Bicolored matchings in some classes of graphs

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1 Introduction

Given a graph $G = (V, E)$ and a set $\mathcal{P} = \{p_o, p_1, \ldots, p_s\}$ of integers $0 \leq p_o < p_1 < \ldots < p_s \leq \lfloor |V|/2 \rfloor$, we want to color a subset $R \subseteq E$ of edges of G, say in red, in such a way that for any i $(0 \leq i \leq s)$ G contains a maximum matching M_i with exactly p_i red edges, i.e., $|M_i \cap R| = p_i$. We shall in particular be interested in finding a smallest subset R for which the required maximum matchings do exist.

A subset R will be $\mathcal{P} - feasible$ for G if for every p_i in $\mathcal{P}$ there is a maximum matching M_i in G with $|M_i \cap R| = p_i$. Notice that for some $\mathcal{P}$ there may be no $\mathcal{P}$-feasible set R (take $P = \{0, 1, 2\}$ in $G = K_{2,2}$).

2 Regular bipartite graphs

We will state some basic results concerning $\mathcal{P}$-feasible sets in regular bipartite graphs.

Proposition 2.1 *In a $\triangle$-regular bipartite graph G for any $\mathcal{P}$ with $|\mathcal{P}| \leq \triangle$ there exists a $\mathcal{P}$-feasible set R.*

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This follows from the fact that the edge set of G can be partitioned into $\triangle$ perfect (and hence maximum) matchings (König theorem).

Let us now briefly consider a special case for a $\triangle$-regular bipartite graph.

Theorem 2.2 *Let $G = (X, Y, E)$ with $|X| = |Y| = n$ be a $\triangle$-regular bipartite graph and let $\mathcal{P} = \{p, q\}$ with $1 \leq p < q \leq n$. The minimum cardinality of a $\mathcal{P}$-feasible set R is given by $|R| = q + \max\{0, p - n + |\mathcal{C}|/2\}$ where $\mathcal{C}$ is a collection of node disjoint cycles which are alternating with respect to a perfect matching and which have a minimum total length $|\mathcal{C}|$ satisfying $|\mathcal{C}|/2 \geq q - p$.*

Notice that if $p \geq q - 2$, we can use a single alternating cycle C instead of the family $\mathcal{C}$ since in any alternating cycle C we have $|C|/2 \geq 2 \geq q - p$.

Corollary 2.3 *Let $G = (X, Y, E)$ with $|X| = |Y| = n$ be a $\triangle$-regular bipartite graph and let $\mathcal{P} = \{q - a, q\}$ with $1 \leq q \leq n$ and $1 \leq a \leq 2$. The minimum cardinality of a $\mathcal{P}$-feasible set R is given by*

$$\min |R| = q + \max\{0, q - n + |C|/2 - a\}$$

where C is a shortest cycle which is alternating with respect to some maximum matching in G.

Surprisingly the complexity of finding in a graph G a shortest possible alternating cycle with respect to some maximum matching (not given) is unknown even if G is a 3-regular bipartite graph. For reference purposes, this problem will be called the SAC problem (Shortest Alternating Cycle); it is formally defined as follows :

INSTANCE : a graph $G = (V, E)$ and a positive integer $L \leq |V|$

QUESTION : is there a maximum matching M and a cycle C with $|C| \leq L$ and $|C \cap M| = \frac{1}{2}|C|$?

Notice that the problem is easy if either a cycle C or a perfect matching M is given.

We give a sufficient condition for a regular graph $G = (X, Y, E)$ with $|X| = |Y| = n$ to have a $\mathcal{P}$-feasible set R with $|R| = n + 1$ for $\mathcal{P} = \{0, 1, \ldots, n\}$.

Theorem 2.4 *Let $G = (X, Y, E)$ be a $\triangle$-regular simple bipartite graph with $|X| = |Y| = n \geq 4$ and $\triangle \geq \frac{1}{2}(n + 2\lceil \frac{n}{4} \rceil + 1)$. Let $\mathcal{P} = \{0, 1, \ldots, n\}$; then there exists a $\mathcal{P}$-feasible set R with $|R| = n + 1$.*

A tedious but not difficult enumeration of cases shows the following:

Theorem 2.5 *For a 3-regular bipartite graph $G = (X, Y, E)$ with $|X| = |Y| = n \leq 7$, there exists a set $R \subseteq E$ with $|R| \leq n + 2$ which is $\mathcal{P}$-feasible for*

$\mathcal{P} = \{0, 1, \ldots, n\}$.

This result is best possible in the sense that there exists a bipartite 3-regular graph on $2n = 14$ nodes for which the minimum value of $|R|$ is $n + 2 = 9$; this is the so-called Heawood graph (or $(3, 6)$-cage).

In 3-regular bipartite graphs $G = (X, Y, E)$ with $|X| = |Y| = n \geq 8$ the minimum cardinality of a $\mathcal{P}$-feasible set R for $\mathcal{P} = \{0, 1, \ldots, n\}$ is not known.

Finally if we restrict $\mathcal{P}$ to $\{0, 1, \ldots, p\}$ with $p \leq 4$, we can state the following:

Theorem 2.6 *For $p \leq 4$ and for a 3-regular bipartite graph $G = (X, Y, E)$, with $|X| = |Y| = n \geq 2(p - 1)$ there exists a set $R \subseteq E$ with $|R| = p$ which is $\mathcal{P}$-feasible for $\mathcal{P} = \{0, 1, \ldots, p\}$.*

3 The interval property (IP)

We consider the case where $\mathcal{P}$ is a set of consecutive integers and we will characterize graphs which have a property related to such a $\mathcal{P}$. We will denote by $\nu(G)$ the cardinality of a maximum matching in G.

A *cactus* is a graph where any two (elementary) cycles have at most one common node. A cactus is *odd* if all its (elementary) cycles are odd. Notice that a tree is a special (odd) cactus.

We shall say that G has property IP (interval property) if whenever there are maximum matchings M_k, M_ν in G with $|M_k \cap M_\nu| = k < \nu = \nu(G)$, there are also maximum matchings M_i with $|M_i \cap M_\nu| = i$ for $i = k, k+1, \ldots, \nu$. In other words, when G has property IP and there is some k and two maximum matchings M_k, M_ν with $|M_k \cap M_\nu| = k \leq \nu(G)$, then $R = M_\nu$ is $\mathcal{P}$-feasible for $\mathcal{P} = \{k, k + 1, \ldots, \nu = \nu(G)\}$ and clearly R has minimum cardinality. We define a *IP-perfect* graph G as a graph in which every partial subgraph has property IP.

Theorem 3.1 *G is an odd cactus $\Leftrightarrow$ G is IP-perfect*

It follows that if we want to find the largest sequence of consecutive integers $\mathcal{P} = \{p_o, p_1, \ldots, p_s\}$ such that a set $R = M_\nu$ is $\mathcal{P}$-feasible for an odd cactus G, we have to find in G two maximum matchings M_k, M_ν such that $|M_k \cap M_\nu|$ is minimum. Let us examine first the case of bipartite graphs (that include trees but not odd cacti).

Theorem 3.2 *If $G = (X, Y, E)$ is a bipartite graph, there exists a polynomial time algorithm to construct two maximum matchings M, M' with a minimum value of $|M \cap M'|$.*

Notice that in the case of trees we can design a more efficient algorithm (linear time). From Theorems 3.1 and 3.2 we can deduce.

Theorem 3.3 *If $G = (V, E)$ is a forest, we can determine in polynomial time a minimum k and a minimum set R of edges to be colored in red in such a way that for $i = k, k+1, \ldots, \nu(G)$ G has a maximum matching M_i with $|M_i \cap R| = i$.*

Remark 3.4 In a graph G with the IP property, there exists a set R with $|R| = \nu(G)$ such that for $i = 0, 1, \ldots, \nu(G)$ G has a maximum matching M_i with $|M_i \cap R| = i$ if and only if G has two disjoint maximum matchings.

It should be noticed that finding in a graph two maximum matchings that are as disjoint as possible is NP-complete. This is an immediate consequence of the NP-completeness of deciding whether a 3-regular graph has an edge 3-coloring.

D. Hartvigsen has developed an algorithm for constructing in a graph a partial graph H with $d_H(v) \leq 2$ for each node v, which contains no triangle and which has a maximum number of edges. Such an algorithm can be used in graphs where the only odd cycles are triangles, so called line-perfect graphs. We obtain the following:

Theorem 3.5 *If G is a line-perfect graph, one can determine in polynomial time whether G contains two disjoint maximum matchings.*

From Theorems 3.1 and 3.5 we obtain:

Corollary 3.6 *If G is a cactus where all cycles are triangles, one can determine in polynomial time whether there exists a minimum set R of edges that is $\mathcal{P}$-feasible for $\mathcal{P} = \{0, 1, \ldots, \nu(G)\}$.*

4 Conclusion

We have examined the problem of finding a minimum subset R of edges for which there exist maximum matchings M_i with $|M_i \cap R| = p_i$ for some given values of p_i. Partial results have been obtained for some classes of graphs (regular bipartite graphs, trees, odd cacti with triangles only,...). Our problem requires the determination of a shortest alternating cycle (SAC problem) whose complexity status is open. Further research is needed to extend our results to other classes.

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