The quadratic assignment polytope

Alain Billionnet [(1)], Sourour Elloumi [(2)], Alain Faye [(1)]

(1) CEDRIC-IIE 18 allée Jean Rostand 91025 Evry cedex FRANCE
(2) CEDRIC-CNAM 292 rue Saint-Martin 75141 Paris cedex FRANCE

Abstract.
We study the quadratic assignment problem (with n^2 variables) from a polyhedral point of view by considering the quadratic assignment polytope that is defined as the convex hull of the solutions of the linearized problem (with $n^2+\frac{1}{2}n^2(n-1)^2$ variables). We give the dimension of the polytope and a minimal description of its affine hull. We also propose a family of facets with a separation algorithm.

Key words: quadratic assignment problem, polytope, facets, separation problem.

1. Introduction, notations

The quadratic assignment problem (QAP) can be stated as:

$$\min_{p\in\Pi}\sum_{i=1}^{n} q_{i\,p(i)}+\sum_{i=1}^{n}\sum_{\substack{h=1\\ h\neq i}}^{n} q_{i\,p(i)\;h\,p(h)}$$

where Π is the set of all permutations of $\{1,2,\ldots,n\}$ and where $q_{ij}, q_{ij\;hk}$ are given real numbers. In some cases the coefficients have the special form $q_{ij\;hk}=a_{ih}b_{jk}$ where a_{ih} is a flow and b_{jk} a distance. For example, consider the following facility location problem: *n* plants are to be assigned to *n* locations. a_{ih} is the flow of material between the facility *i* and the facility *h*, b_{jk} is the distance between location *j* and location *k*. $a_{ih}b_{jk}$ is the cost induced by the flow between facilities *i* and *h* when they are assigned to locations *j* and *k*. q_{ij} can be interpreted as the cost to pay to assign facility *i* to location *j*. The classical linear assignment problem involves only these last costs. The traveling salesman problem is a special QAP in which *a* is the adjency matrix of an hamiltonian cycle.

QAP can be reformulated by the following mathematical program (QP):

$$\min\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_{ij}+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\ h\neq i}}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n} q_{ij\;hk}x_{ij}x_{hk}$$

subject to

$$\sum_{i=1}^{n} x_{ij}=1\;\; j=1,\ldots,n,\;\sum_{j=1}^{n} x_{ij}=1\;\; i=1,\ldots,n\;,\;\; x_{ij}\in\{0,1\}\;\; i=1,\ldots,n, j=1,\ldots,n$$

The objective function of this 0-1 programming formulation of QAP is quadratic; hence the term "quadratic assignment problem". The equality constraints are called assignment constraints. In the context of the previous example, $x_{ij}=1$ if and only if facility *i* is assigned to location *j* and the assignment constraints stipulate that a facility is assigned to one and only one location and conversely.

Many linearizations are proposed for QAP and can be shared out into two families. The first one relies on the structure of the feasible solutions and the second one, on the structure of the objective function. The first approach consists in modelizing each product $x_{ij}x_{hk}$ by a new variable $y_{ij\,hk}$. $\mathrm{O}(n^4)$ variables are so added: n^4 if one considers all the products without any simplifications, $n^2(n-1)^2$ if one takes into account the equalities $x_{ij}x_{ik}=x_{ij}x_{hj}=0$ and $\frac{1}{2}n^2(n-1)^2$ if moreover one takes into account the equalities $x_{ij}x_{hk}=x_{hk}x_{ij}$. The type of constraints used to ensure the equality $y_{ij\,hk}=x_{ij}x_{hk}$ leads to different models. The equality constraints used in these models come from the agregation of equality constraints generated by multiplying the assignment constraints by the variables x_{ij} following a method proposed by Adams and Sherali (1986). Lawler (1963) proposes a linearization with n^4 additional variables and

n^4+1 additional constraints (compared with QP). The linearization proposed by Christofides, Mingozzi and Toth (1980) uses n^4 additional variables and n^4 additional constraints. This linearization does not use constraints generated by products and therefore is equivalent to the initial model only if the coefficients of the quadratic terms are non negative. These authors propose some rules for decreasing the size of the problem by fixing variables. The linearization of Bazaraa and Sherali (1980) requires $\frac{1}{2}n^2(n-1)^2$ additional variables and $2n^2$ additional constraints. The solution of this program is carried out by a Benders' decomposition. Frieze and Yadegar (1983) propose a linearization with $n^2(n-1)^2$ additional variables and $4n^3$ additional constraints. The continuous relaxation of the program is computed by a dual method. Assad and Xu (1985) propose a linearization with n^4 additional variables and $3n^2$ additional constraints. The lagrangian dual optimum is obtained (with a given precision) by generating a non decreasing suite of lower bounds. Adams and Johnson (1994) propose a linearization with $n^2(n-1)^2$ additional variables and $2n^2(n-1)+\frac{1}{2}n^2(n-1)^2$ additional constraints. $\frac{1}{2}n^2(n-1)^2$ constraints ensure that the equality $x_{ij}x_{hk}=x_{hk}x_{ij}$ is satisfied while the other $2n^2(n-1)$ constraints are the constraints generated by product. Once more the solution of the continuous relaxation is obtained by a dual method in which the dual function is computed by solving n^2+1 linear assignment problems. Recently, Resende, Ramakrishnan and Drezner (1995) consider an equivalent linearization using $\frac{1}{2}n^2(n-1)^2$ additional variables (the equality $x_{ij}x_{hk}=x_{hk}x_{ij}$ is taken into account in the definiton of the variables) and $2n^2(n-1)$ additional constraints. They report extended computational results for solving the continuous relaxation by an interior point method.

In the second family of linearizations, two approaches can be distinguished. The first one works on the structure of the objective function and the second exploits the special form of the costs when $q_{ij\,hk}=a_{ih}b_{jk}$. Kaufman and Broeckx (1978) modelize each term $x_{ij}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}q_{ij\,hk}x_{hk}$ by a single variable w_{ij} and so create n^2 additional variables. n^2 additional constraints are required for ensure the equality $w_{ij}=x_{ij}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}q_{ij\,hk}x_{hk}$. Through an equivalent approach, Burkard and Bönniger (1983) propose an approximate algorithm. In the case where $q_{ij\,hk}=a_{ih}b_{jk}$, Rosenblatt and Golany (1992) propose an heuristic method based on the solution of the assignment of the pairs of plants to distances. This problem, which is a relaxation of QAP, can be stated as a flow problem. The number of variables depends on the number, D, of distinct inter-sites distances $\left(D\leq\frac{1}{2}n(n-1)\right)$ and is exactly equal to n^2D. The solution of this relaxation (which is an integer one because of the total-unimodularity of the constraints matrix) and the reduced costs are exploited in order to build an assignment of the plants to the sites. Always in the case of costs $q_{ij\,hk}=a_{ih}b_{jk}$, Ball, Kaku and Vakhutinsky (1998) propose two modelizations based on integer flows in a graph. In the first one, the graph involves $O\left(n^3\right)$ vertices and $O\left(n^4\right)$ arcs, and these numbers depend on the density of the matrix a. In the second one, the graph involves $4n$ vertices and $2n^2+n(n-1)$ arcs. However, this last formulation requires $O\left(n^4\right)$ additional constraints plus the flow conservation equations. The authors give computational experiments concerning the continuous relaxation of these programs in which valid inequalities are dynamically generated. When the structure of the positive flows of a is a tree, Christofides and Benavent (1989) design a formulation with $O\left(n^3\right)$ additional variables and $O\left(n^2\right)$ additional constraints. Then they use a lagrangian relaxation of this formulation as a bound in a branch-and-bound algorithm.

All these linearizations are equivalent to (QP) only if some of the variables are forced to be integer. On the other hand, their continuous relaxations give diffferent lower bounds. Adams and Johnson (1994) show that the continuous relaxation of their linear formulation dominates the continuous relaxation of many linearizations belonging to the first family. Indeed, in this family the equality constraints are only agregation of the equality constraints considered by Adams and Johnson. Following the results of Adams and Sherali (1986), they prove that the continuous relaxation of their formulation also dominates the continuous relaxation of the linear formulation of Kaufman and Broeckx (1978), a linearization of the second family.

Gilmore (1962) and Lawler (1963) propose a lower bound based on the solutions of n^2+1 linear assignment problems. This bound, easy to compute, is used in many branch-and-bound algorithms: see Kaku and Thompson (1986), Mautor and Roucairol (1994) for example. Frieze and Yadegar (1983) use the assignment constraints to reformulate the objective function and apply the bound of Gilmore and Lawler. They show that the search of the best reformulation can be stated as finding the solution of a lagrangian dual problem associated with their linearization. Assad and Xu (1985) use a similar approach which is less general but they can obtain in this way a non decreasing suite of lower bounds. They show too that their method can be viewed as a method to solve a lagrangian dual problem associated with their linearization. Carraresi and Malucelli (1992) propose a method which produces a non decreasing suite of lower bounds whose first one is that of Gilmore-Lawler. Adams and Johnson (1994) show how this new method can be interpreted as the adjustment of the multipliers in a lagrangian relaxation of their linearization.

In this paper, we consider the polyhedral properties of the most compact (for the number of variables) linearization belonging to the first family in which the products $x_{ij}x_{hk}$ are replaced by the variables $y_{ij\,hk}$ and where the equalities $x_{ij}x_{hk}=x_{hk}x_{ij}$ and $x_{ij}x_{ik}=x_{ij}x_{hj}=0$ are integrated into the definition of the variables.

The 0-1 quadratic program (QP) can be reformulated as a (continuous) linear program in the following way. First we substitute the non zero products $x_{ij}x_{hk}$ (i.e. $i\neq h, j\neq k$) by the variables $y_{ij\;hk}$. Next we define

$$Q_n := \left\{ (x,y) : \begin{bmatrix} \sum_{i=1}^{n} x_{ij} = 1 \;\; j=1,\dots,n, \sum_{j=1}^{n} x_{ij} = 1 \;\; i=1,\dots,n, \\ y_{ij\;hk} = x_{ij}x_{hk} \;\; i\neq h, j\neq k,\; x_{ij}\in\{0,1\} \end{bmatrix} \right\}$$

Here a point must be specified. The variables $y_{ij\;hk}$ and $y_{hk\;ij}$ take place for the same product and then can be identified i.e. $y_{ij\;hk}\equiv y_{hk\;ij}$, so $Q_n\subseteq\{0,1\}^{n^2+\frac{n^2(n-1)^2}{2}}$. This is equivalent to index the variables y by the sets of two ordered pairs (couples) of the following form $\{(i,j),(h,k)\}$. In order to simplify the notation, we will write $y_{ij\;hk}$ or equally $y_{hk\;ij}$ in place of $y_{\{(i,j),(h,k)\}}$. At last we define the polytope $P_n := Conv(Q_n)$ (the convex hull of Q_n).

So we obtain the following problem (LP):

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} q_{ij\;hk}y_{ij\;hk} \text{ subject to } (x,y)\in P_n$$

In order to solve this problem by linear programming methods (simplex, interior points for example) it is necessary to describe P_n with a set, if possible minimal, of linear equalities and inequalities.
Resende, Ramakrishnan and Drezner (1995) report numerical tests on the following relaxation of (LP):

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} q_{ij\;hk}y_{ij\;hk} \text{ subject to } (x,y)\in P_n'$$

where

$$P_n' := \left\{ (x,y): \begin{bmatrix} \sum_{i=1}^{n} x_{ij} = 1 \;\; j=1,\ldots,n, \sum_{j=1}^{n} x_{ij} = 1 \;\; i=1,\ldots,n, \\ \sum_{\substack{h=1 \\ h\neq i}}^{n} y_{ij\,hk} = x_{ij} \;\; i,j,k=1,\ldots,n,\; k\neq j \\ \sum_{\substack{k=1 \\ k\neq j}}^{n} y_{ij\,hk} = x_{ij} \;\; i,j,h=1,\ldots,n,\; h\neq i \\ y_{ij\,hk} \geq 0, x_{ij} \geq 0 \;\; i\neq h \;\; j\neq k \end{bmatrix} \right\}$$

The set of equality constraints of this polyhedron contains on one side the assignment constraints and on the other side the constraints obtained in multiplying each of these assignment constraints by each of the variables x_{ij} and in substituting the generated products $x_{ij}x_{hk}$ $h \neq i, k \neq j$ by the adequate variables *y*. So the constraints of this polyhedron are valid for Q_n that is to say $Q_n \subseteq P_n'$ and finally $P_n \subseteq P_n'$.

In this paper, we propose to study the polytope P_n which will be called the quadratic assignment polytope of order *n*. We give its dimension and a minimal description of its affine hull (section 3). We propose an exponentially large family of inequalities inducing facets (including the non-negativity inequalities of the variables *y* i.e. $y_{ij\,hk} \geq 0$) and a polynomial separation algorithm (section 4).

Previously let us establish some notations.

A couple is an ordered pair (i,j).
The first (resp. second) element of the couples which are indices of the variables *x* or *y* is called row index or more concisely row (resp. column). For example in x_{ij}, $y_{ij\,hk}$ i, h, are row indices (or rows) and *j*, *k* are column indices (or columns).
The transposition of row indices and column indices in the variables *x* or *y* consists in the permutation of the row indices and the column indices in the couples which are indices of the variables. For example for x_{ij}, $y_{ij\,hk}$ the transposition gives x_{ji}, $y_{ji\,kh}$.
Given a polytope P, $\dim(P)$ designates the dimension of *P*.
$Aff(.)$, $Conv(.)$ designate respectively the affine and convex hulls.
Given a matrix or a vectors family *M*, $rank(M)$ designates the rank of *M*.
The rank of an equations system $Ax = b$ is the rank of *A*.
Let *I* be a finite set and $v = (v_i)_{i\in I}$ be a vector, the projection of *v* on the coordinates $J \subset I$ is the vector $v' = (v_i)_{i\in J}$.
The support of a vector *v* is the set of indices *i* s.t. $v_i \neq 0$. The support of a set of vectors is the union of the supports of the vectors of this set.
A linear form $f(x) = \sum_{i=1}^{m} a_i x_i$ can be considered as a linear application $f: R^m \to R$ or as a vector $(a_i)_{i=1,\ldots,m}$. The definitions of rank, projection, support of a linear form or of a set of linear forms follow.
Let *m* be a positive integer, I_m designates the identity matrix of order *m*.
Given a set *E*, the cardinality (number of elements) of *E* is denoted $card(E)$ or $|E|$.

2. Results on linear assignment problems

In this section, we recall some results on linear assignment and related problems.

RESULT 1.
Let *H*, *K* be 2 sets of integers with respective cardinality $m \geq 1$ and $n \geq 1$, the following set of linear forms $\begin{Bmatrix} c_k(x) := \sum_{h\in H} x_{hk} & k \in K \\ l_h(x) := \sum_{k\in K} x_{hk} & h \in H \end{Bmatrix}$ has a rank equal to $m+n-1$. These forms satisfy the equation

$\sum_{k\in K} c_k(x) = \sum_{h\in H} l_h(x)$ and any one of them can be removed from the set without reducing the rank.

Arbitrarily choose $l_{h_{\sup}}$ where $h_{\sup} = \sup H$, then $T_{HK}(x) := \left\{ \begin{array}{ll} c_k(x) := \sum_{h\in H} x_{hk} & k \in K \\ l_h(x) := \sum_{k\in K} x_{hk} & h \in H - \{h_{\sup}\} \end{array} \right\}$ has the same rank $m+n-1$. Observe that $T_{HK}(x)$ is a set of linearly independent forms (see theorem p.120 in Luenberger 1989).

RESULT 2.

For $n \geq 1$, let $X_n := \left\{ (x_{hk})_{(h,k)\in\{1,\ldots,n\}^2} : \left[\begin{array}{l} \sum_{h=1}^{n} x_{hk} = 1 \quad k \in \{1,\ldots,n\} \\ \sum_{k=1}^{n} x_{hk} = 1 \quad h \in \{1,\ldots,n\} \\ x_{hk} \in \{0,1\} \quad \forall (h,k) \end{array} \right] \right\}$

Remark that $(x_{hk})_{(h,k)\in\{1,\ldots,n\}^2}$ can be represented as a matrix or as a column vector by ordering the couples (h,k) following the lexicographic order.
Consider $(X_n)_{x_{ij}=1}$, the points of X_n verifying $x_{ij} = 1$ then the projection of $(X_n)_{x_{ij}=1}$ on the coordinates $(\{1,\ldots,n\} - \{i\}) \times (\{1,\ldots,n\} - \{j\})$ is equal to X_{n-1}.
A linear description of $Conv(X_n)$ is obtained in relaxing the condition $x_{hk} \in \{0,1\}$ to $x_{hk} \geq 0$ $(\forall (h,k))$ in the definition of X_n. This polytope is the (linear) assignment polytope of order n. Its dimension is $(n-1)^2$ (see 1.7 p.97 and 1.11 p.99-100 in Marcus and Minc 1964). As linear and affine independences are equivalent notions in affine spaces which do not contain 0, the (linear) rank of X_n is equal to $(n-1)^2 + 1$.

3. Dimension and affine hull of the quadratic assignment polytope

LEMMA 0.
Suppose $n\geq 3$. Consider (i,j), (h,k) two row-column couples s.t. $i \neq h$ and $j \neq k$. The projection of the points of Q_n (vertices of P_n) s.t. $x_{ij} = x_{hk} = 1$ on the coordinates of the variables $y_{ij\,h'k'}$ $h' \in \{1,\ldots,n\} - \{i,h\}$, $k' \in \{1,\ldots,n\} - \{j,k\}$ is a set of vectors of rank $(n-3)^2 + 1$.

PROOF.
Let V be a set of vectors and Cond a condition, the vectors of V satisfying Cond are noted V_{Cond} or $(V)_{\text{Cond}}$.
Let $\Pi \subseteq Q_n$, let $H \subseteq \{1,\ldots,n\} - \{i\}$, $K \subseteq \{1,\ldots,n\} - \{j\}$. Let us note $pr_{y_{ij\,HK}}$ the projection on the coordinates of the variables $y_{ij\,h'k'}$ $(h',k') \in H \times K$ and $pr_{x_{HK}}$ the projection on the coordinates of the variables $x_{h'k'}$ $(h',k') \in H \times K$. It follows from the definition of Q_n that $pr_{y_{ij\,HK}}\left(\Pi_{x_{ij}=1}\right) = pr_{x_{HK}}\left(\Pi_{x_{ij}=1}\right)$.
Now consider Π, the points of Q_n verifying $x_{hk} = 1$ and $H = \{1,\ldots,n\} - \{i,h\}$, $K = \{1,\ldots,n\} - \{j,k\}$.
Recall that X_n (the vertices of the linear assignment polytope) is the projection of Q_n on the coordinates of the variables x. Then:

$$pr_{y_{ij\,HK}}\left(\Pi_{x_{ij}=1}\right) = pr_{x_{HK}}\left(\Pi_{x_{ij}=1}\right) = pr_{x_{HK}}\left((Q_n)_{x_{ij}=x_{hk}=1}\right) = pr_{x_{HK}}\left((X_n)_{x_{ij}=x_{hk}=1}\right) = X_{n-2}$$

The rank of X_{n-2} is $(n-3)^2 + 1$ (Result 2).

□

LEMMA 1.

The quadratic assignment polytope P_n of order $n \geq 3$ has at least $\frac{1}{2}n(n-1)(n-2)(n-3)+(n-1)^2+2$ linearly independent vertices.

PROOF.

We proceed by recurrence on n.

For n=3, the quadratic assignment polytope has 6 vertices which are linearly independent (the matrix made up of their variables y contains as sub-matrix the identity matrix of order 6).

For n≥4, let us consider the matrices built according to the following method:

let V be a set of variables of P_n and Π be a set of vertices of P_n, we build the matrix M such that each column is the projection of a point of Π on the coordinates of the variables of V. We will denote $M = pr_V(\Pi)$.

1) let $M_1 = pr_{V_1}(\Pi_1)$ built with:

- V_1 the variables x, y whose indices are in $\{2,\ldots,n\}$
- Π_1 the vertices of P_n s.t. $x_{11} = 1$

The columns of M_1 are the vertices of P_{n-1}. Then, by recurrence hypothesis, $rank(M_1) \geq \frac{1}{2}(n-1)(n-2)(n-3)(n-4)+(n-2)^2+2$.

2) a) let $M_{2a} = pr_{V_{2a}}(\Pi_{2a})$ built with $V_{2a} = \bigcup_{j=2}^{n} V_{2a}^j$, $\Pi_{2a} = \bigcup_{j=2}^{n} \Pi_{2a}^j$ where :

$$\left.\begin{array}{l} \text{- } V_{2a}^j \text{ the variables } y_{1j\,hk} \;\; h \in \{2,\ldots,n-1\}, k \in \{2,\ldots,n\} \setminus \{j\} \\ \text{- } \Pi_{2a}^j \text{ the vertices of } P_n \text{ s.t. } x_{1j} = x_{n1} = 1 \end{array}\right\} j = 2,\ldots,n$$

Let $M_{2a}^j = pr_{V_{2a}^j}(\Pi_{2a}^j)$ $j = 2,\ldots,n$. Taking Π_{2a}^j (resp. V_{2a}^j) in the order $j = 2,\ldots,n$, the blocks M_{2a}^j $(j = 2,\ldots,n)$ appear on the diagonal of M_{2a}. As the vertices of P_n satisfy $x_{1j} = 1 \Rightarrow (x_{1j'} = 0 \;\; j' \neq j) \Rightarrow (y_{1j'h'k'} = 0 \;\; h' \neq 1, k' \neq j')$, M_{2a} has zero elements elsewhere and is block diagonal that is to say $M_{2a} = \begin{pmatrix} M_{2a}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M_{2a}^n \end{pmatrix}$. The rank of M_{2a}^j is $(n-3)^2+1$ (apply Lemma 0 with $(i,j) = (1,j)$ and $(h,k) = (n,1)$) and the rank of M_{2a} is $(n-1)\left((n-3)^2+1\right)$.

2)b) let $M_{2b} = pr_{V_{2b}}(\Pi_{2b})$ built with $V_{2b} = \bigcup_{i=2}^{n-1} V_{2b}^i$, $\Pi_{2b} = \bigcup_{i=2}^{n-1} \Pi_{2b}^i$ where:

$$\left.\begin{array}{l} \text{- } V_{2b}^i \text{ the variables } y_{i1\,hk} \;\; h \in \{1,\ldots,n-1\} \setminus \{i\}, k \in \{3,\ldots,n\} \\ \text{- } \Pi_{2b}^i \text{ the vertices of } P_n \text{ s.t. } x_{i1} = x_{n2} = 1 \end{array}\right\} i = 2,\ldots,n-1$$

In a similar way to a), the blocks $M_{2b}^i = pr_{V_{2b}^i}(\Pi_{2b}^i)$ $(i = 2,\ldots,n-1)$ appear on the diagonal of M_{2b} . As the vertices of P_n satisfy $x_{i1} = 1 \Rightarrow (x_{i'1} = 0 \;\; i' \neq i) \Rightarrow (y_{i'1h'k'} = 0 \;\; h' \neq i', k' \neq 1)$, M_{2b} is block diagonal. The rank of M_{2b}^i is $(n-3)^2+1$ (apply Lemma 0 with $(i,j) = (i,1)$ and $(h,k) = (n,2)$) and the rank of M_{2b} is $(n-2)\left((n-3)^2+1\right)$.

2)c) let $M_{2c} = pr_{V_{2c}}(\Pi_{2c})$ built with $V_{2c} = V_{2c}^1 \cup V_{2c}^2$, $\Pi_{2c} = \Pi_{2c}^1 \cup \Pi_{2c}^2$ where:

$$\begin{cases} \text{- } V_{2c}^1 \text{ the variables } \begin{cases} y_{i1\,h2} & h \in \{2,\ldots,n-1\} \setminus \{i\} \\ y_{i1\,12} \end{cases} \Bigg] \; i \in \{2,\ldots,n-1\} \\ \text{- } \Pi_{2c}^1 \text{ the vertices of } P_n \text{ s.t. } \begin{cases} x_{i1} = x_{n3} = x_{1n} = x_{h2} = 1 & h \in \{2,\ldots,n-1\} \setminus \{i\} \\ x_{i1} = x_{n3} = x_{12} = 1 \end{cases} \Bigg] \; i \in \{2,\ldots,n-1\} \end{cases}$$

$$\begin{cases} \text{- } V_{2c}^2 \text{ the variables } y_{i1\,nk} \quad i \in \{2,\ldots,n-1\},\ k \in \{4,\ldots,n\} \\ \text{- } \Pi_{2c}^2 \text{ the vertices of } P_n \text{ s.t. } x_{i1} = x_{12} = x_{nk} = 1 \quad i \in \{2,\ldots,n-1\},\ k \in \{4,\ldots,n\} \end{cases}$$

In Π_{2c}^1, for each condition (corresponding to each value of i, h), we keep one and only one vertex satisfying it. So $\left|\Pi_{2c}^1\right| = (n-2)(n-2)$. We proceed analogously for Π_{2c}^2 and $\left|\Pi_{2c}^2\right| = (n-2)(n-3)$.

The diagonal of M_{2c} can be partitionned into 2 blocks: the first one M_{2c}^1 built with rows corresponding to variables V_{2c}^1 and columns corresponding to points Π_{2c}^1 that is to say $M_{2c}^1 = pr_{V_{2c}^1}\left(\Pi_{2c}^1\right)$, the second one M_{2c}^2 with rows corresponding to variables V_{2c}^2 and columns corresponding to points Π_{2c}^2 that is to say $M_{2c}^2 = pr_{V_{2c}^2}\left(\Pi_{2c}^2\right)$.

The vertices of P_n satisfy $x_{i1} = x_{h2} = 1 \Rightarrow y_{i'1h'2} = x_{i'1}x_{h'2} = 0$ for $i' \neq i$ or $h' \neq h$. Hence arranging rows and columns in the order they appear in the definition, we have $M_{2c}^1 = I_{(n-2)(n-2)}$.

The vertices of P_n satisfy $x_{i1} = x_{nk} = 1 \Rightarrow y_{i'1nk'} = x_{i'1}x_{nk'} = 0$ for $i' \neq i$ or $k' \neq k$. Hence arranging rows and columns in the order they appear in the definition, we have $M_{2c}^2 = I_{(n-2)(n-3)}$.

The condition $x_{n3} = 1$ satisfied by the points of Π_{2c}^1 implies the (upper) triangular structure of the whole that is to say we have $M_{2c} = \begin{pmatrix} I_{(n-2)(n-2)} & \\ 0 & I_{(n-2)(n-3)} \end{pmatrix}$. The rank of M_{2c} is $(n-2)(2(n-3)+1)$.

2)d) let $M_{2d} = pr_{V_{2d}}\left(\Pi_{2d}\right)$ built with:

- V_{2d} the variables $\begin{cases} y_{1j\,nk} & j \in \{3,\ldots,n\}, k \in \{4,\ldots,n\} \setminus \{j\} \\ y_{1j\,n3} & j \in \{4,\ldots,n-1\} \end{cases}$
- Π_{2d} the vertices of P_n s.t. $\begin{cases} x_{1j} = x_{nk} = 1 & j \in \{3,\ldots,n\}, k \in \{4,\ldots,n\} \setminus \{j\} \\ x_{1j} = x_{n3} = 1 & j \in \{4,\ldots,n-1\} \end{cases}$

In Π_{2d}, for each condition (corresponding to each value of j, k), we keep one and only one vertex satisfying it. So $\left|\Pi_{2d}\right| = (n-3)^2 + (n-4)$.

The vertices of P_n satisfy $x_{1j} = x_{nk} = 1 \Rightarrow y_{1j'nk'} = x_{1j'}x_{nk'} = 0$ for $j' \neq j$ or $k' \neq k$. Hence arranging rows (variables V_{2d}) and columns (vertices Π_{2d}) in the order they appear in the definition, we have $M_{2d} = I_{(n-3)^2+(n-4)}$. The rank of M_{2d} is $(n-3)^2 + (n-4)$.

Now let us consider the matrix M built from the union of the variables described above and from the union of the vertices described above, in ordering the rows and the columns in such a way that M_1, M_{2a} ,..., M_{2d} are on the diagonal of M and in this order. Let us prove that M is block (upper) triangular that is to say of the following form:

$$M = \begin{pmatrix} M_1 & & & & \\ 0 & M_{2a} & & & \\ 0 & 0 & M_{2b} & & \\ 0 & 0 & 0 & M_{2c} & \\ 0 & 0 & 0 & 0 & M_{2d} \end{pmatrix}$$

(See the example figure1).

The vertices of P_n satisfy the following points:

- $x_{11} = 1 \Rightarrow x_{1j} = x_{i1} = 0 \ (i,j) \in \{2,\ldots,n\}^2$. Consequently the support of Π_1 does not intersect the indices of the variables of $V_{2a} \cup V_{2b} \cup V_{2c} \cup V_{2d}$.

- $x_{n1}=1 \Rightarrow \begin{cases} x_{i1}=0 & i \neq n \\ x_{nk}=0 & k \neq 1 \end{cases}$. Consequently the support of Π_{2a} does not intersect the indices of the variables of $V_{2b} \cup V_{2c} \cup V_{2d}$.

- $x_{n2}=1 \Rightarrow \begin{cases} x_{nk}=0 & k \neq 2 \\ x_{h2}=0 & h \neq n \end{cases}$. Consequently the support of Π_{2b} does not intersect the indices of the variables of $V_{2c} \cup V_{2d}$.

The points of Π_{2c} are divided into two groups: those s.t. $x_{12}=1$ and those s.t. $x_{n3}=x_{1n}=1$.
- $x_{12}=1 \Rightarrow x_{1j}=0 \;\; j \neq 2$. Consequently the support of the first group does not intersect the indices of the variables of V_{2d}.
- $x_{n3}=1 \Rightarrow x_{nk}=0 \;\; k \neq 3$ and $x_{1n}=1 \Rightarrow x_{1j}=0 \;\; j \neq n$. Consequently the support of the second group does not intersect the indices of the variables of V_{2d}.

Therefore one has:

$$
\begin{aligned}
&rank(M) \geq rank(M_1) + rank(M_{2a}) + \ldots + rank(M_{2d}) \\
&\geq \frac{1}{2}(n-1)(n-2)(n-3)(n-4) + (n-2)^2 + 2 \\
&+(n-1)\left((n-3)^2+1\right) + (n-2)\left((n-3)^2+1\right) + (n-2)\left(2(n-3)+1\right) + (n-3)^2 + (n-4) \\
&= \frac{1}{2}n(n-1)(n-2)(n-3) + (n-1)^2 + 2
\end{aligned}
$$

The figure1 illustrates the structure of the matrix *M* in the case *n*=4. Note that two vertices are candidate to enter Π_{2d}. We have arbitrarily kept the vertex s.t. $x_{31}=1$.

$$
\begin{array}{c|cccccc|ccc|ccc|cccccc|c}
 & \multicolumn{6}{c|}{\Pi_1} & \multicolumn{3}{c|}{\Pi_{2a}} & \multicolumn{3}{c|}{\Pi_{2b}} & \multicolumn{6}{c|}{\Pi_{2c}} & \Pi_{2d} \\
\hline
y_{2233} & 1&0&0&0&0&0 & 0&0& & 0&0& & 0&0&0&0&0&0 & 0 \\
y_{2234} & 0&1&0&0&0&0 & \vdots&\vdots& & \vdots&\vdots& & \vdots&\vdots&\vdots&\vdots&\vdots&\vdots & \vdots \\
y_{2332} & 0&0&1&0&0&0 & & & & & & & & & & & & & \\
y_{2334} & 0&0&0&1&0&0 & 1& & & & & & & & & & & & \\
y_{2432} & 0&0&0&0&1&0 & 0& & & & & & & & & & & & \\
y_{2433} & 0&0&0&0&0&1 & 0&1& & 0&0& & 0&0&0&0&0&0 & 0 \\
\vdots & \multicolumn{6}{c|}{\cdots} & & & & & & & & & & & & & \\
\hline
y_{1223} & & & & & & & 1&0& & 0&0& & 0&0&0&0&0&1 & 0 \\
y_{1224} & & & & & & & 0&1& 0 & \vdots&\vdots& & \vdots&\vdots&\vdots&1&0&0 & \vdots \\
y_{1233} & \multicolumn{6}{c|}{0} & 0&1& & & & & & & &0&1&0 & \\
y_{1234} & & & & & & & 1&0& & 0&0& & 0&1&0&0&0&0 & 0 \\
\vdots & & & & & & & \multicolumn{2}{c}{0} & \ddots & & & & & & & & & & \\
\hline
y_{2113} & & & & & & & & & & 1&0& & 0&0&0&0&0&0 & 0 \\
y_{2114} & & & & & & & & & & 0&1& 0 & 1&\vdots&\vdots&\vdots&0&\vdots & \vdots \\
y_{2133} & \multicolumn{6}{c|}{0} & \multicolumn{3}{c|}{0} & 0&1& & 0& & & &1& & \\
y_{2134} & & & & & & & & & & 1&0& & 0&1&0&0&0&0 & 0 \\
\vdots & & & & & & & & & & \multicolumn{2}{c}{0} & \ddots & & & & & & & \\
\hline
y_{2132} & & & & & & & & & & & & & 1&0&0&0&0&0 & 0 \\
y_{2112} & & & & & & & & & & & & & 0&1&0&0&1&\vdots & 0 \\
y_{3122} & \multicolumn{6}{c|}{0} & \multicolumn{3}{c|}{0} & \multicolumn{3}{c|}{0} & 0&0&1&0&0& & 1 \\
y_{3112} & & & & & & & & & & & & & 0&0&0&1&0&1 & 0 \\
y_{2144} & & & & & & & & & & & & & \multicolumn{4}{c}{0} &1&0 & 0 \\
y_{3144} & & & & & & & & & & & & & & & & &0&1 & 1 \\
\hline
y_{1344} & \multicolumn{6}{c|}{0} & \multicolumn{3}{c|}{0} & \multicolumn{3}{c|}{0} & \multicolumn{6}{c|}{0} & \boxed{1}
\end{array}
$$

figure1

Matrix M for n=4 (proof of Lemma 1)

□

COROLLARY.

For $n \geq 3$, $\dim(P_n) \geq \frac{1}{2} n(n-1)(n-2)(n-3) + (n-1)^2 + 1$.

DEFINITION 1.

For all $i,h = 1,\ldots,m \;\; (h \neq i) \;\; j,k = 1,\ldots,n \;\; (k \neq j)$, we define $c_{ijk}(y) := \sum_{\substack{h=1\\h\neq i}}^{m} y_{ijhk}$ and $l_{ijh}(y) := \sum_{\substack{k=1\\k\neq j}}^{n} y_{ijhk}$.

$c_{ijk}(y)$ (respectively $l_{ijh}(y)$) can be viewed as the linearization of the product of $c_k(x) := \sum_{h=1}^{m} x_{hk}$ (respectively $l_h(x) := \sum_{k=1}^{n} x_{hk}$) by the variable x_{ij} in which the products $x_{ij}x_{ik}$, $x_{ij}x_{hj}$ vanish (for example in the case m=n, the products $x_{ij}x_{ik}$, $x_{ij}x_{hj}$ are zero on Q_n).

LEMMA 2.
For $m \geq 2$ et $n \geq 3$ (m rows, n columns), the rank of the following linear forms F:

$$\begin{cases} c_{1j\,k}(y) := \sum_{h=2}^{m} y_{1j\,hk} & k=1,\ldots,n,\ k\neq j\ ,\ j=1,\ldots,n-2 \\ c_{1(n-1)\,k}(y) := \sum_{h=2}^{m} y_{1(n-1)\,hk} & k=1,\ldots,n-2 \\ l_{1j\,h}(y) := \sum_{\substack{k=1\\k\neq j}}^{n} y_{1j\,hk} & h=2,\ldots,m-1, \quad j=1,\ldots,n-1 \\ l_{ij\,1}(y) := \sum_{\substack{k=1\\k\neq j}}^{n} y_{ij\,1k} & i=2,\ldots,m,\ j=1,\ldots,n \end{cases}$$

is at least $(m-1)(n-1)+(n-1)(m+n-3)+(m-2)$.

PROOF.
The variables of the linear forms F are such that a row index i or h is equal to 1. Consider the partition of the variables of the linear forms F in the n following subsets $Y_j := \left\{ y_{ij'\,hk} : (1,j) \in \{(i,j'),(h,k)\},\ i\neq h,\ j'\neq k \right\}$ for $j=1,\ldots,n$. Consider the partition of the set F in the following subsets $G_j := \left\{ c_{1j\,k} \quad k=1,\ldots,n,\ k\neq j,\ \ l_{1j\,h} \quad h=2,\ldots,m-1 \right\}$ for $j=1,\ldots,n-2$, $G_{n-1} := \left\{ c_{1(n-1)\,k} \quad k=1,\ldots,n-2,\ \ l_{1(n-1)\,h} \quad h=2,\ldots,m-1 \right\}$, $H_j := \left\{ l_{ij\,1},\ i=2,\ldots,m \right\}$ for $j=1,\ldots,n$.

Remark that for all j the support of G_j coincides with the indices of the variables of Y_j and that the support of H_n does not encounter the indices of the variables of Y_n.

a) For $j=1,\ldots,n-2$, $G_j = T_{HK}\left(y_{1j}\right)$ where $H=\{2,\ldots,m\}$, $K=\{1,\ldots,n\}\setminus\{j\}$, therefore $rank\left(G_j\right) = m+n-3$ (see Result 1).
b) Consider the following subset of Y_{n-1} defined as $Y_{n-1}^{n} := Y_{n-1} \cap \left\{ y_{ij'\,hk} : n \in \{j',k\},\ i\neq h,\ j'\neq k \right\}$. Consider H'_n the projections of the linear forms of H_n on the coordinates of the variables of Y_{n-1}, we have: $H'_n = \left\{ y_{in\,1(n-1)},\ i=2,\ldots,m \right\}$ and therefore $rank\left(H'_n\right) = m-1$. Remark that the variables of the forms of H'_n are in Y_{n-1}^{n}.
Consider G'_{n-1} the projections of the linear forms of G_{n-1} on the coordinates of the variables of $Y_{n-1} \setminus Y_{n-1}^{n}$, we have:

$$G'_{n-1} = \left\{ c'_{1(n-1)\,k}(y) = c_{1(n-1)\,k}(y) \quad k=1,\ldots,n-2,\ l'_{1(n-1)\,h}(y) = \sum_{k=1}^{n-2} y_{1(n-1)\,hk} \quad h=2,\ldots,m-1 \right\} = T_{HK}\left(y_{1(n-1)}\right)$$

where $H=\{2,\ldots,m\}$, $K=\{1,\ldots,n-2\}$, K non empty for $n>2$, therefore $rank\left(G'_{n-1}\right) = m+n-4$ (see Result 1).
c) For $j\neq n$, consider H'_j, the projections of the linear forms of H_j on the coordinates of the variables of Y_n, we have: $H'_j = \left\{ y_{ij\,1n}, i=2,\ldots,m \right\}$. Therefore $rank\left(\bigcup_{j=1}^{n-1} H'_j\right) = (m-1)(n-1)$.

Finally, consider the matrix M whose columns are indiced by the variables $Y_1,\ldots,Y_{n-2}, Y_{n-1}^{n}, Y_{n-1}\setminus Y_{n-1}^{n}, Y_n$ (and in this order) and whose rows are the forms $G_1,\ldots,G_{n-2}, H_n, G_{n-1}, \bigcup_{j=1}^{n-1} H_j$ (and in this order). M is block (lower) triangular that is to say

$$M=\begin{pmatrix} G_1 & 0 & 0 & 0 & 0 & 0 \\ & \ddots & & & & \\ & & G_{n-2} & 0 & 0 & 0 \\ & & & H'_n & 0 & 0 \\ & & & & G'_{n-1} & 0 \\ & & & & & \bigcup_{j=1}^{n-1} H'_j \end{pmatrix}$$ (see the example figure2) and its rank is greater or equal to the sum of the ranks of the diagonal blocks.

$$rank(F)=rank(M)\geq (n-2)(m+n-3)+(m+n-4+m-1)+(m-1)(n-1)$$

The figure2 illustrates the construction of the matrix M in the case $m=2, n=3$. For convenience of notation, the linear forms are represented as linear functions of the variables y instead of row vectors of their coefficients.

$$\begin{array}{c|c|c|c|c|}
 & Y_1 & Y_2^3 & Y_2 \setminus Y_2^3 & Y_3 \\
\hline
G_1 & y_{1122} & & & \\
 & y_{1123} & & & \\
\hline
H_3 & y_{2311}\; + & \boxed{y_{2312}} & & \\
\hline
G_2 & & & \boxed{y_{1221}} & \\
\hline
H_1 & & & y_{2112}\; + & y_{2113} \\
H_2 & y_{2211}\; + & & & y_{2213}
\end{array}$$

figure2

Matrix M for $m=2, n=3$ (proof of Lemma 2)

$\square$

REMARK 1. As $rank(F)\leq card(F)=(m-1)(n-1)+(n-1)(m+n-3)+(m-2)$, Lemma 2 could be stated with a strict equality.

LEMMA 3.
For $m\geq 3$ and $n\geq 2$ (m rows, n columns), the rank of the following linear forms $\overline{F}$:

$$\begin{cases} l_{i1\,h}(y):=\sum_{k=2}^{n} y_{i1\,hk} & h=1,\ldots,m,\ h\neq i\ , i=1,\ldots,m-2 \\ l_{(m-1)1\,h}(y):=\sum_{k=2}^{n} y_{(m-1)1\,hk} & h=1,\ldots,m-2 \\ c_{i1\,k}(y):=\sum_{\substack{h=1\\h\neq i}}^{m} y_{i1\,hk} & k=2,\ldots,n-1, \quad i=1,\ldots,m-1 \\ c_{ij\,1}(y):=\sum_{\substack{h=1\\h\neq i}}^{m} y_{ij\,h1} & j=2,\ldots,n,\ i=1,\ldots,m \end{cases}$$

is at least $(m-1)(n-1)+(m-1)(m+n-3)+(n-2)$.

PROOF. The transposition of row indices and column indices in the variables of the forms F of Lemma 2 does not change the rank and gives $\overline{F}$.

$\square$

LEMMA 4.
For n≥3, let

$$F_{r=1,\ldots,n-2}:=\left\{\begin{array}{ll} c_{rj\ k}(y):=\sum_{\substack{h=1\\ h\neq r}}^{n} y_{rj\ hk} & k=r,\ldots,n,\ k\neq j\ ,j=r,\ldots,n-2 \\ c_{r(n-1)\ k}(y):=\sum_{\substack{h=1\\ h\neq r}}^{n} y_{r(n-1)\ hk} & k=r,\ldots,n-2 \\ l_{rj\ h}(y):=\sum_{\substack{k=1\\ k\neq j}}^{n} y_{rj\ hk} & h=r+1,\ldots,n-1,\quad j=r,\ldots,n-1 \\ l_{ij\ r}(y):=\sum_{\substack{k=1\\ k\neq j}}^{n} y_{ij\ rk} & i=r+1,\ldots,n,\ j=r,\ldots,n \end{array}\right\}$$

and let

$$\overline{F}_{r=1,\ldots,n-3}:=\left\{\begin{array}{ll} l_{ir\ h}(y):=\sum_{\substack{k=1\\ k\neq r}}^{n} y_{ir\ hk} & h=r+1,\ldots,n,\ h\neq i\ ,i=r+1,\ldots,n-2 \\ l_{(n-1)r\ h}(y):=\sum_{\substack{k=1\\ k\neq r}}^{n} y_{(n-1)r\ hk} & h=r+1,\ldots,n-2 \\ c_{ir\ k}(y):=\sum_{\substack{h=1\\ h\neq i}}^{n} y_{ir\ hk} & k=r+1,\ldots,n-1,\quad i=r+1,\ldots,n-1 \\ c_{ij\ r}(y):=\sum_{\substack{h=1\\ h\neq i}}^{n} y_{ij\ hr} & j=r+1,\ldots,n,\ i=r+1,\ldots,n \end{array}\right\}$$

$$\overline{F}_{n-2}:=\left\{\ c_{ijk}(y):=\sum_{\substack{h=1\\ h\neq i}}^{n} y_{ijhk}\quad i=n-1,n\quad (j,k)\in\{(n-2,\ n-1),\ (n-1,\ n),\ (n,\ n-2)\}\right\}$$

Let $G_{r=1,\ldots,n-2}:=F_r\cup\overline{F}_r$, let $S_n:=\bigcup_{r=1}^{n-2}G_r$ then $rank(S_n)\geq n(n-1)(2n-3)-1$.

PROOF.
We proceed by recurrence on n .

For n=3, one has $S_3=G_1$. Consider the partition of the variables of G_1 in Y_1, the set of variables with a row index equal to 1, and $\overline{Y}_1$ the complementary set that is to say the set of variables whose row indices are not equal to 1. Remark that the support of F_1 does not encounter the indices of the variables of $\overline{Y}_1$. Applying Lemma 2 with m:=3 and n:=3 we obtain $rank(F_1)=11$. Consider $\overline{F}_1'$ the projections of the linear forms of $\overline{F}_1$ on the coordinates of the variables of $\overline{Y}_1$. These projections are $\overline{F}_1':=\{c'_{i1\ 2}(y)=y_{i1\ h2}, c'_{i2\ 3}(y)=y_{i2\ h3}, c'_{i3\ 1}(y)=y_{i3\ h1}\ i=2,3\ h=2,3\ i\neq h\}$ and therefore $rank(\overline{F}_1')=6$. Finally consider M_1 the matrix whose columns are indiced by the variables Y_1 and $\overline{Y}_1$, and whose rows are the forms F_1 and $\overline{F}_1$. The rank of the linear forms of G_1 is equal to the rank of the matrix $M_1=\begin{pmatrix} F_1 & 0 \\ & \overline{F}_1' \end{pmatrix}$. Therefore $rank(G_1)\geq rank(F_1)+rank(\overline{F}_1')=17$.

For n≥4, let Y be the set of the variables of S_n and y_{ijhk} be any one element of Y. Consider the partition of Y in Y_1, the set of variables with at least one index i, j, h or k equal to 1, and $\overline{Y}_1$ the complementary set that is to say the set of variables whose indices i, j, h and k are not equal to 1. Remark that the support of G_1 does not encounter the indices of the variables of $\overline{Y}_1$. Consider $G'_{r=2,\ldots,n-2}$, the projections of the linear forms of $G_{r=2,\ldots,n-2}$ on the coordinates with all indices i, j, h and k different from 1 (coordinates

of variables of $\overline{Y}_1$). These projections are the following forms $c'_{ij\ k}(y) = \sum_{\substack{h=2 \\ h \neq i}}^{n} y_{ij\ hk}$ and $l'_{ij\ h}(y) = \sum_{\substack{k=2 \\ k \neq j}}^{n} y_{ij\ hk}$ in which the row indices and column indices range from 2 to n. Decrementing the row indices and column indices of each variable of $\bigcup_{r=2}^{n-2} G'_r$ that is to say applying the renumbering $y_{ij\,hk} \to y_{(i-1)(j-1)(h-1)(k-1)}$ to all the variables of the forms of $\bigcup_{r=2}^{n-2} G'_r$ we obtain S_{n-1} and conclude that $rank\left(\bigcup_{r=2}^{n-2} G'_r\right) = rank\left(S_{n-1}\right)$.
Consider M_1 the matrix whose columns are indiced by the variables Y_1 and $\overline{Y}_1$, and whose rows are the forms G_1 and $\bigcup_{r=2}^{n-2} G_r$. M_1 is block triangular $M_1 = \begin{pmatrix} G_1 & 0 \\ & \bigcup_{r=2}^{n-2} G'_r \end{pmatrix}$ and its rank is greater or equal to the sum of the ranks of the diagonal blocks. The rank of $\bigcup_{r=2}^{n-2} G'_r$ is equal to the rank of S_{n-1} and by recurrence is at least $(n-1)(n-2)(2n-5)-1$. Now let us prove that the rank of G_1 is at least $2(n-1)(3n-5)$.
Always for n≥4, consider the partition of the variables of Y_1 in Y_{11}, the set of variables with a row index i or h equal to 1, and $\overline{Y}_{11}$ the complementary set (of Y_1) that is to say the set of variables whose row indices i and h are not equal to 1 (therefore a column j or k is equal to 1). Remark that the support of F_1 does not encounter the indices of variables of $\overline{Y}_{11}$. Consider $\overline{F}_1'$ the projections of the linear forms of $\overline{F}_1$ on the coordinates with row indices i and h different from 1 (coordinates of variables of $\overline{Y}_{11}$). These projections are the following forms $c'_{ij\ k}(y) = \sum_{\substack{h=2 \\ h \neq i}}^{n} y_{ij\ hk}$ and $l'_{i1\ h}(y) = l_{i1\ h}(y)$ in which the row indices range from 2 to n. The rank of the linear forms of G_1 is equal to the rank of the matrix $M_{11} = \begin{pmatrix} F_1 & 0 \\ & \overline{F}_1' \end{pmatrix}$.
Therefore $rank\left(G_1\right) \geq rank\left(F_1\right) + rank\left(\overline{F}_1'\right)$.
α) Applying Lemma 2 with m:=n we obtain that $rank\left(F_1\right) \geq (n-1)(2n-3)+(n-2)+(n-1)^2$.
β) Decrementing the row indices of each variable of $\overline{F}_1'$ that is to say applying the renumbering $y_{ij\,hk} \to y_{(i-1)j(h-1)k}$ to all the variables of the forms of $\overline{F}_1'$ we obtain $\overline{F}$ of Lemma 3 with m:=n-1. Applying Lemma 3 with m:=n-1 we obtain that $rank\left(\overline{F}_1'\right) \geq (n-2)(2n-4)+(n-2)+(n-2)(n-1)$.
Therefore

$$
\begin{aligned}
rank\left(G_1\right) &\geq (n-1)(2n-3)+(n-2)+(n-1)^2+(n-2)(2n-4)+(n-2)+(n-2)(n-1) \\
&= 2(n-1)(3n-5)
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
rank\left(S_n\right) &\geq (n-1)(n-2)(2n-5)-1+2(n-1)(3n-5) \\
&= n(n-1)(2n-3)-1
\end{aligned}
$$

□

REMARK 2. The cardinality of the set S_n is $n(n-1)(2n-3)-1$. This can be verified by the following recurrence.
The cardinality of S_3 is 17.
For n≥4,

$$
\begin{aligned}
card\left(S_n\right) &= card\left(G_1\right) + \sum_{r=2}^{n-2} card\left(G_r\right) = card\left(G_1\right) + card\left(S_{n-1}\right) \\
&= card\left(G_1\right) + (n-1)(n-2)(2n-5)-1 \quad \text{(applying the hypothesis of recurrence)} \\
&= card\left(F_1\right) + card\left(\overline{F}_1\right) + (n-1)(n-2)(2n-5)-1 \\
&= 2(n-1)(3n-5)+(n-1)(n-2)(2n-5)-1 = n(n-1)(2n-3)-1
\end{aligned}
$$

As $rank\left(S_n\right) \leq card\left(S_n\right)$, Lemma 4 could be stated with a strict equality.

LEMMA 5.
For $n \geq 3$, the rank of the system

$$\begin{cases} c_j(x) := \sum_{i=1}^{n} x_{ij} = 1 & j = 1,\ldots,n \\ l_i(x) := \sum_{j=1}^{n} x_{ij} = 1 & i = 1,\ldots,n-1 \\ c_{ijk}(y) = x_{ij} & \forall c_{ijk}(y) \in S_n \\ l_{ijh}(y) = x_{ij} & \forall l_{ijh}(y) \in S_n \end{cases}$$

is equal to $2(n-1) + n(n-1)(2n-3)$ and the system does not contain a redundant equality.

PROOF.

The system can be written: $\begin{cases} c_j(x) & & = 1 & j = 1,\ldots,n \\ l_i(x) & & = 1 & i = 1,\ldots,n-1 \\ c_{ij\,k}(y) & -x_{ij} & = 0 & \forall c_{ijk}(y) \in S_n \\ l_{ij\,h}(y) & -x_{ij} & = 0 & \forall l_{ijh}(y) \in S_n \end{cases}$

It appears a block triangular structure and the rank of the system is greater or equal to the sum of the rank of the linear forms $\left\{c_j, l_i \quad j = 1,\ldots,n,\ i = 1,\ldots,n-1\right\}$ and the rank of S_n.
The rank of the linear forms $\left\{c_j, l_i \quad j = 1,\ldots,n,\ i = 1,\ldots,n-1\right\}$ is equal to $2n$-1 (Result 1).
The rank of S_n is at least $n(n-1)(2n-3)-1$ (Lemma 4).
Therefore the rank of the system is at least $2(n-1) + n(n-1)(2n-3)$.

The number of equalities of the system is equal to $2(n-1) + n(n-1)(2n-3)$ (use Remark 2) and finally the system is of full rank.

$\square$

THEOREM 1.
For $n \geq 3$, $\dim(P_n) = \frac{1}{2}n(n-1)(n-2)(n-3) + (n-1)^2 + 1$ and

$$Aff(P_n) = M_n := \left\{ (x,y) : \begin{bmatrix} c_j(x) := \sum_{i=1}^{n} x_{ij} = 1 & j = 1,\ldots,n \\ l_i(x) := \sum_{j=1}^{n} x_{ij} = 1 & i = 1,\ldots,n-1 \\ c_{ijk}(y) = x_{ij} & \forall c_{ijk}(y) \in S_n \\ l_{ijh}(y) = x_{ij} & \forall l_{ijh}(y) \in S_n \end{bmatrix} \right\}$$

PROOF.
Let us denote by $A(x,y) = b$ the equalities that appear in the expression of M_n and by m the total number of variables minus the rank of A. From Lemma 5 we have:

$$m = \left(n^2 + \tfrac{n^2(n-1)^2}{2}\right) - \left(2(n-1) + n(n-1)(2n-3)\right) = \frac{1}{2}n(n-1)(n-2)(n-3) + (n-1)^2 + 1$$

$$Q_n \subseteq M_n \Rightarrow Aff(Q_n) = Aff(P_n) \subseteq M_n$$

Therefore $\dim(P_n) \leq \dim(M_n) = \left(n^2 + \tfrac{n^2(n-1)^2}{2}\right) - rank(A) = m$

From Corollary of Lemma 1 we observe that $m \leq \dim(P_n)$.
Therefore $\dim(P_n) = \dim(M_n) = m$.

By inclusion, we deduce that $Aff\left(P_n\right) = M_n$.

∎

REMARK 3. Consider M'_n the affine set defined as the solutions of the system of equalities used in the description of P'_n, the polyhedron used by Resende, Ramakrishnan and Drezner (1995) (see Section 1), that is to say:

$$M'_n := \left\{ (x,y) : \begin{bmatrix} \sum_{i=1}^{n} x_{ij} = 1 & j = 1,\ldots,n \\ \sum_{j=1}^{n} x_{ij} = 1 & i = 1,\ldots,n \\ \sum_{\substack{h=1 \\ h \neq i}}^{n} y_{ij\,hk} = x_{ij} & i,j,k = 1,\ldots,n,\ k \neq j \\ \sum_{\substack{k=1 \\ k \neq j}}^{n} y_{ij\,hk} = x_{ij} & i,j,h = 1,\ldots,n,\ h \neq i \end{bmatrix} \right\}$$

The equalities used to describe M_n is a subset of those used to describe M'_n , therefore $P_n \subseteq M'_n \subseteq M_n$. Then from Theorem 1 $Aff\left(P_n\right) = M_n = M'_n$. Resende and al. have therefore described $Aff\left(P_n\right)$ with $2n + 2n^2(n-1)$ equalities. Theorem 1 tells us that we can describe it with $2(n-1) + n(n-1)(2n-3)$ equalities which provides a gain of $3n(n-1)+2$ equalities. Moreover this description is minimal, no redundant equality being used (Lemma 5).

4. A facets family

First we state two useful principles which follow from the symmetry properties of the quadratic assignment problem.

Let $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} c_{ij\,hk}y_{ij\,hk} \geq 0$ be a valid inequality (resp. a facet) for P_n then:

PRINCIPLE 1: renumbering rows and columns.
$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{L(i)C(j)} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} c_{ij\,hk}y_{L(i)C(j)\,L(h)C(k)} \geq 0$ is a valid inequality (resp. a facet) for P_n, where L and C are two permutations of $\{1,\ldots,n\}$.

PRINCIPLE 2: transposition of rows and columns.
$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{h=1\\h\neq i}}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} c_{ij\,hk}y_{ji\,kh} \geq 0$ is a valid inequality (resp. a facet) for P_n.

DEFINITION 2.
Consider $n \geq 4$.
Let i, j, h, k $i \neq h$ and $j \neq k$ (i, h row indices and j, k column indices)
Let l $l \neq i$ and $l \neq h$ (l row index)
Let $B = \{1,\ldots,n\} - \{j,k\}$
Let $C \subseteq B$ and define the inequality $I_C \equiv \sum_{c \in C} y_{ij\,lc} \leq y_{ij\,hk} + \sum_{c \in C}\sum_{\substack{b \in B\\b\neq c}} y_{lc\,hb}$

In the following (up to Definition 3), if no other precision is given, i, j, h, k, l will refer to Definition 2.

PROPOSITION 1.
I_C is a valid inequality for P_n.

PROOF.
From the definition of P_n, it is sufficient to verify the proposition on Q_n (the vertices of P_n). Let $(x,y)\in Q_n$, let us show that (x,y) verifies I_C. First remark that the right member of the inequality is non negative. Moreover one, at most, variable of the left member can take the value 1 (because of the constraint $\sum_{c=1}^{n} x_{lc} = 1$). It suffices to verify that in this case (left member not zero) the right member is equal to 1. Suppose now that the left member is not zero. Then there exist one (and only one) column $c\in C$ such that $x_{ij} = x_{lc} = 1$, this implies $x_{hj} = x_{hc} = 0$ and $\sum_{k'=1}^{n} x_{hk'} = \sum_{\substack{b\in B\\ b\neq c}}^{n} x_{hb} + x_{hk} = 1$ since $B\cup\{k\}-\{c\} = \{1,\ldots,n\}-\{j,c\}$ and finally (x,y) satisfies $x_{lc}\sum_{\substack{b\in B\\ b\neq c}}^{n} x_{hb} + x_{ij}x_{hk} = \sum_{\substack{b\in B\\ b\neq c}}^{n} y_{lc\,hb} + y_{ij\,hk} = 1$.

□

PROPOSITION 2.
The vertices of P_n which saturate I_C (i.e. satisfy I_C with equality) satisfy one of the 5 following conditions:

$$\begin{cases} - \ (x_{ij}=1)\wedge(\exists c\in C\ x_{lc}=1) \\ - \ (x_{ij}=0)\wedge(\forall c\in C\ x_{lc}=0) \\ - \ (x_{ij}=0)\wedge(x_{hk}=1) \\ - \ (x_{hk}=0)\wedge(\forall c\in C\ x_{lc}=0) \\ - \ x_{hj}=1 \end{cases}$$

PROOF.
The vertices of P_n saturate I_C in two manners: either the left member is equal to 1 and then the right member is just equal to 1 (see the proof of Proposition 1), or the right member is zero. This leads to 2 conditions:
- the vertices s.t. $x_{(n-1)1} = 1 \wedge \exists c\in C\ x_{nc} = 1$ give the value 1 at the left member of I_C
- the vertices s.t. $(x_{ij}=0 \vee x_{hk}=0)\wedge(\forall c\in C\ x_{lc}=0 \vee \forall b\in B\ x_{hb}=0)$ give the value 0 at the right member of I_C . Since the vertices satisfy the constraint $\sum_{k'=1}^{n} x_{hk'} = 1$ the condition $\forall b\in B\ x_{hb}=0$ can be written $x_{hj}+x_{hk}=1$ that is to say $x_{hj}=1$ or $x_{hk}=1$. Now if we developp $(x_{ij}=0 \vee x_{hk}=0)\wedge(\forall c\in C\ x_{lc}=0 \vee x_{hj}=1 \vee x_{hk}=1)$ in taking into account that the vertices of P_n satisfy $x_{hj}=1 \Rightarrow x_{ij}=0 \wedge x_{hk}=0 \quad (i\neq h, k\neq j)$, we obtain the announced result.

□

THEOREM 2.
Let $n\geq 4$. If $0\leq |C|\leq n-3$ then I_C induces a facet of P_n.

PROOF.
Without loss of generality (Principle 1) let us choose $(i,j)=(n-1,1)$, $(h,k)=(1,2)$, $l=n$, $B=\{3,4,\ldots,n\}$, $C\subseteq\{4,5,\ldots,n\}$ and prove that $I_C \equiv \sum_{c\in C} y_{(n-1)1nc} \leq y_{(n-1)112} + \sum_{c\in C}\sum_{\substack{b\in B\\ b\neq c}} y_{nc1b}$ induces a facet of P_n. Let $\overline{C}=\{4,\ldots,n\}-C$.

We have to exhibit $\dim(P_n)$ linearly independant vertices of P_n that saturate I_C (i.e. I_C induces a face of dimension $\dim(P_n)-1$). To do this, let us consider the matrices M_1, M_{2a} ,..., M_{2d} built in the proof of Lemma 1 and apply the following modifications in the construction of M_{2c} and M_{2d} (see the example figure 3).

As before, in Π_{2c} and Π_{2d}, for each of the specified conditions, we keep one and only one vertex satisfying it.

The set Π_{2c}^2 is now divided into two sets:

- $\Pi_{2c\alpha}^2$ the vertices of P_n s.t. $x_{i1} = x_{12} = x_{nk} = 1 \quad i \in \{2,\ldots,n-1\},\ k \in C$
- $\Pi_{2c\beta}^2$ the vertices of P_n s.t. $x_{i1} = x_{13} = x_{nk} = 1 \quad i \in \{2,\ldots,n-1\},\ k \in \overline{C}$

Note that this does not alter the rank of M_{2c} .

In V_{2d} , the variables $y_{1j\,nk} \quad j \in \{3,\ldots,n\}, k \in \{4,\ldots,n\} \setminus \{j\}$ are now divided into two groups
$\begin{cases} y_{1j\,nk} & j \in \{3,4,\ldots,n\}, k \in C \setminus \{j\} \\ y_{1j\,nk} & j \in \{2,4,\ldots,n\}, k \in \overline{C} \setminus \{j\} \end{cases}$

We then build $V_{2d} = V_{2d\alpha} \cup V_{2d\beta}$ with

- $V_{2d\alpha}$ the variables $\begin{cases} y_{1j\,nk} & j \in \{3,\ldots,n\}, k \in C \setminus \{j\} \\ y_{1j\,n3} & j \in \{4,\ldots,n-1\} \end{cases}$
- $V_{2d\beta}$ the variables $y_{1j\,nk} \quad j \in \{2,4,\ldots,n\}, k \in \overline{C} \setminus \{j\}$

The resulting matrix *M* is built with "old" points and variables (i.e. the ones used in the proof of Lemma 1) and "new" points and variables (i.e. the ones introduced here). Let us show that *M* is again block triangular.

- As in the proof of Lemma 1 one can verify that the support of $\Pi_1 \cup \Pi_{2a} \cup \Pi_{2b}$ does not encounter the indices of the "new" variables of V_{2d} i.e. $V_{2d\beta}$.

- Let us verify that the supports of the "old" points of Π_{2c} i.e. $\Pi_{2c}^1 \cup \Pi_{2c\alpha}^2$ do not encounter the indices of the "new" variables of V_{2d} i.e. $V_{2d\beta}$:
the vertices of P_n satisfy $x_{nk} = 1 \ k \in C \cup \{3\} = 1 \Rightarrow x_{nk} = 0 \ k \in \overline{C}$. Therefore the support of $\Pi_{2c}^1 \cup \Pi_{2c\alpha}^2$ does not encounter the indices of the variables of $V_{2d\beta}$.

- Let us verify that the supports of the "new" points of Π_{2c} i.e. $\Pi_{2c\beta}^2$ do not encounter the indices of the variables of V_{2d}:
the vertices of P_n satisfy $x_{13} = 1 \Rightarrow x_{1j} = 0 \ j \in \{2,4,\ldots,n\}$. Therefore the support of $\Pi_{2c\beta}^2$ does not encounter the indices of the variables of $V_{2d\beta}$;
the vertices of P_n satisfy $x_{nk} = 1 \ k \in \overline{C} \Rightarrow x_{nk} = 0 \ k \in C \cup \{3\}$. Therefore the support of $\Pi_{2c\beta}^2$ does not encounter the indices of the variables of $V_{2d\alpha}$.

Now let us modify Π_{2d} in such a way that the rank of M_{2d} remains identical.
The vertices of Π_{2d} s.t. $x_{1j} = x_{nk} = 1 \ j \in \{3,\ldots,n\}, k \in \{4,\ldots,n\} \setminus \{j\}$ are divided into two groups
$\begin{cases} x_{1j} = x_{nk} = x_{(n-1)1} = 1 & j \in \{3,4,\ldots,n\}, k \in C \setminus \{j\} \\ x_{1j} = x_{nk} = x_{(n-2)1} = 1 & j \in \{2,4,\ldots,n\}, k \in \overline{C} \setminus \{j\} \end{cases}$

We then build

- Π_{2d} the vertices of P_n s.t. $\begin{cases} x_{1j} = x_{nk} = x_{(n-1)1} = 1 & j \in \{3,4,\ldots,n\}, k \in C \setminus \{j\} \\ x_{1j} = x_{n3} = 1 & j \in \{4,\ldots,n-1\} \\ x_{1j} = x_{nk} = x_{(n-2)1} = 1 & j \in \{2,4,\ldots,n\}, k \in \overline{C} \setminus \{j\} \end{cases}$

M_{2d} is again equal to the identity matrix of order $(n-3)^2 + (n-4)$ (use the same construction as in Lemma 1).

In Π_{2d} we will distinguish $\Pi_{2d\alpha}$, $\Pi_{2d\beta}$, $\Pi_{2d\gamma}$ corresponding respectively to the first, the second and the third line in the above definition.

In Π^1_{2c} let us distinghuish:

- $\Pi^1_{2c\alpha}$ the vertices of P_n s.t. $x_{i1} = x_{n3} = x_{12} = 1 \quad i \in \{2,\ldots,n-1\}$
- $\Pi^1_{2c\beta}$ the vertices of P_n s.t. $x_{i1} = x_{n3} = x_{1n} = x_{h2} = 1 \quad i \in \{2,\ldots,n-1\}\ h \in \{2,\ldots,n-1\} \setminus \{i\}$

The vertices of P_n which saturate I_C must satisfy one of the 5 conditions of Proposition 2. This the case for all the vertices used to build M except for the point of $\Pi^1_{2c\alpha}$ with $i = n-1$ i.e. s.t. $x_{(n-1)1} = x_{n3} = x_{12} = 1$ (see table 1). The ranks of the blocks $M_1, M_{2a}, \ldots, M_{2d}$ being unchanged (compared to Lemma 1), I_C induces a facet.

$\exists c \in C\ x_{(n-1)1} = x_{nc} = 1$	$\forall c \in C\ x_{(n-1)1} = x_{nc} = 0$	$x_{(n-1)1} = 0 \wedge x_{12} = 1$	$\forall c \in C\ x_{nc} = x_{12} = 0$	$x_{11} = 1$
$\Pi^2_{2c\alpha}$ s.t. $i = n-1$	Π_{2a}	$\Pi^1_{2c\alpha}$ s.t. $i \neq n-1$	Π_{2b}, $\Pi^1_{2c\beta}$	Π_1
$\Pi_{2d\alpha}$	$\Pi_{2d\gamma}$	$\Pi^2_{2c\alpha}$ s.t. $i \neq n-1$	$\Pi^2_{2c\beta}$, $\Pi_{2d\beta}$	

table 1: correspondences between conditions of Proposition 2 and points satisfying them.

For $n = 4$, the figure 3 illustrates the modified matrix M in the case $C = \emptyset$, $\overline{C} = \{4\}$ i.e. $I_C \equiv 0 \leq y_{3112}$. In the case $C = \{4\}$, $\overline{C} = \emptyset$ i.e. $I_C \equiv y_{3144} \leq y_{3112} + y_{4413}$ the matrix M is unchanged and is given in figure1.

$$
\begin{array}{c|cccccc|ccc|ccc|cccc|cc|c}
 & \multicolumn{6}{c|}{\Pi_1} & \multicolumn{3}{c|}{\Pi_{2a}} & \multicolumn{3}{c|}{\Pi_{2b}} & \multicolumn{6}{c|}{\Pi_{2c}} & \Pi_{2d} \\
\hline
y_{2233} & 1&0&0&0&0&0 & 0&0& & 0&0& & 0&0&0&0 & 0&0 & 0 \\
y_{2234} & 0&1&0&0&0&0 & \vdots&\vdots& & \vdots&\vdots& & \vdots&\vdots&\vdots&\vdots & \vdots&\vdots & \vdots \\
y_{2332} & 0&0&1&0&0&0 & &&& &&& &&&& && \\
y_{2334} & 0&0&0&1&0&0 & 1&&& &&& &&&& && \\
y_{2432} & 0&0&0&0&1&0 & 0&&& &&& &&&& && \\
y_{2433} & 0&0&0&0&0&1 & 0&1& & 0&0& & 0&0&0&0 & 0&0 & 0 \\
\vdots & &&\cdots&&& & &&& &&& &&&& && \\
\hline
y_{1223} & &&&&& & 1&0& & 0&0& & 0&0&0&0 & 0&0 & 0 \\
y_{1224} & &&&&& & 0&1& & \vdots&\vdots& & \vdots&\vdots&\vdots&1 & \vdots&\vdots & 0 \\
y_{1233} & &&0&&& & 0&1&0 & &&& &&&0 & && 1 \\
y_{1234} & &&&&& & 1&0& & 0&0& & 0&1&0&0 & 0&0 & 0 \\
\vdots & &&&&& & 0&&\ddots & &&& &&&& && \\
\hline
y_{2113} & &&&&& & &&& 1&0& & 0&0&0&0 & 1&0 & 0 \\
y_{2114} & &&&&& & &&& 0&1& & 1&\vdots&\vdots&\vdots & 0&\vdots & 0 \\
y_{2133} & &&0&&& & &0& & 0&1&0 & 0&&& & 0& & 1 \\
y_{2134} & &&&&& & &&& 1&0& & 0&1&0&0 & 0&0 & 0 \\
\vdots & &&&&& & &&& 0&&\ddots & &&&& && \\
\hline
y_{2132} & &&&&& & &&& &&& 1&0&0&0 & 1&0 & 0 \\
y_{2112} & &&&&& & &&& &&& 0&1&0&0 & 0&0 & 1 \\
y_{3122} & &&0&&& & &0& & &0& & 0&0&1&0 & 0&1 & 0 \\
y_{3112} & &&&&& & &&& &&& 0&0&0&1 & 0&0 & 0 \\
y_{2144} & &&&&& & &&& &&& &0&& & 1&0 & 1 \\
y_{3144} & &&&&& & &&& &&& &&& & 0&1 & 0 \\
\hline
y_{1244} & &&0&&& & &0& & &0& & &0&& & & & \boxed{1} \\
 & &&&&& & &&& &&& &&&\uparrow & &&
\end{array}
$$

<u>figure3</u>

Matrix M for n=4 and $C = \varnothing$, $\overline{C} = \{4\}$ (proof of Theorem 2)

The arrow indicates the only vertex which does not saturate $I_C \equiv 0 \le y_{3112}$

$\square$

REMARK 4. The number of subsets of $B = \{1,\ldots,n\} - \{j,k\}$ of size strictly smaller than $n-2$ is $2^{n-2} - 1$. So the total number of inequalities I_C inducing facets is $\Omega\left(2^{n-2}\right)$.

REMARK 5.
Let n≥4. If $|C| = n-2$ then I_C is the sum of all inequalities $y_{i'jh'k} \ge 0$ $i' \notin \{i,h,l\}$, $h' \notin \{h,l,i'\}$.

PROOF.
Renumbering rows and columns, we can always choose $(i,j) = (n-1,1)$, $(h,k) = (1,2)$, $l = n$. Then $C = B = \{3,4,\ldots,n\}$. Using equalities satisfied by the points of P_n (see Remark 3 and Definition 1 with m=n), we state:

$\sum_{i=2}^{n-2}\sum_{\substack{h=2\\h\neq i}}^{n-1} y_{i1h2} = \sum_{i=2}^{n-2}\left(x_{i1} - y_{i112} - y_{i1n2}\right)$ using the equalities $c_{i12}(y) = x_{i1}$

$= \left(1 - x_{11} - x_{(n-1)1} - x_{n1}\right) + \left(y_{(n-1)112} + y_{n112} - x_{12}\right) + \left(y_{(n-1)1n2} + y_{11n2} - x_{n2}\right)$ using the equalities $c_1(x) = 1$, $c_{121}(y) = x_{12}$, $c_{n21}(y) = x_{n2}$

$= y_{(n-1)112} + \left(1 - x_{n1} - x_{n2}\right) + \left(y_{n112} - x_{12}\right) + \left(y_{(n-1)1n2} - x_{(n-1)1}\right) + \left(y_{11n2} - x_{11}\right)$ reordering the terms

$= y_{(n-1)112} + \sum_{c=3}^{n} x_{nc} - \sum_{c=3}^{n} y_{nc12} - \sum_{c=3}^{n} y_{(n-1)1nc} - \sum_{c=3}^{n} y_{11nc}$ using the equalities $l_n(x) = 1$, $l_{12n}(y) = x_{12}$, $l_{(n-1)1n}(y) = x_{(n-1)1}$, $l_{11n}(y) = x_{11}$

$= y_{(n-1)112} + \sum_{c=3}^{n} \sum_{\substack{b=3 \\ b \neq c}}^{n} y_{nc1b} - \sum_{c=3}^{n} y_{(n-1)1nc}$ using the equalities $l_{nc1}(x) = x_{nc}$

Adding the non-negativity condition to this last one term, we obtain I_C .

□

DEFINITION 3 and THEOREM 3.
Consider $n \geq 4$.
Let i, j, h, k $i \neq h$ and $j \neq k$ (i, h row indices and j, k column indices)
Let l $l \neq j$ and $l \neq k$ (l column index)
Let $B = \{1,\ldots,n\} - \{i,h\}$
Let $L \subseteq B$ then $I_L \equiv \sum_{c \in L} y_{ij\,cl} \leq y_{ij\,hk} + \sum_{c \in L} \sum_{\substack{b \in B \\ b \neq c}} y_{cl\,bk}$ is a valid inequality for P_n and induces a facet if $0 \leq |L| \leq n-3$.

PROOF. Apply Principle 2 to I_C.

□

REMARK 6.
For n =3 the inequalities $y_{ij\,hk} \geq 0$ are sufficient to describe the quadratic assignment polytope, more precisely one has:

$$P_3 = P_3' := M_3 \cap \left\{(x,y) \colon y_{ij\,hk} \geq 0 \;\; i,j,h,k = 1,2,3 \;\; i \neq h \;\; j \neq k\right\}$$

PROOF.
One has: $P_3' \subseteq M_3 = Aff(P_3)$, the equality resulting from Theorem 1. Let $(x,y) \in P_3'$ then $(x,y) \in Aff(P_3) = Aff\left(\left\{V_{p=1,\ldots,6}\right\}\right)$ where $\left\{V_{p=1,\ldots,6}\right\}$ are the 6 vertices of P_3. Therefore $\exists \lambda_{p=1,\ldots,6}$ s.t. $\sum_{p=1}^{6} \lambda_p = 1$ and $(x,y) = \sum_{p=1}^{6} \lambda_p V_p$. The matrix whose columns are $\left\{V_{p=1,\ldots,6}\right\}$ contains as sub-matrix the identity of order 6: for example extract the rows corresponding to the variables $\left\{y_{1122}, y_{1123}, y_{1221}, y_{1223}, y_{1321}, y_{1322}\right\}$. Therefore each λ_p is equal to one of these variables and consequently λ_p is not negative. It follows that $P_3' \subseteq Conv\left(\left\{V_{p=1,\ldots,6}\right\}\right) = P_3$.
The converse inclusion is verified because $\left\{V_{p=1,\ldots,6}\right\} \subseteq P_3'$.

□

Since the number of inequaltities I_C and I_L inducing facets is exponential (see Remark 4), from a practical point of view they cannot be put all together into a linear program. That suggests a cut generation algorithm with the associated separation problem: given $(x,y) \in M_n$ (the affine hull of P_n) find the most violated facet by (x,y).
First consider the separation problem for the I_C facets. Given the distinct row indices i, h, l, and the distinct column indices j, k, the separation problem consists to find the set of columns $C \subseteq B = \{1,\ldots,n\} - \{j,k\}$ that gives the most violated facet. This problem can be stated as the following 0-1 linear program:

$\min \sum_{c \in B} q_c z_c$ subject to $\sum_{c \in B} z_c \leq n-3$, $z \in \{0,1\}^{|B|}$

where $q_c = \sum_{\substack{b \in B \\ b \neq c}} y_{lc\,hb} - y_{ij\,lc}$ and z_c is a boolean variable equal to 1 if and only if the column c is in the set C.

Let z^* be the optimal solution of this problem. If $\sum_{c \in B} q_c z_c^* + y_{ij\,hk} \geq 0$ then there is no I_C facets violated by (x, y), otherwise the facet I_C with $C = \left\{c \in B\colon z_c^* = 1\right\}$ is violated by (x, y).

The 0-1 linear program is easy to solve. If $q_c \geq 0$ then $z_c^* = 0$. Let $N = \left\{c \in B\colon q_c < 0\right\}$; if $|N| > n - 3$ then $z_{c_{\max}}^* = 0$ where $c_{\max}$ satisfies $q_{c_{\max}} = \max\left\{q_{c \in N}\right\}$ and $z_c^* = 1$ for $c \in N - \left\{c_{\max}\right\}$ the n-3 other elements of N, otherwise if $|N| \leq n - 3$ then $z_c^* = 1$ $\forall c \in N$. Therefore the solution is obtained in $O(n)$ time.

As (x, y) belongs to the affine hull of P_n, it satisfies the equalities $l_{lc\,h}(y) = x_{lc}$ $(c \in C)$ (see Remark 3 and Definition 1 with m=n) and then $q_c = x_{lc} - y_{lc\,hj} - y_{lc\,hk} - y_{ij\,lc}$ $(c \in C)$. Therefore the q_c $(c \in B)$ can be computed in $O(n)$ time.

Since the inputs and the solution of the 0-1 linear program are computed in $O(n)$ time, we finally obtain by enumerating all the values of the indices i, j, h, k, l, a separation algorithm in $O\left(n^6\right)$ time.

The most violated I_L facet can be found by a similar algorithm.

5. Conclusion

We have initiated the study of the quadratic assignment polytope and have obtained the following results:

- the minimal description of the affine hull of the polytope. Compared to the linearization of Resende, Ramakrishnan and Drezner (1995), this description enables us to save $\Omega\left(n^2\right)$ constraints;
- the definition of a family of $\Omega\left(2^{n-2}\right)$ inequalities inducing facets. This family includes the classical non negativity constraints of the variables y and new inequalities which seem to be never used in any linearization.

It would be interesting to evaluate, from an efficiency point of view, the consequence of these results for solving QAP or at the very least for obtaining lower bounds. We have proposed a polynomial-time algorithm in $O\left(n^6\right)$ for determining the most violated facet. In order to build a cut generation algorithm it would be perhaps better to find a "good" violated facet by an heuristic method less time consuming.

References

W.P. Adams, T.A. Johnson. *Improved linear programming-based lower bounds for the quadratic assignment problem*. DIMACS Series in Discrete Mathematics and Theorical Computer Science 16 (1994) 43-75.

W.P. Adams, H.D. Sherali. *A tight linearization and an algorithm for zero-one quadratic programming problems*. Management Science 32 n°10 (1986) 1274-1290.

A.A. Assad, W. Xu. *On lower bounds for a class of quadratic 0-1 programs*. Operations Research Letters 4 n°4 (1985) 175-180.

M.O. Ball, B.K. Kaku, A. Vakhutinsky. *Network-based formulations of the quadratic assignment problem*. European Journal of Operational Research 104 (1998) 241-249.

M.S. Bazaraa, H.D. Sherali. *Benders' partitioning scheme applied to a new formulation of the quadratic assignment problem*. Naval Research Logistics Quaterly 27 (1980) 29-41.

E. Burkard, T. Bönniger. *A heuristic for quadratic Boolean programs with applications to quadratic assignment problems*. European Journal of Operational Research 13 (1983) 374-386.

P. Carraresi, F. Malucelli. *A new lower bound for the quadratic assignment problem*. Operations Research 40 (Supplement 1) (1992) S22-S27.

N. Christofides, E. Benavent. *An exact algorithm for the quadratic assignment problem on a tree*. Operations Research 37 n°5 (1989) 760-768.

N. Christofides, A. Mingozzi, P. Toth. *Contributions to the quadratic assignment problem*. European Journal of Operational Research 4 (1980) 243-247.

A.M. Frieze, J. Yadegar. *On the quadratic assignment problem*. Discrete Applied Mathematics 5 (1983) 89-98.

P.C. Gilmore. *Optimal and suboptimal algorithms for the quadratic assignment problem*. SIAM J. Appl. Math. 10 (1962) 305-313.

B.K. Kaku, G.L. Thompson. *An exact algorithm for the general quadratic assignment problem*. European Journal of Operational Research 23 (1986) 382-390.

L. Kaufman, F. Broeckx. *An algorithm for the quadratic assignment problem using Benders' decomposition*. European Journal of Operational Research 2 (1978) 207-211.

E.L. Lawler. *The quadratic assignment problem*. Management Science 9 (1963) 586-599.

D.G. Luenberger. *Linear and nonlinear programming*. Second edition. Addison-Welsey publishing compagny (1989).

M. Marcus, H. Minc. *A survey of matrix theory and matrix inequalities*. Allyn and Bacon, Inc., Boston (1964).

T. Mautor, C. Roucairol. *A new exact algorithm for the solution of quadratic assignment problems*. Discrete Applied Mathematics 55 (1994) 281-293.

M.G.C. Resende, K.G. Ramakrishnan, Z. Drezner. *Computing Lower Bounds for the Quadratic assignment problem with an interior point algorithm for linear programming*. Operations Research Vol.43, n°5, September-October (1995) 781-791.

M.J. Rosenblatt, B. Golany. *A distance assignment approach to the facility layout problem*. European Journal of Operational Research 57 (1992) 253-270.