T,\alpha_{i\in T}}(z_1) = hyp^{S,T,\alpha_{i\in T}}(z_2) = 0$ then $f(z_1) - f(z_2) = 0$ which gives

$$a_h + a_{ih} + a_{hU} + a_{hR} = a_j + a_{ji} + a_{jU} + a_{jR}\ (1)$$

Substituting R by $R' = R - h' + h''$ in (1) leads to $a_h + a_{ih} + a_{hU} + a_{hR'} = a_j + a_{ji} + a_{jU} + a_{jR'}$ (1') and (1) - (1') gives

$$a_{hh'} - a_{hh''} = a_{jh'} - a_{jh''}\ (2)$$

which is valid $\forall j \in S, \forall h, h', h'' \in N - S \cup T$ since the sets U, R are arbitrarily chosen.
Substituting U by $U' = U - j' + j''$ in (1) leads to $a_h + a_{ih} + a_{hU'} + a_{hR} = a_j + a_{ji} + a_{jU'} + a_{jR}$ (1") and (1)-(1") gives

$$a_{hj'} - a_{hj''} = a_{jj'} - a_{jj''} \quad (3)$$

which is valid $\forall j, j', j'' \in S$, $\forall h \in N - S \cup T$ since the sets U, R are arbitrarily chosen.

Moreover, [C4.4.2] implies the existence of subset R of $N - (S \cup T)$ such that $|R| = k - 1$ and $0 \le |R| \le |N - S \cup T| - 1$. Then let $h \in (N - S \cup T) - R$. Let $j \in S$ and consider $z_1, z_2 \in BP_{n,k}$ such that $\begin{cases} x(z_1) = R + h \\ x(z_2) = R + j \end{cases}$. As $hyp^{S,T,\alpha_{i \in T}}(z_1) = hyp^{S,T,\alpha_{i \in T}}(z_2) = 0$ then $f(z_1) - f(z_2) = 0$ which gives

$$a_h + a_{hR} = a_j + a_{jR} \quad (4)$$

Without lost of generality (see Lemma.3.1) let s, s' be two elements of S (recall $|S| \ge 2$) and 1, 2, 3 be three elements of $N - S \cup T$ (recall $|N - S \cup T| \ge 3$). We will now denote $\overline{S \cup T} = N - (S \cup T)$.

Let $\lambda_1, \lambda_2, \lambda_3$ be solution of the non-singular system $\begin{cases} a_{12} = \lambda_1 + \lambda_2 \\ a_{13} = \lambda_1 + \lambda_3 \\ a_{23} = \lambda_2 + \lambda_3 \end{cases}$

and then let	$\lambda_h \; h \in \overline{S \cup T} - \{1,2,3\}$	defined through	$a_{1h} = \lambda_1 + \lambda_h$
and	$\lambda_j \; j \in S$	defined through	$a_{1j} = \lambda_1 + \lambda_j$
and	$\lambda_i \; i \in T$	defined through	$a_{1i} = \lambda_1 + \lambda_i$
and finally	μ	defined through	$a_{ss'} = \lambda_s + \lambda_{s'} - \mu$

Then

(2) with	$h'' = 1, h' = 2, h = 3$	gives	$a_{j2} = a_{j1} + a_{23} - a_{13} = \lambda_j + \lambda_2$	$\forall j \in S$
(2) with	$h'' = 2,\ h = 1$	gives	$a_{jh'} = a_{j2} + a_{1h'} - a_{12} = \lambda_j + \lambda_{h'}$	$\forall j \in S, \forall h' \in \overline{S \cup T} - \{1,2\}$
(2) with	$h'' = 1$	gives	$a_{hh'} = a_{jh'} - a_{j1} + a_{1h} = \lambda_h + \lambda_{h'}$	$\forall h, h' \in \overline{S \cup T} - \{1\}$

(3) with	$j = s, j'' = s'$	gives	$a_{sj'} = a_{hj'} - a_{hs'} + a_{ss'} = \lambda_{j'} + \lambda_s - \mu$	$\forall j' \in S - \{s, s'\}$
(3) with	$j'' = s$	gives	$a_{jj'} = a_{sj} + a_{hj'} - a_{hs} = \lambda_j + \lambda_{j'} - \mu$	$\forall j, j' \in S - \{s\}$

Now, since the sets U, R corresponding to (1) are such that $|U| + |R| = k - 2$ and $|U| = \alpha_i$, (1) becomes $a_h + a_{ih} + (k-2)\lambda_h = a_j + a_{ij} + (k-2)\lambda_j - \alpha_i \mu$.
Likewise, since the set R corresponding to (4) is such that $|R| = k - 1$, (4) becomes $a_h + (k-1)\lambda_h = a_j + (k-1)\lambda_j$
As the sets R, U are arbitrarily chosen, theses two relations are available $\forall i \in T$, $\forall h \in \overline{S \cup T}$ and $\forall j \in S$.
Now (1)-(4) gives $a_{ih} - \lambda_h = a_{ij} - \lambda_j - \alpha_i \mu$ (5)

(5) with	$h = 1$	gives	$a_{ij} = a_{i1} - \lambda_1 + \lambda_j + \alpha_i \mu = \lambda_i + \lambda_j + \alpha_i \mu$	$\forall j \in S, \forall i \in T$
(5)		gives	$a_{ih} = a_{ij} + \lambda_h - \lambda_j - \alpha_i \mu = \lambda_i + \lambda_h$	$\forall h \in \overline{S \cup T}, \forall i \in T$

(4) implies there exists a constant λ_0 such that $a_h + (k-1)\lambda_h = a_j + (k-1)\lambda_j = \lambda_0$ $\forall h \in \overline{S \cup T}$, $\forall j \in S$ and then $\begin{cases} a_h = \lambda_0 - (k-1)\lambda_h & \forall h \in \overline{S \cup T} \\ a_j = \lambda_0 - (k-1)\lambda_j & \forall j \in S \end{cases}$.

The coefficients which are still unknown are a_0, a_i, $a_{ii'}$ for $i, i' \in T$. For each of them, we construct and solve an equation where they appear alone.
For the coefficient a_0, consider z of $BP_{n,k}$ such that $x(z) = R$ where $R \subset \overline{S \cup T}$ and $|R| = k$ (such set exists see z_1 used to built (4)). Then $hyp^{S,T,\alpha_{i \in T}}(z) = 0$ and a_0 is the only unknown of the equation f(z)=0.
For a coefficient a_i $i \in T$, consider z of $BP_{n,k}$ such that $x(z) = i + U + R$ where $U \subset S$, $R \subset \overline{S \cup T}$ and $|U| = \alpha_i$, $|R| = k - \alpha_i - 1$ (such sets exist see z_1 used to built (1)). Then $hyp^{S,T,\alpha_{i \in T}}(z) = 0$ and, as a_0 is now determined, a_i is the only unknown of the equation f(z)=0.
For a coefficient $a_{ii'}$ $i, i' \in T$, [C4.4.3] implies the existence of subsets $U \subset S$, $R \subset \overline{S \cup T}$ such that $|U| = \alpha_i + \alpha_{i'}$ and $|R| = k - \alpha_i - \alpha_{i'} - 2 \ge 0$. Consider z of $BP_{n,k}$ such that $x(z) = i + i' + U + R$. Then

$hyp^{S,T,\alpha_{i\in T}}(z) = 0$ and, as a_0, a_i $i \in T$ are now determined, $a_{ii'}$ is the only unknown of the equation f(z)=0.

The resolution of these equations necessarily gives $\begin{cases} a_0 = -\lambda_0 k \\ a_i = \lambda_0 - (k-1)\lambda_i - \frac{1}{2}\alpha_i(\alpha_i+1)\mu & \forall i \in T \\ a_{ii'} = \lambda_i + \lambda_{i'} - \alpha_i\alpha_{i'}\mu & \forall i, i' \in T \end{cases}$

Now let us verify that the face induced by the hypermetric inequality is strictly contained in $QP_{n,k}$. As [C4.4.1] implies $|S| \geq 2$ and [C4.4.2] implies $k \leq |N - S \cup T|$, there exists subsets U of S and R of $N - S \cup T$ such that $|U| = 2$ and $|R| = k-2$. Let z of $BP_{n,k}$ such that x(z)=U+R then $hyp^{S,T,\alpha_{i\in T}}(z) \neq 0$.

Finally we have exhibited $\lambda = \begin{pmatrix} \lambda_0 \\ \lambda_{i\in N} \end{pmatrix}$ and μ such that $f = \mu hyp^{S,T,\alpha_{i\in T}} + \lambda^t A_n^k$. As $hyp^{S,T,\alpha_{i\in T}}(z) \neq 0$ for some z of $QP_{n,k}$, the hypermetric inequality $hyp^{S,T,\alpha_{i\in T}}(x,y) \leq 0$ induces a facet of $QP_{n,k}$.

Remark.4.4.1. Here we discuss the necessity of the conditions involved in the previous proposition. In this paragraph, for a given valid inequality f(x,y)≤0 of $QP_{n,k}$, F(f) denotes the face induced on $QP_{n,k}$. Recall it is sufficient to verify the validity of the following inclusions on the vertices of $QP_{n,k}$.

If $|T| = 1$, let $T = \{i\}$. Suppose $|S| = 1$. If $\alpha_i \geq 2$ then $F\left(hyp^{S,T,\alpha_i}\right)$ is strictly included in the face induced by the valid inequality $-x_i \leq 0$ and if $\alpha_i = 1$ then $hyp^{S,T,\alpha_i} = cut^{T,S}$. Hence we can assume $|S| \geq 2$. Let us denote $R = N - S \cup T$. We look at three cases:

1. $\alpha_i \geq |S| - 1$. Let h, l be two elements of S. Then $F\left(hyp^{S,T,\alpha_i}\right) \subset F\left(cut^{\{i\},\{h,l\}}\right)$. If $|S| \geq 3$ then the inclusion is strict.
2. $k > |R|$. Suppose R non empty and let h be an element of R. Then $F\left(hyp^{S,T,\alpha_i}\right) \subset F\left(clq^{\{i,h\},1}\right)$. If $k > |R| + 1$ or $|R| \geq 2$, then the inclusion is strict.
3. $\alpha_i \geq k - 2$ and $|R| = k$. We can assume $|R| \geq 2$ (k≥2) and let h, l be two elements of R. Then $F\left(hyp^{S,T,\alpha_i}\right) \subset F\left(clq^{\{i,h,l\},1}\right)$. If $|R| \geq 3$ then the inclusion is strict.

If $|T| \geq 2$. If $\exists i, j \in T$ such that $\alpha_i + \alpha_j > k - 2$ or $\alpha_i + \alpha_j > |S|$ then $F\left(hyp^{S,T,\alpha_{i\in T}}\right) \underset{\neq}{\subset} F\left(clq^{\{i,j\},0}\right)$.
Moreover note that $\alpha_i + \alpha_j \leq k - 2 \Rightarrow \alpha_i \leq k - 3$ and this last condition is then necessary and that if $hyp^{S,T,\alpha_{i\in T}}$ is not $cut^{T,S}$ i.e. there exists $\alpha_j \geq 2$ then $\alpha_i + \alpha_j \leq |S| \Rightarrow \alpha_i \leq |S| - 2$ and this last condition is then necessary for any i≠j.

5. Connection with some related polytopes

We show in this section, two linear transformations that map our polytope, for some special value of k, on two polytopes of the literature. As a consequence, we can deduce from the results of the previous section, families of facets (some of them new) for these two polytopes.

5.1. Connection with the polytope of the cardinality inequality constrained problem

K.Park et al. [17], A.Mehrotra [14] consider the polytope defined by the convex hull of the binary points satisfying the cardinality inequality constraint that is to say $QP_{n,k^{\leq}} = \mathrm{Conv}\left(BP_{n,k^{\leq}}\right)$ where $BP_{n,k^{\leq}} = \left\{(x,y) \in BP_n : \sum_{i \in N} x_i \leq k\right\}$. This polytope is the projection of our polytope on some variables. Indeed, let us denote by Proj_N, the projection on the variables x_i, y_{ij} with indices i, j in N and consider our polytope $QP_{n+k,k} = \mathrm{Conv}\left(BP_{n+k,k}\right)$ defined with k additionnal variables x_i. It can be easily checked that $BP_{n,k^{\leq}} = \mathrm{Proj}_N\left(BP_{n+k,k}\right)$. Then by linearity of the projection we have $QP_{n,k^{\leq}} = \mathrm{Proj}_N\left(QP_{n+k,k}\right)$.

This observation can lead to some new valid inequalities inducing facet for $QP_{n,k^{\leq}}$. Indeed any valid inequality with non zero coefficients only for the variables x_i, y_{ij} with indices i, j in N, which induces a facet of $QP_{n+k,k}$ induces a facet of $QP_{n,k^{\leq}}$ (see [1] Proposition 3.4 and Corollary 3.3). Now from Propositions 4.1.1, 4.2.1, 4.3.1 and 4.4.1, we can deduce families of facets (some of them new) of $QP_{n,k^{\leq}}$. Note that the conditions which were necessary to give facets of $QP_{n+k,k}$ might fail for $QP_{n,k^{\leq}}$ since a valid inequality not inducing a facet of $QP_{n+k,k}$ can induce a facet of the projected polyhedron. For example we can observe that the conditions obtained from Proposition 4.2.1 for the cut inequality are slightly stronger that those given in theorem 3.2 of [17]. If $\mathrm{cut}^{S,T}(z) \leq 0$ induces a facet of $QP_{n+k,k}$ with $S \cup T \subseteq N$ then $\left|\{1,\ldots,n+k\} - (S \cup T)\right| \geq k$ and C4.2.3 is always true and C4.2.4 implies $|T| \geq 2$ if $\left|\{1,\ldots,n+k\} - (S \cup T)\right| = k$. Hence with regard to the conditions of Theorem 3.2 of [17] the only one additionnal condition is C4.2.5 i.e. $\left|\{1,\ldots,n+k\} - (S \cup T)\right| \geq 4$ which is always true except for k=3 and $S \cup T = N$. So $\mathrm{cut}^{S,T}(z) \leq 0$ with $|S| = 1, |T| = n-1$ does not induce a facet of $QP_{n+3,3}$ but induces a facet of $QP_{n,3^{\leq}}$.

For the clique inequality the analysis of the conditions of Proposition 4.1.1 leads to the same conditions of Theorem 3.2 of [17]. Under these observations one can think that the conditions of Propositions 4.3.1 and 4.4.1 for the generalized cut and the hypermetric inequalities give conditions that are not too strong for the projected polytope.

Note that if one considers the polytope $QP_{n,k^{\leq}}$ defined with the variables x_i, with indices i in N, and y_{ij}, with pairs i,j in E where E is a proper subset of the set of all pairs i<j in N^2 (see [14]), then the previous result holds for any valid inequality inducing a facet of $QP_{n+k,k}$ with non zero coefficients only for the variables x_i with indices i in N and y_{ij} with pairs i,j in E.

5.2. Connection with the polytope of the equicut problem

Here we focus on the special case n=2k.

Let K_n be the undirected complete graph with n=2k nodes and let V={1,2,...,n} be the set of nodes of K_n. For any $U \subset V$, let $\delta(U)$ be the set of edges of K_n with exactly one extremity in U. $\delta(U)$ is called a cut. Note that U and V-U give the same cut.

A cut $\delta(U)$ is an equicut if $|U| = k = \frac{n}{2}$. Since $|U| = \frac{n}{2}$ iff $|V - U| = \frac{n}{2}$, U and V-U give the same equicut.

Let us define $B_{EQ}(K_n) = \left\{u \in B^{n(n-1)/2}\text{: u is the incidence vector of an equicut of } K_n\right\}$ and $Q_{EQ}(K_n) = \mathrm{Conv}(B_{EQ}(K_n))$. $Q_{EQ}(K_n)$ is the equicut polytope [9].

In order to see the relationship between our polytope and the equicut polytope, let $x_i = \begin{cases} 1 \text{ if node } i \in U \\ 0 \text{ else} \end{cases}$ and $u_{ij} = x_i(1 - x_j) + x_j(1 - x_i)$ for $1 \le i < j \le n$

then $u_{ij} = \begin{cases} 1 \text{ if edge } ij \in \delta(U) \\ 0 \text{ else} \end{cases}$

As U and V-U give the same equicut, for any equicut $\delta(U)$ we can suppose that the node 1 (arbitrarily chosen) belongs to V-U.

Letting $x_1=0$ and $y_{ij}=x_i x_j$, one gets the following expression for u_{ij}: $\begin{cases} u_{1i} = x_i & i = 2,\ldots,n \\ u_{ij} = x_i + x_j - 2y_{ij} & 2 \le i < j \le n \end{cases}$

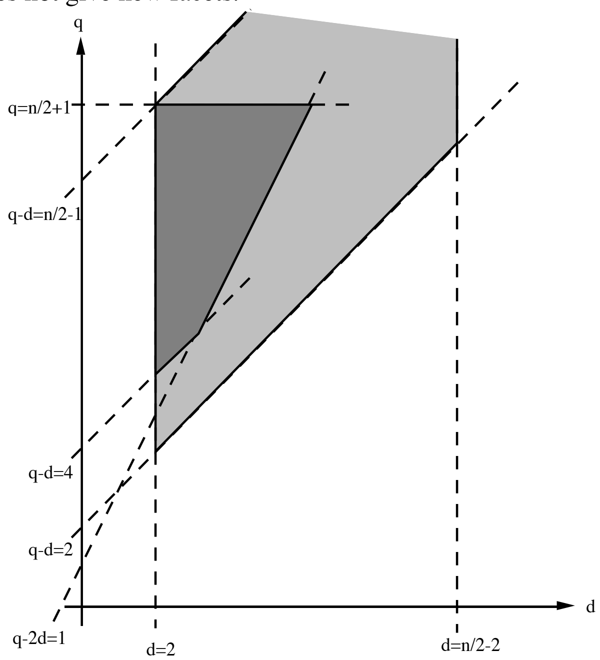
Now it is easy to check that the mapping $\varphi: (x,y) \in \Re^{n(n-1)/2} \mapsto u \in \Re^{n(n-1)/2}$ defined by the above expression is a bijection between $BP_{n-1,n/2}$ and $B_{EQ}(K_n)$. As φ is affine, φ is a bijection between $QP_{n-1,n/2}$ and $Q_{EQ}(K_n)$, and a facet of $QP_{n-1,n/2}$ maps onto a facet of $Q_{EQ}(K_n)$ and vice versa. This observation allows us to construct valid inequalities inducing facet of $Q_{EQ}(K_n)$ through Propositions 4.1.1, 4.2.1, 4.3.1, 4.4.1. For example if $clq^{S,\alpha}(x,y) \le 0$ (with $S \subset N - \{1\}$) induces a facet of $QP_{n-1,n/2}$ then $clq^{S,\alpha}\left(\varphi^{-1}(u)\right) = \left(\alpha - \frac{|S|-1}{2}\right)\sum_{i\in S} u_{1i} + \frac{1}{2}\sum_{\substack{i,j\in S \\ i<j}} u_{ij} - \frac{\alpha(\alpha+1)}{2} \le 0$ induces a facet of $Q_{EQ}(K_n)$.

Letting $q = |S| + 1, d = |S| - \alpha$ and K_q the complete graph with nodes $S \cup \{1\}$, we see that this inequality corresponds to the one of Corollary 7.4 of [10]. The conditions of Proposition 4.1.1 for the polytope $QP_{n-1,n/2}$ expressed with q and d are for $|S| \ge 3$: $\begin{cases} 2 \le d \le q - 2 & [C4.1.2] \\ q - \frac{n}{2} + 1 \le d \le \frac{n}{2} - 2 & [C4.1.3] \end{cases}$

and the conditions (see Theorem7.3 [10]) for which the inequality of Corollary 7.4 induces a facet of $Q_{EQ}(K_n)$ (n=2k) are, following the Remark p.59 in [9]: $q \ge 2d + 1, d \ge 2, q \le \frac{n}{2} + 1, q - d \ge 4$. The representation of these two sets of conditions in the plane d, q (figure 1) shows that the facets built from Proposition 4.1.1 enlarge the family given in Corollary 7.4 of [10].

Exchanging x_1, arbitrarily chosen, with another variable x_i, one gets through the same method, other valid inequalities inducing facet for $Q_{EQ}(K_n)$ where the role of indices 1 and i are permuted.

At last letting $x_1=1$ (instead of 0), one can construct (in a similar way) a bijection $\overline{\varphi}$ between $QP_{n-1,n/2-1}$ and $Q_{EQ}(K_n)$. If $f(x,y) \le 0$ is facet defining for $QP_{n-1,n/2-1}$ then $f\left(\overline{\varphi}^{-1}(u)\right) \le 0$ is facet defining for $Q_{EQ}(K_n)$; but using the transformation ψ of Lemma3.2 that maps an inequality inducing a facet of $QP_{n-1,n/2-1}$ onto an inequality inducing a facet of $QP_{n-1,n/2}$, one can deduce this inequality by $f \circ \overline{\varphi}^{-1} = \psi(f) \circ \varphi^{-1}$ and this approach does not give new facets.



integral points in the region correspond to facets built from Proposition 4.1.1
integral points in the region correspond to facets of Corollary 4.7 of [10]

figure 1

6. Numerical results.

6.1. A cutting plane algorithm.

The initial general problem we study is

$$IP_{n,k} \equiv \left| \begin{array}{l} \text{Min/Max}\left(f(x) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} y_{ij} \right) \\ \text{s.t:}\ (x,y) \in QP_{n,k} \end{array} \right.$$

Where $2 \le k \le n-2$ is the given size restriction. We will solve the following relaxation of $IP_{n,k}$:

$$LP_{n,k} \equiv \left| \begin{array}{l} \text{Min/Max}\left(L_f(x,y) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} y_{ij} \right) \\ \text{s.t:} \begin{cases} A_n^k(x,y) = 0 \\ (x,y) \text{ satisfies } F \end{cases} \end{array} \right.$$

where F is the set of following inequalities:

- Trivial inequalities, built with the class of $clq^{S,0}$, $clq^{S,1}$, $cut^{S,T}$ where $|T| = 1$ and $|S| = 1$ for the cuts but $|S| = 2$ for the cliques, see Proposition.4.1.1. and Proposition.4.2.1.
- Great cliques, built with the class of $clq^{S,\alpha}$ where $|S| \ge 2$ and $\alpha \ge 1$, see Proposition.4.1.1.
- Great cuts, built with the class of $cut^{S,T}$ where $|S| \ge 1$ and $|T| \ge 2$, see Proposition.4.2.1.
- Complement cuts, built with the class of $gct^{S,T}$, see Proposition.4.3.1.
- Simple hypermetrics, built with the class of $hyp^{S,T,\alpha_{i\in T}}$ where $|T| \in \{1,2\}$, see Proposition.4.4.1.

We notice that each optimal solution (x^*,y^*) of $LP_{n,k}$ is, in particular, an optimal solution of $IP_{n,k}$ whenever (x^*,y^*) has binary components.

As usual, we will use a cutting plane method where the inequalities of F are dynamically introduced in the linear programming formulation. Our cutting plane algorithm can be sketched as follows

0. Initialize the linear program

$$\overline{LP}_{n,k} \equiv \left| \begin{array}{l} \text{Min/Max}\left(L_f(x,y) = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} y_{ij} \right) \\ \text{s.t:} \begin{cases} A_n^k(x,y) = 0 \\ C = \varnothing \end{cases} \end{array} \right.$$

1. Determination of an optimal solution (x^*,y^*) for $\overline{LP}_{n,k}$.
2. If (x^*,y^*) is a binary point of $BP_{n,k}$, then output the optimum value of f and goto step 6.
3. Search for inequalities in F that are violated by (x^*,y^*).
4. If search in 3 is successful, update $\overline{LP}_{n,k}$.
5. If the maximum number of iterations and the maximum CPU time are not reached and the matrix is not full goto step 1.
6. Stop the algorithm.

We execute step 3 in the following manner: we start exhaustively searching in Trivial inequalities until we have found the maximum number of MaxTriangle violated inequalities. For the other inequalities we search with a kind of simulated annealing method. For each class of functions considered, we proceed in the following manner. Given some current function f, we randomly choose a function g in the neighbourhood of f. If the inequality $g(x,y) \le 0$ is violated, we iterate the search from g, otherwise we randomly choose f or g as the new current function to continue the search. If f is constructed with sets S, T, it is considered as neighbour of f, a function that can be attained from f by exchanging an element of $S \cup T$ and another element. The families of inequalities are explored in the following order.

For the Great cliques, with each feasible couple (s,α) where $s \le \text{Maxclq}$, we find at most MaxPad violated inequalities $clq^{S,\alpha}(z) \le 0$ where $|S| = s$.

For the Great cuts, with each feasible couple (s,t) where $s + t \le \text{Maxcut}$, we find at most MaxPad violated inequalities $cut^{S,T}(z) \le 0$ where $|S| = s$ and $|T| = t$.

For the Complement cuts we do the same than the Great cuts with the inequalities $gct^{S,T}(z) \le 0$.

At least, for the Simple hypermetrics, for each feasible triplet $(s,t,\{\alpha,\beta\})$, we find at most Maxhyp violated inequalities $hyp^{S,T,\alpha_{i\in T}}(z) \le 0$, where $\{\alpha_i\} = \{\alpha,\beta\}, |T| = t, |S| = s$.

The number of violated inequalities searched depends of course on the space which remains in the memory.

The update in step 4, is organized as follows. Except for the case of the Trivial inequalities, we use ε as the violation value such that if $f\left(x^*, y^*\right) \geq \varepsilon > 0$ then f is considered as violated. The magnitude of the violation is chosen to be at least proportionnal to the number of x_i variables used in the inequalities. If this criterion is verified by an inequality, it is then added to C. On the other hand, among the inequalities of C, we remove the inactive inequalities i.e. inequalities whose slack value is greater than MinSlack, and we add them to D. D is a buffer in which non violated constraints removed from C, stay during one iteration. At the next iteration, if they are again violated, they are replaced in C and discarded otherwise. The goal is to limit the number of inequalities in C.
MaxTriangle, Maxclq, Maxcut, Maxpad, Maxhyp, MinSlack are parameters assigned with different values depending on the size of the problem. We want to insist on the fact that we didn't use any heuristic to search the violated facets, and we didn't explore entirely the families of facets, especially the family of hypermetrics.
For implementing our algorithm we used the LP solver Cplex version 3.0, and the programs ran on a station HP9720 Appolo series 700.

6.2. The studied problems.

We tested the facets on three problems categories:

- Looking for a clique of given cardinality and with minimal cost in a non-negative edge-weighted complete graph (see [18]).
- Looking for a vertices bi-partition $(k, n-k)$ with minimal cost in a non-negative edge-weighted graph, where k is given and n is the number of nodes (see [3]).
- Optimizing a qpBf with non-negative coefficients (see [3]).

For our experiments we need to introduce the following definitions:
The quality indicates the integrality of the solution obtained by our cutting-plane algorithm. If for all the instances of a given problem an integer solution has been found, the quality is called Integer. The quality is said to be Not Integer in the other case.

The relative error is the ratio: $\text{GAP} = \dfrac{(\text{Upper bound}) - (\text{Cutting plane solution value})}{\text{Upper bound}} \times 100$.

The upper bound is the value of the solution found by a simulated annealing method or the value of the optimal solution found by the integer version of the LP solver Cplex if this computation does not consume a prohibitive time.
Nb_iter is the number of iterations done by the algorithm.
CPU_LP is the time in seconds spent by the LP solver.
CPU_TT is the time in seconds consumed by the whole algorithm.
Nclq is the total number of clique inequalities generated.
Ncut is the total number of cut inequalities generated.
Ngct is the total number of generalized cut inequalities generated.
Nhyp is the total number of hypermetric inequalities generated.

Up to n=40, each entry (GAP, Nb_iter, CPU_LP, CPU_TT, Nclq, Ncut, Ngct, Nhyp) of the following tables (except for table2.b) is the average of the values obtained by five different instances of a problem. Upwards, because of the more important CPU time consumed, three instances are used. In table 2.b each row corresponds to one instance problem.

<u>The edge-weighted clique problem.</u>

The edge-weighted clique problem is defined as follow: Given an edge-weighted complete graph, find a subset of k vertices such that the sum of the weights of the induced clique is minimum. This graph problem can be easily translated in the quadratic formulation of the section 1; Given c_{ij} the cost of the edge between the nodes i and j, then the qpBf associated to this problem is $f = \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j$

In our generated problems, the weights of the edges are integers equally distributed in $[1;1000]$ (this corresponds to the parameter $w = 2$ in Späth's experiments [18]). The values of k are taken between 2 and $n-2$.

table1: edge-weighted clique problem

n	k	Quality	GAP (%)	Nb_iter	CPU_LP	CPU_TT	Nclq	Ncut	Ngct	Nhyp
25	5	Integer	0	171	15	30	433	817	6	273
25	10	Integer	0	38	41	43	1210	1079	52	279
25	12	Integer	0	46	66	67	1565	1186	96	277
25	13	Integer	0	60	76	80	1568	1192	84	261
25	15	Integer	0	34	53	56	1332	1014	94	219

25	20	Integer	0	10	18	20	686	687	50	134
30	5	Integer	0	208	74	136	660	1708	22	133
30	10	Integer	0	175	431	366	2149	2376	71	1374
30	15	Integer	0	136	297	308	3032	2582	185	581
30	20	Integer	0	36	106	110	1799	1578	141	302
30	25	Integer	0	9	43	44	988	956	67	157
35	5	Integer	0	425	187	325	773	2473	17	199
35	10	Integer	0	390	1445	1599	3363	4310	125	422
35	15	Integer	0	279	805	843	4234	4092	273	1018
35	20	Integer	0	120	531	549	4139	3531	367	668
35	25	Integer	0	55	189	197	2121	1912	197	341
35	30	Integer	0	13	88	89	1392	1260	138	200
40	5	Integer	0	316	195	558	659	2317	32	1130
40	10	Not Int	0.6	309	2171	2309	3845	6264	62	636
40	15	Integer	0	565	2470	2577	5422	7050	253	1892
40	20	Not Int	0.1	688	3602	3729	8001	7585	483	1737
40	25	Integer	0	239	1203	1241	5778	5171	508	885
40	30	Integer	0	118	424	441	2893	2827	231	442
40	35	Integer	0	13	153	155	1737	1475	126	258
50	10	Not Int	6	423	6644	7007	5278	8841	89	918
50	20	Not Int	3	384	6120	6273	9028	12306	444	3377
50	25	Not Int	2	259	5792	5872	11465	12152	850	2806
50	30	Not Int	0.6	343	5303	5350	13712	11944	964	376
50	40	Integer	0	51	751	765	3841	3531	382	647
60	40	Not Int	1	308	8971	9028	18982	17896	1649	3
60	50	Integer	0	101	1891	1905	5651	5451	627	1
70	40	Not Int	6	121	10836	10906	17599	15130	1383	7
70	50	Not Int	1	223	10750	10823	18565	18501	1834	3
70	60	Integer	0	225	2908	2967	6428	7255	356	1
80	60	Not Int	0.7	278	14230	14438	17517	22216	1208	1668
80	70	Integer	0	127	4169	4246	7117	7446	310	516

Up to $n = 35$ all the problems are solved by our cutting plane algorithm. As a point of comparison, Park et al.[17] consider the problems of size $n = 25$ and their algorithm seems to solve them in a larger CPU time. This gain is probably due to the fact that the formulation of the problem with a strict equality leads to a better lower bound than the formulation (used in [17]) with a large inequality. For $n = 40$ the problems are solved for some values of k and when the integer solution is not obtained the value of GAP remains very small. For n=50 and upwards, integer solutions or at least small GAPs are attained espescially when k is large enough compared with n $(n - k \leq 20)$.

The bi-partition problem.

Given a non-oriented and non-negative edge-weighted graph with n nodes and k an integer, the bi-partition problem consists in separating the set of nodes into two distinct sets whose cardinalities are k and $n - k$ such the sum of the costs of the edges having an extremity in one set and the other in the other set is minimum. The quadratic formulation (see section 1) of this graph problem is achieved with the following transformation:
Let c_{ij} the cost of the edge between the nodes i and j, then the qpBf associated to this problem is

$$f = \sum_{1 \leq i \leq n} \left(\sum_{1 \leq j < i} c_{ji} + \sum_{i < j \leq n} c_{ij} \right) x_i - \sum_{1 \leq i < j \leq n} 2 c_{ij} x_i x_j$$

The bipartition problems $(k, n-k)$ and $(n-k, k)$ being the same, the values of k are taken between 2 and $\left\lfloor \frac{n}{2} \right\rfloor$. Bipartition problems with $k < \left\lfloor \frac{n}{2} \right\rfloor$ are called inequicut problems and with $k = \left\lfloor \frac{n}{2} \right\rfloor$ equicut problems.
We first consider inequicut problem. The graphs are randomly generated. The non zero costs of the edges are integers equally distributed in $[1;100]$. The density of the graph, i.e the ratio $\frac{\text{Number of non zero costs}}{\text{Number of edges of the complete graph of order n}}$, is taken in $\{1, 0.25, 0,5, 0.75\}$.

table2.a: inequicut problem

n	k	Density	Quality	GAP (%)	Nb_iter	CPU_LP	CPU_TT	Nclq	Ncut	Ngct	Nhyp
30	7	0.25	Integer	0	45	196	239	1127	1381	63	249
30	7	0.50	Integer	0	140	201	307	1191	1679	50	1032

30	7	0.75	Integer	0	36	71	123	1223	1774	59	797
30	7	1.00	Integer	0	37	81	117	1014	1531	49	711
40	10	0.25	Integer	0	591	4039	5491	3637	5909	65	4523
40	10	0.50	Not Int	0.4	940	4421	5019	2995	6124	48	2473
40	10	0.75	Integer	0	628	2439	2740	2704	5501	50	2027
40	10	1.00	Integer	0	240	1811	2529	2809	5174	59	2559
50	15	0.25	Not Int	16	1160	10443	10806	8285	13956	132	5837
50	15	0.5	Not Int	11	498	10652	10829	6244	12963	92	2667
50	15	0.75	Not Int	6	189	9292	10063	5518	10864	194	12745
50	15	1.00	Not Int	4	334	10708	10838	6425	14087	81	1783
60	10	0.25	Integer	0	1940	14437	15791	2221	11312	45	1409
60	10	0.5	Not Int	1.7	1995	9241	10453	2005	9921	39	1686
60	10	0.75	Not Int	0.7	1484	10294	11344	2017	9961	24	973
60	10	1.00	Not Int	0.7	1435	9703	10687	2082	10305	36	951

The cutting plane approach solves the problems or at least gives tight lower bounds when k is small (k≤10). For k=15, the provided lower bound is not so tight but however GAP is reduced as the density of the graph increases.

We now consider equicut problem. The problem instances are those used by L.Brunetta et al. in [8] (accessible via ftp anonymous). Each row of table 2.b corresponds to one instance problem. There are five categories of randomly generated instances: pure random, planar grid, toroidal grid, mixed grid, negative weights instances (see [8] for more details). In table 2.b the size of an instance is given by the number of nodes n for pure random and negative weights instances, and by the dimension of the grid (hxk) for the grid instances. The size of an instance is ended with a ‘g’, a ‘t’, a ‘m’ or a ‘n’ for respectivly planar, toroidal, mixed grid and negative weights instances. Density is the percentage of nonzero weight edges.

Following the remarks of section 5.2, we fix $x_1 = 0$ that is we consider the polytope $QP_{n-1,n/2}$. L.Brunetta et al. use four facets families: triangle, clique, cycles, matching. The correspondences, via the one-to-one mapping φ of section 5.2, between these facets and our facets are the following:
- triangle inequalities correspond to inequalities $clq^{S,\alpha}(z) \le 0$, $cut^{S,T}(z) \le 0$ with $|S \cup T| \le 3$ and $\alpha \in \{0,1\}$
- clique inequalities correspond to inequalities $clq^{S,\alpha}(z) \le 0$ with $\alpha = \left\lfloor \frac{|S|}{2} \right\rfloor$
- cycles and matching inequalities have no correspondence in the set of inequalities we use and conversely the generalized cut and hypermetric inequalities have no correspondence in the set of inequalities used by L.Brunetta et al.

table2.b: equicut problem

n	Density(%)	Quality	GAP(%)	Nb_iter	CPU_LP	CPU_TT	Nclq	Ncut	Ngct	Nhyp
40	20	Integer	0	551	5856	6737	13406	13725	1018	649
40	30	Integer	0	452	6000	6706	15488	16247	1039	872
40	70	Not Int	2	441	6535	7213	14206	16739	958	892
40	80	Not Int	2	471	6453	7211	15535	17789	1077	1056
40	90	Not Int	3	461	6315	7216	15319	17464	1056	1013
40	100	Integer	0	337	7640	8804	17570	14629	1081	1344
50	10	Integer	0	332	15576	16331	23697	27194	2373	2940
50	20	Not Int	8	1501	52781	60320	56246	39587	2964	5115
50	30	Not Int	8	1501	70383	77945	57647	40583	2621	4213
50	70	Not Int	2	1501	83905	91437	46347	37175	2163	3084
50	90	Not Int	0.5	1209	65916	72078	41759	34699	2031	3182
50	100	Integer	0	1392	82206	91555	38125	41867	1878	5773
52	10	Integer	0	317	37471	39029	34228	29461	3485	7716
54	10	Integer	0	672	64816	68350	53577	42611	3900	8843
56	10	Integer	0	590	79998	83394	55598	45540	4444	10183
5x8g	10	Integer	0	108	3715	3939	8755	9372	942	1182
3x14g	10	Integer	0	126	3637	3874	7123	7914	737	513
11x4g	10	Integer	0	146	28943	30419	16986	15609	2812	8264
22x2g	10	Integer	0	89	35340	36158	16720	16005	2848	8465
23x2g	10	Integer	0	301	117152	128554	26441	22680	5076	20558
4x12g	10	Integer	0	88	54674	57215	25409	20011	4819	11978
6x8g	10	Integer	0	77	94798	96998	24532	19664	4977	14869
8x5t	10	Integer	0	115	3931	4168	9006	9539	1079	1116
21x2t	10	Integer	0	129	4704	4985	11026	11253	1187	1318
22x2t	10	Integer	0	145	4712	5040	9441	10638	1076	1140

11x4t	10	Integer	0	69	19198	20027	11850	10897	1813	4851
23x2t	10	Integer	0	159	32321	34398	17846	15809	2834	6992
4x12t	10	Integer	0	74	110769	112584	18833	15532	2893	13027
6x8t	10	Integer	0	93	61588	63865	21521	18056	3251	15653
5x10t	10	Integer	0	98	80213	82850	21821	17798	3320	13326
10x4m	100	Integer	0	84	3787	3952	7154	7558	841	846
21x2m	100	Integer	0	304	37247	40335	15026	14234	2585	7552
14x3m	100	Integer	0	64	21379	21934	11211	10626	1801	4611
11x4m	100	Integer	0	88	26217	27021	13298	12805	2174	5070
23x2m	100	Integer	0	200	75652	79744	19224	15571	2720	11730
12x4m	100	Integer	0	81	64526	66102	18901	16245	3477	14591
8x6m	100	Integer	0	97	88246	90104	21826	17943	3928	16632
5x10m	100	Integer	0	76	194587	199006	26271	24916	5580	20704
40n	20	Integer	0	184	3627	3887	11228	14255	1299	1226
40n	30	Integer	0	606	4865	5870	13847	15429	1189	801
40n	50	Not Int	12	669	6096	7205	13991	15440	1104	580
40n	70	Integer	0	277	3895	4320	11352	13344	781	504
40n	80	Integer	0	266	3942	4377	10327	12409	862	401
40n	90	Integer	0	472	5851	6625	11823	14153	923	555
50n	30	Integer	0	906	36228	42059	39605	40434	2328	6788
50n	50	Integer	0	1497	59069	69187	38343	39582	1965	5240
50n	90	Not Int	4	1001	70742	77247	41950	43870	1881	5109
50n	100	Not Int	12	879	97180	108088	46064	42283	4329	5820
60n	20	Integer	0	824	69421	71975	57924	71689	6538	7142

Cplex encounters degenerescence problems on grid graphs (planar, toroidal, mixed) for n>40. In order to avoid these difficulties, we are obliged to use large facets (that is with a large number of variables). The drawback is that the CPU time needed by Cplex to solve the problem, drasticly increases with the size of the inequalities. For pure random graphs, sizes of 12 or 16 variables are sufficient but for grid graphs sizes up to 50 are sometimes necessary. This induces a noticeable difference of CPU time between pure random (with negative weights or not) and grid graphs problems.
The values of the lower bound are not given in [8] and a comparison on this point cannot be done. We observe that we are able to solve problems (i.e. Quality=Integer) for which L.Brunetta et al. have resorted to branching. However the number of LP calls (Nb_iter) is always greater in our algorithm. We point out that our separation algorithm is not specifically devoted to the equicut problem and is probably less efficient than the L.Brunetta et al.'s algorithms that are suited to each family of facets.

Optimizing a qpBf with non-negative coefficients.

Given n and k integers, the problem consists in minimizing a qpBf, with n boolean variables x_i, submitted to the constraint $\sum_{1\le i\le n} x_i = k$.

The qpBf $f = q_0 + \sum_{1\le i\le n} q_i x_i + \sum_{1\le i<j\le n} q_{ij} x_i x_j$ satisfies the conditions:

- The linear coefficients q_i are all strictly positive.
- All the quadratic coefficients q_{ij} are positive or zero and the m non zero quadratic coefficients are randomly chosen among the $n(n-1)/2$ possible with $m = \lceil \text{density} \times (n(n-1)/2) \rceil$.

The density is a value taken in $\{1, 0.25, 0{,}5, 0.75\}$.

The non zero coefficients are integers equally distributed in $[1;100]$

table3: minimizing a qpBf with nonnegative coefficients

n	k	Density	Quality	GAP (%)	Nb_iter	CPU_LP	CPU_TT	Nclq	Ncut	Ngct	Nhyp
30	10	0.25	Integer	0	7	38	40	1146	782	120	101
30	10	0.5	Integer	0	23	112	117	1804	1992	153	192
30	10	0.75	Integer	0	35	138	145	1889	2026	140	207
30	10	1.00	Integer	0	58	237	250	2176	2564	133	269
30	15	0.25	Integer	0	36	341	343	2601	2578	228	613
30	15	0.5	Integer	0	45	116	119	2309	2247	176	525
30	15	0.75	Integer	0	67	135	142	2124	1774	127	448
30	15	1.00	Integer	0	153	276	290	3159	2606	186	636
30	20	0.25	Integer	0	35	422	424	3261	2964	280	642
30	20	0.5	Integer	0	23	67	69	1488	1369	98	284
30	20	0.75	Integer	0	19	60	61	1363	1181	95	269
30	20	1.00	Integer	0	36	90	94	1791	1588	110	329

40	10	0.25	Integer	0	17	30	32	963	680	40	104
40	10	0.5	Integer	0	149	751	1346	3213	5120	123	4912
40	10	0.75	Integer	0	155	764	1408	2841	4734	119	4709
40	10	1.00	Integer	0	259	1521	1620	3690	5961	108	647
40	20	0.25	Integer	0	108	1011	1020	5665	7018	344	695
40	20	0.5	Integer	0	131	787	803	5917	7007	334	737
40	20	0.75	Integer	0	149	778	797	4639	5043	258	623
40	20	1.00	Not Int	0.02	1267	4051	4270	8483	8868	448	1358
40	30	0.25	Integer	0	35	218	220	2136	2678	146	231
40	30	0.5	Integer	0	25	208	210	2353	2121	167	448
40	30	0.75	Integer	0	32	190	194	2191	1863	181	371
40	30	1.00	Integer	0	47	291	297	2788	2709	247	498
50	15	0.25	Not Int	2	207	2124	2170	5579	6632	161	1243
50	15	0.5	Not Int	3	501	2479	2642	7369	15677	321	4683
50	15	0.75	Not Int	4	348	3853	3996	6991	10326	233	1804
50	15	1.00	Not Int	5	409	5697	5858	7624	11184	180	2355
50	25	0.25	Not Int	4	232	5882	5923	10593	14066	580	1438
50	25	0.5	Integer	0	229	2834	2883	9766	13753	495	1294
50	25	0.75	Not Int	2	200	5191	5259	12597	10930	778	1530
50	25	1.00	Not Int	1	224	7135	7219	16566	14143	1004	2150
50	35	0.25	Integer	0	94	1202	1216	4911	6599	332	547
50	35	0.5	Integer	0	92	1027	1037	5017	6263	335	519
50	35	0.75	Not Int	0.3	85	2189	2212	9232	7782	1073	977
50	35	1.00	Integer	0	159	2999	3043	10162	8246	1070	1073
60	20	0.25	Not Int	20	228	8959	9053	7654	14305	314	1054
60	20	0.5	Not Int	22	501	7109	7353	9381	24139	347	3392
60	20	0.75	Not Int	10	501	8240	8526	8441	18719	266	3191
60	20	1.00	Not Int	9	440	8792	9016	7874	16970	256	2911
60	30	0.25	Not Int	17	124	9050	9076	7378	12109	510	851
60	30	0.5	Not Int	7	270	8931	9014	15378	23698	919	1847
60	30	0.75	Not Int	4	231	7156	7231	12554	17122	669	7
60	30	1.00	Not Int	5	215	7177	7240	12111	16199	694	7
60	40	0.25	Integer	0	168	4292	4316	9949	13164	682	2
60	40	0.5	Not Int	0.3	419	8772	8872	20572	28118	1464	2095
60	40	0.75	Integer	0	356	7504	7600	13271	15132	837	459
60	40	1.00	Not Int	0.3	477	10630	10810	18906	19889	1318	2247
70	50	0.25	Integer	0	247	10753	10837	13039	16951	920	981
70	50	0.75	Integer	0	282	9423	9554	14342	16679	926	923
80	60	0.25	Not Int	0.5	206	19180	19263	14093	17733	938	886
80	60	0.75	Not Int	0.4	408	17967	18290	18689	21931	1299	1667

For $n = 30$ all the problems are solved. The CPU times are competitive with the ones given in [3] where the problems are solved using the integer version of the Cplex software with a lower bound involving exclusively the "Trivial inequalities" defined above. Here the more general facets used are decisive to solve the problems at least for this size. For $n \geq 50$ the magnitude of GAP increases for the low and medium values of k. However when the value of k is not too low compared with n $(n - k \leq 20)$, large problems (n=70, 80) are solved or at least small GAPs are achieved.

A general remark on the above results is that, when n increases, bad GAPs seem to coincide with low density of the quadratic terms (see table3 n=60). In our polytope $QP_{n,k}$, all the quadratic pairs are present and zero coefficients must be introduced in the objective function if it is necessary. For low density problems too many such zero coefficients are needed and this approach leads to inefficient models. In this case it would perhaps be better to use polytopes involving exclusively the initial variables.

7. Conclusion.

In this paper we have introduced the facial study of the polytope $QP_{n,k}$ which is included in QP_n the Padberg's Boolean quadric polytope. Having proved our polytope $QP_{n,k}$ is a face of QP_n, we have exhibited some subsets of known facets families of QP_n that also induce facets of $QP_{n,k}$. For $k = 2$ and $k = n - 2$, we have given the full description of $QP_{n,k}$. For particular values of k, we have shown how facets of our polytope $QP_{n,k}$ can be used to give facets for two related polytopes of the literature.
The experiments have shown that the studied families of facets are in practice sufficient to solve some optimization problems in $QP_{n,k}$ for medium values of n and for higher values of n when k belongs to some restricted intervals.

In order to enlarge these limits it could be useful to implement some more efficient heuristics in the choices of the facets, or/and to use some others facets families.

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