On Liveness and Controlled Siphons in Petri Nets

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Abstract. Structure theory of Petri nets investigates the relationship between the behavior and the structure of the net. Contrary to linear algebraic techniques, graph based techniques fully exploit the properties of the flow relation of the net (pre and post sets). Liveness of a Petri net is closely related to the validation of certain predicates on siphons. In this paper, we study thoroughly the connections between siphons structures and liveness. We define the controlled-siphon property that generalizes the well-known Commoner's property, since it involves both traps and invariants notions. We precise some structural conditions under which siphons cannot be controlled implying the structural non-liveness. These conditions based on local synchronization patterns cannot be captured by linear algebraic techniques. We establish a graph-theoretical characterization of the non-liveness under the controlled-siphon property. Finally, we prove that the controlled-siphon property is a necessary and sufficient liveness condition for simple nets and asymmetric choice nets. All these results are illustrated by significant examples taken from literature.

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1 Introduction

Place/Transition nets [12] are a mathematical tool well suited for the modeling and analyzing systems exhibiting behaviors such as concurrency, conflict, and causal dependency between events. However, the high degree of complexity of the analysis is considered as the key obstacle that limits the applicability of Petri nets to real-world problems. The reachability graph of such systems is actually unmanageable, thus it is crucial to enforce the analysis power of techniques based on the net structure. Two paradigms have been proposed to tackle with this explosion problem: the relevant properties of the entire system are gained from the corresponding properties of its smaller and simpler components; the structure theory [4] consists in investigating the relationship between the behavior of the net and its flow relation while the initial marking is considered as a parameter. This paper presents new results in this second direction.

Background

Liveness is an important behavioral property of nets. It corresponds to the absence of global or local deadlock situations. The liveness of a Petri net is closely related to the satisfiability of some predicates on siphons. A siphon is a subset of places once "unsufficiently marked", will never again get new tokens. A siphon is said to be controlled if for each reachable marking the siphon remains "sufficiently" marked. When all siphons are controlled, the net is said to be satisfying the controlled-siphon property (cs-property for short). One major goal of structure theory is to propose necessary and sufficient structural conditions ensuring the controlled-siphon property.

It is showed that the cs-property is a necessary and sufficient liveness condition for some classes of Petri nets. Moreover, under the boundedness hypothesis, liveness, i.e. cs-property, of these sub-classes can be checked in polynomial time. For all algorithms proposed, the controlled-siphon property has been expressed either using graph theoretical structures such as traps, conflict-free paths in [3, 1], or using linear algebraic properties such as conservativeness, consistency and rank condition in [6, 8, 11].

Contribution of the paper

The paper is organized as follows: we present in the next section, the basic concepts and notations used, we introduce in section 3 the notions of min and max controlled siphon. After defining the controlled siphon property (cs-property), we show how this property can be checked structurally using in a refined manner both traps [9] and invariants [13]. Hence we extend the decision power of the Commoner's property. In section 4, we highlight new structural non-liveness conditions related to the close structure of siphons containing no trap. We show through a significant example how this characterization escapes to linear algebraic techniques. In section 5, we state a graph-theoretical characterization of the structural non-liveness under the cs-property hypothesis. From these results we prove in section 6 that the cs-property is a necessary and sufficient liveness condition for simple and asymmetric choice nets. We can then determine for such nets initial markings under which the net is live. Finally, we discuss worthwhile future work and conclude.

2 Basic Definitions and Notations

We briefly define Petri nets. A complete definition can be found in [12].

Definition 1. A Petri net N is a 4-tuple $N = \langle P, T, F, V \rangle$ where :

- $G = \langle P, T, F, V \rangle$ is a weighted bipartite digraph :
  P is the set of node places and T is the set node transitions with
  $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$
  F is the flow relation : $F \subseteq (P \times T) \cup (T \times P)$
  V is the weight application (valuation) : $V \in [F \rightarrow \mathbb{N}^+]$

Definition 2. A marking M of a Petri net $N = \langle P, T, F, V \rangle$ is a mapping from P to $\mathbb{N}$ where $M(p)$ denotes the number of tokens contained in place p.

A marked Petri net is a couple (N, M_0) where N is a Petri net and M_0 a marking of N called the initial marking.

We denote for a node $s \in P \cup T$, ${}^\bullet s$ (resp. $s^\bullet$) the set of nodes s' such that $(s', s) \in F$ (resp. $(s, s') \in F$).

Given a place p, we denote $Max_{t \in p^\bullet}\{V(p,t)\}$ by $max_{p^\bullet}$, $Min_{t \in p^\bullet}\{V(p,t)\}$ by $min_{p^\bullet}$, $Max_{t \in {}^\bullet p}\{V(t,p)\}$ by $max_{{}^\bullet p}$ and $Min_{t \in {}^\bullet p}\{V(t,p)\}$ by $min_{{}^\bullet p}$.

In all the paper, we denote by N a Petri net $N = \langle P, T, F, V \rangle$ as defined above and by (N, M_0) a marked Petri net.

Basic Structural Objects of Petri Nets

Definition 3. Let N be a Petri net. The subnet induced by (P', T') with $P' \subseteq P$, $T' \subseteq T$, is the net $N' = \langle P', T', F', V' \rangle$ where : $F' = F \cap ((P' \times T') \cup (T' \times P'))$ and V' is the restriction of V on F'.

Definition 4. Let N' be a subnet of a Petri net N. The valuation V' of N' is said to be :

- *homogeneous* if and only if : $\forall p \in P', \forall t_1, t_2 \in p^\bullet, V'(p, t_1) = V'(p, t_2)$
- *non-blocking* if and only if : $\forall p \in P', p^\bullet \neq \emptyset \Rightarrow min_{{}^\bullet p} \geq min_{p^\bullet}$
- *strongly non-blocking* if and only if : $\forall p \in P', p^\bullet \neq \emptyset \Rightarrow min_{{}^\bullet p} \geq max_{p^\bullet}$

The valuation V of a Petri net N can be extended to the application W from $(P \times T) \cup (T \times P) \rightarrow \mathbb{N}$ defined by : $\forall u \in (P \times T) \cup (T \times P), W(u) = V(u)$ if $u \in F$ and $W(u) = 0$ otherwise. The matrix C indexed $P \times T$ and defined by $C(p,t) = W(t,p) - W(p,t)$ is called the incidence matrix of the net.

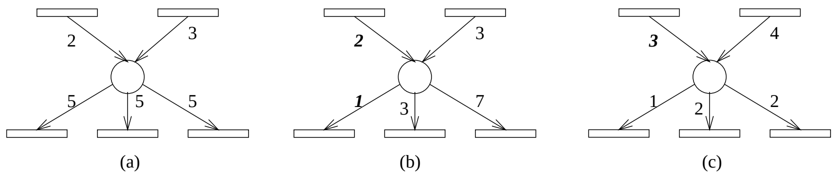

Fig. 1. A subnet with : (a) homogeneous valuation, (b) non-blocking (but not strongly non-blocking) valuation, (c) strongly non-blocking valuation

Definition 5. A integer vector f ($f \neq 0$) indexed by P ($f \in \mathbb{Z}^P$) is a place invariant if it satisfies ${}^t f.C = 0_T$. The positive support of f is the set of places defined by : $||f||^+ = \{p \in P \mid f(p) > 0\}$. and the negative support of f is the set of places defined by : $||f||^- = \{p \in P \mid f(p) < 0\}$.

Definition 6. Let N be a Petri net. Let $A \subseteq P, A \neq \emptyset$.

- A is called a *siphon* if and only if ${}^\bullet A \subseteq A^\bullet$. The siphon A is *minimal* if and only if it contains no other siphon as a proper subset and A is *maximal* if and only if none other siphon includes it.
- A is called a *trap* if and only if $A^\bullet \subseteq {}^\bullet A$. The trap A is *minimal* if and only if it contains no other trap as a proper subset and A *maximal* if and only if none other trap includes it.
- A is said to be *max-marked* (resp. *min-marked*) at a marking M if and only if $\exists p \in A$ such that $M(p) \geq max_{p^\bullet}$ (resp. $M(p) \geq min_{p^\bullet}$).
- we denote by $M(A)$ the sum of tokens contained in A at the marking M : $M(A) = \sum_{p \in A} M(p)$

Some Behavioral Properties of Petri Nets

Definition 7. Let (N, M_0) be a marked Petri net.

- A transition $t \in T$ is enabled at a marking M if and only if $\forall p \in {}^\bullet t, M(p) \geq W(p,t)$. The marking M' reached by firing t at M is defined by :

$$\forall p \in P, M'(p) = M(p) - W(p,t) + W(t,p)$$

  (denoted by $M[t > M')$
- By extension, a marking M' is said to be reachable from a marking M if there exists a sequence of transitions $s = t_0.t_1.\ldots.t_n$ and a series of marking $m_1, \ldots, m_n$ such that $M[t_0 > m_1, m_1[t_1 > m_2, \ldots, m_n[t_n > M'$ (denoted $M[s > M')$.
- The set of the markings of N reachable from a marking M is denoted by $Acc(N, M)$.

Definition 8. Let (N, M_0) be a marked Petri net.

- A transition t of N is *live* if and only if: $\forall M \in Acc(N, M_0), \exists M' \in Acc(N, M)$ such that $M'[t >$. A transition that is not live is a *dead* transition.
- (N, M_0) is *live* if and only if all transitions of N are live.
- (N, M_0) is *deadlock-free (or weakly-live)* if and only if: $\forall M \in Acc(N, M_0)$, $\exists t \in T$ such that $M[t >$
- (N, M_0) is *deadlockable* if and only if: $\exists M^* \in Acc(N, M_0)$ such that $\nexists t \in T$ with $M^*[t >$ (M^* is called a "dead marking").
- N is *structurally non-live* if and only if: $\nexists M_0 \in \mathbb{N}^P$ such that (N, M_0) is a live Petri net.

3 The Controlled-Siphon Property

In this section we first establish two propositions linking liveness or weak-liveness of a net to the initial marking of its siphons. We deduce of these propositions the notion of controlled siphons. We then give two structural ways ensuring the cs-property.

Definition 9. A siphon S of a marked Petri net (N, M_0) is said to be *min-controlled* if and only if S is min-marked at any reachable marking;

$$\forall M \in Acc(N, M_0), \exists p \in S \text{ such that } M(p) \geq min_{p^\bullet}$$

N satisfies the *min cs-property* when all siphons of N are min-controlled.

Definition 10. A siphon S of a marked Petri net (N, M_0) is said to be *max-controlled* if and only if S is max-marked at any reachable marking:

$$\forall M \in Acc(N, M_0), \exists p \in S \text{ such that } M(p) \geq max_{p^\bullet}$$

N satisfies the *max cs-property* when all siphons of N are max-controlled.

Definition 11. A Petri net (N, M_0) is said to be satisfying the *controlled-siphon property* (cs-property) if and only if each minimal siphon of (N, M_0) is min or max controlled.

Remark. Obviously, a max-controlled siphon is also a min-controlled siphon and, when the valuation of the net is homogeneous, a min-controlled siphon is also a max-controlled siphon.

Proposition 12. *If a marked Petri net (N, M_0) is live then it satisfies the min cs-property.*

Proof. We prove that each minimal siphon is min-controlled.

Let (N, M_0) be a live Petri net and S a siphon of N. Suppose that there exists a reachable marking M with $\forall p \in S, M(p) < min_{p^\bullet}$.

At this marking, all transitions of $S^\bullet$ are not enabled. Since S is a siphon $({}^\bullet S \subseteq S^\bullet)$, all transitions of $S^\bullet$ will never be enabled at any marking reachable from M, which contradicts the liveness of N. □

Proposition 13. *If a marked Petri net (N, M_0) satisfies the max cs-property, then it is weakly-live.*

Proof. If (N, M_0) is not weakly-live then there exists a reachable dead marking M. At M, $\forall t \in T, \exists p_t \in {}^\bullet t \mid M(p_t) < V(p_t, t) \leq max_{p_t^\bullet}$. Let $A = \{p_t\}_{t \in T}$ be the set of such places. A is by construction not a max-controlled siphon. qed. □

Two structural conditions can ensure that a siphon is min or max controlled. For the first one, the control is internal to the siphon and involves trap structure and, for the second one, the control is external to the siphon and is related to place invariants.

Proposition 14. *Let (N, M_0) be a marked Petri net and S a siphon of N. If one of the two following conditions holds then S is min-controlled.*

1. *there exists a trap R included in S such that: R is min-marked at M_0 and the valuation of the subnet induced by $(R, R^\bullet)$ is non blocking.*
2. *there exists a place invariant f such that: $\forall p \in (||f||^- \cap S)$, $min_{p^\bullet} = 1$, $||f||^+ \subseteq S$, and $\sum_{p \in P}[f(p).M_0(p)] > \sum_{p \in S}[f(p).(min_{p^\bullet} - 1)]$*

Proof. Suppose that point 1 of the proposition holds but that S is not min-controlled. In this case, there exists a marking M reachable from M_0 for which $\forall p \in S$, $M(p) < min_{p^\bullet}$. In particular, the trap R included in S would be not min-marked ($\forall p \in R$, $M(p) < min_{p^\bullet}$). Since the valuation of the subnet induced by $(R, R^\bullet)$ is non blocking, and R is min-marked at M_0, the last conclusion is not possible. Then S is necessarily min-controlled.

Suppose now that point 2 holds. If S is not min-controlled, then there exists a marking M reachable from M_0 for which $\forall p \in S$, $M(p) \leq min_{p^\bullet} - 1$. Since $\forall p \in ||f||^- \cap S$, $min_{p^\bullet} = 1$, we obtain $\sum_{p \in S} f(p).M(p) \leq \sum_{p \in S} f(p).(min_{p^\bullet} - 1)$. Moreover as $f(p) < 0$ for $p \in P \backslash S$, we have $\sum_{p \in P} f(p).M(p) \leq \sum_{p \in S} f(p).M(p)$ and thus $\sum_{p \in P} f(p).M(p) \leq \sum_{p \in S} f(p).(min_{p^\bullet} - 1)$. This inequality can not be satisfied because f is a place invariant ($\sum_{p \in P} f(p).M(p) = \sum_{p \in P} f(p).M_0(p)$) and f min-controls S ($\sum_{p \in P}[f(p).M_0(p)] > \sum_{p \in S}[f(p).(min_{p^\bullet} - 1)]$). □

Proposition 15. *Let (N, M_0) be a marked Petri net and S a siphon of N. If one of the following condition holds then S is max-controlled.*

1. *there exists a trap R included in S such that: R is max-marked at M_0 and the valuation of the subnet induced by $(R, R^\bullet)$ is strongly non blocking.*
2. *there exists a place invariant f such that: $\forall p \in (||f||^- \cap S)$, $max_{p^\bullet} = 1$, $||f||^+ \subseteq S$, and $\sum_{p \in P}[f(p).M_0(p)] > \sum_{p \in S}[f(p).(max_{p^\bullet} - 1)]$*

Proof. Suppose that point 1 of proposition holds but that S is not max-controlled. In this case, there exists a marking M reachable from M_0 for which $\forall p \in S$, $M(p) < max_{p^\bullet}$. In particular, the trap R included in S would be not max-marked ($\forall p \in R, M(p) < max_{p^\bullet}$). Since valuation of the subnet defined by R is strongly non blocking ($\forall p \in R, min{\bullet}_p \geq max_{p^\bullet}$), and initially R is max-marked, the last conclusion is impossible. So S is necessary min-controlled.

Suppose now that point 2 holds. If S is not max-controlled, then there exists a marking M reachable from M_0 for which $\forall p \in S, M(p) \leq max_{p^\bullet} - 1$. Since $\forall p \in ||f||^- \cap S, max_{p^\bullet} = 1$, we obtain $\sum_{p \in S} f(p).M(p) \leq \sum_{p \in S} f(p).(max_{p^\bullet} - 1)$. Moreover as $f(p) < 0$ for $p \in P \backslash S$, we have $\sum_{p \in P} f(p).M(p) \leq \sum_{p \in S} f(p).M(p)$ and thus $\sum_{p \in P} f(p).M(p) \leq \sum_{p \in S} f(p).(max_{p^\bullet} - 1)$. This inequality can not be satisfied because f is an invariant ($\sum_{p \in P} f(p).M(p) = \sum_{p \in P} f(p).M_0(p)$) and f max-controls S
($\sum_{p \in P}[f(p).M_0(p)] > \sum_{p \in S}[f(p).(max_{p^\bullet} - 1)]$). □

Remark. When all siphons of a marked Petri net (N, M_0) are min-controlled with respect to condition 1 of proposition 14, (N, M_0) is said to be satisfying the siphon-trap property (Commoner's property) [9], [12].

It is well known that liveness is not a monotonic property (N live under M_0 does not imply N live under $M_0' > M_0$). One must observe that if a siphon S is min or max controlled by a trap under M_0, S remains controlled under $M_0' > M_0$. On the contrary, this control is not necessarily preserved if S is min or max controlled by an invariant. This feature is important since it explains partially the non-monotonicity of the liveness property. Thus, the cs-property is more powerful than the siphon-trap property in the sense that it gives more strict necessary liveness conditions.

Example : Consider the following net N. This net contains two minimal siphons $D_1 = \{p, q\}$ and $D_2 = \{r\}$.

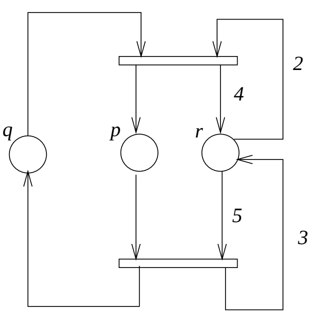

Fig. 2. A Petri net

D_1 is both a trap and the support of a positive invariant. Since valuation of the subnet induced by D_1 is homogeneous D_1 is min or max controlled as soon as p or q is marked.

D_2 is a trap with non blocking valuation (but not strongly non blocking). So D_2 is min-controlled if $M_0(r) \geq 2$. Moreover, $f = r - 2p$ is an invariant such that $||f||^+ = D_2$. So D_2 is min-controlled for initial markings M_0 satisfying $M_0(r) > 1 + 2M_0(p)$ and max-controlled for initial markings M_0 satisfying $M_0(r) > 4 + 2M_0(p)$.

Using the min cs-property, one can conclude that if N is live then necessarily, either $(M_0(q) > 0$ and $M_0(r) > 1)$ (or $M_0(p) > 0$ and $M_0(r) > 3)$. For instance, under $M_0 = p + 2r$, the net is not live although N satisfies the siphon-trap property.

In the same way, if $(M_0(p) > 0$ and $M_0(r) > 4 + 2M_0(p))$ (or $M_0(q) > 0$ and $M_0(r) > 4)$ then N is deadlock free (weakly-live) (for instance, under $M_0 = q + 5r$ or $M_0 = p + 7r$).

Remark. One can note that it may exist a "shadow" interval of initial markings under which cs-property is satisfied but no conclusion about liveness can be deduced.

For instance, under these two following initial markings $M_0 = p + 4r$, or $(M_0 = q + 2r)$ and $M'_0 = p + 5r$ or $(M'_0 = q + 3r)$ the net of fig.2 satisfies the min cs-property but not max cs. However, one can check that N is deadlockable under M_0 but live under M'_0.

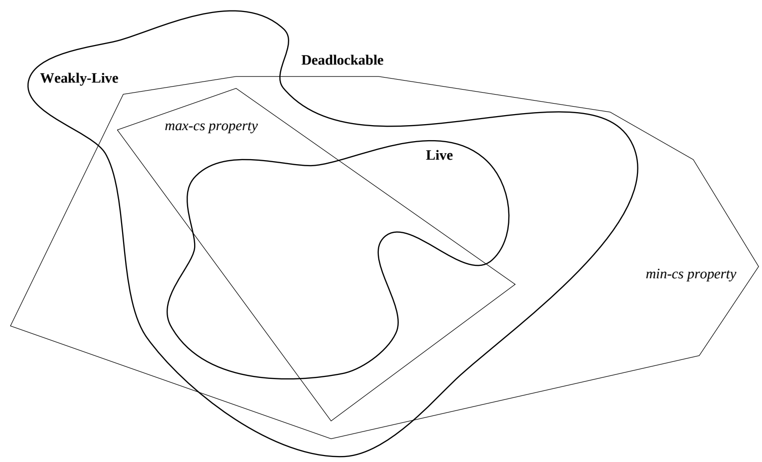

Fig. 3. Relation between properties

4 A Non Controllable Siphons Characterization

The linear algebraic techniques give some structural non-liveness conditions using structural repetitiveness for general nets [5] or using conservativeness [5], consistency [5] and rank theorem [7, 6] for bounded nets.

We establish here a new structural non-liveness condition by exploiting the properties of subnets induced by siphons without trap.

Proposition 16. *Let N be a Petri net and A be a non-empty subset of places which does not contain any trap. There exists a partition $(A_i)_{i=0..I} \subset \mathcal{P}(A)$ of A such that*

$$\forall p \in A_i, \exists t \in p^\bullet \text{ with } t^\bullet \cap \cup_{k \geq i} A_k = \emptyset$$

We denote by $Dec(p) = \{t \in p^\bullet \mid t^\bullet \cap \cup_{k \geq i} A_k = \emptyset\}$

Proof. Let $(S_i) \subset \mathcal{P}(A), (A_i) \subset \mathcal{P}(A), (T_i) \subset \mathcal{P}(T)$ be the series defined by :

- $S_0 = A$,
- $\forall i \geq 0, T_i = S_i^\bullet \setminus {}^\bullet(S_i)$, $A_i = {}^\bullet T_i$ and $S_{i+1} = S_i \setminus A_i$,

Since each S_i is a subset of A, A is finite, and does not contain a trap, then $\exists I > 0$ such that $S_{I+1} = \emptyset$ and $\forall i \leq I, A_i \neq \emptyset, T_i \neq \emptyset$.

Let $i < I, j < I$, and we suppose $i > j$. By construction, $S_i = S_j \setminus \cup_{k=j+1..i} A_i$, and $A_i \subseteq S_i$, so $A_i \cap A_j = \emptyset$. Since $A_i = {}^\bullet T_i$, we also deduce that $T_i \cap T_j = \emptyset$. By construction $S_{I+1} = S_I \setminus \cup_{i=0..I} A_i$, and $S_{I+1} = \emptyset$, so $\cup_{i=0..I} A_i = A$.

Then, $(A_i)_{i=0..I}$ forms a partition of A and $(T_i)_{i=0..I}$ a sub partition of T. Let $p \in A_i$, there exists necessarily $t \in T_i$ such that $t \in p^\bullet$ and $t^\bullet \cap \cup_{k \geq i} A_k = \emptyset$. Indeed, if such a transition does not exist, either p belongs to A_j with $j > i$, or A contains a trap. This contradicts the hypothesis. □

Theorem 17. *Let N be a Petri net with homogeneous valuation. If there exists a siphon S of N containing no trap and for which the following condition holds :*

$$\forall p \in S, \forall t \in p^\bullet \setminus Dec(p), \exists t_{dec} \in Dec(p) \text{ such that } {}^\bullet t_{dec} \subseteq {}^\bullet t \quad [1]$$

then N is structurally not live. Such a siphon is said to be non-controllable.

In order to prove this theorem, we first establish two lemmas.

Lemma 18. *Let A be a finite subset of place and $(A_i)_{i=0..I}$ a partition of A. Let "$\prec_{dec}$" be the order relation on the set of markings of A defined by :*

$$M'_A \prec_{dec} M_A \Leftrightarrow \exists k \in [0, I] \,|\, \forall i > k, M'(A_i) = M(A_i) \text{ and } M'(A_k) < M(A_k)$$

Given an initial (finite) marking of A, M_A, there cannot exist an infinite series of markings of A, starting from M_A and strictly decreasing with respect to $\prec_{dec}$.

Proof. If such an infinite series exists then, either A is infinite, or A is not finitely marked. In both cases, we obtain a contradiction. □

[1] this condition means that the siphon has only very locally (according to $Dec(p)$) the free choice pattern synchronization

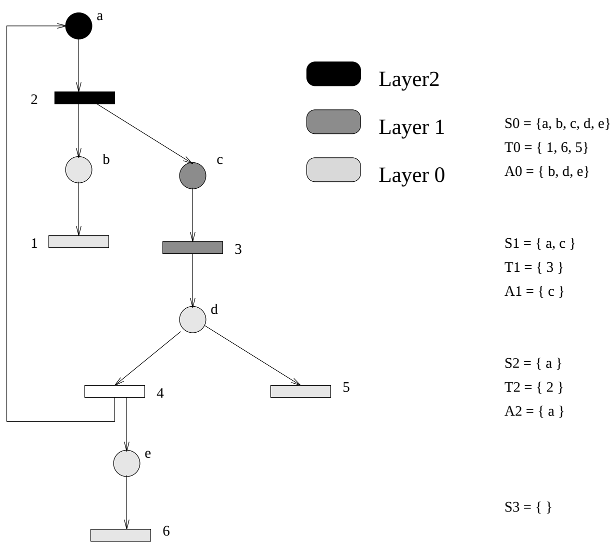

Fig. 4. Example of construction of the partitions $A = \{a, b, c, d, e\}$

Lemma 19. *Let M be a reachable marking and $t \in S^\bullet$ a transition enabled at M. For all $p \in {}^\bullet t \cap S$ there exists a transition $t_{dec} \in S^\bullet \cap Dec(p)$ such that t_{dec} is enabled at M.*

Proof. If $t \in Dec(p)$ then the proof is done. Otherwise, by hypothesis, $\exists t_{dec} \in Dec(p)$ such that ${}^\bullet t_{dec} \subseteq {}^\bullet t$. Since t is enabled at M (${}^\bullet t_{dec} \subseteq {}^\bullet t$) and the valuation is homogeneous, t_{dec} is also enabled at M. $\square$

Proof of theorem 17. Let M_0 be an initial marking of N and suppose that N is live under M_0.

Let $(A_i)_{i<I}$ the partition defined by S (cf. prop 16). Let m_0 be the restriction of M_0 on places of S and let t_0 be the first transition belonging to $S^\bullet$ which is enabled at a marking M. Let $p_0 \in {}^\bullet t_0 \cap S$.

Consequently to the previous lemma, there exists $t_{dec} \in Dec(p_0)$ such that t_{dec} is also enabled at M. Let then M' be the marking obtained by firing t_{dec} at M. By definition of the order relation " $\prec_{dec}$ ", the restriction of M' on places of S, m_1, is such that $m_1 \prec_{dec} m_0$.

If we suppose that N is live under M_0, we generate an infinite series of markings of S, starting from m_0, strictly decreasing with respect to the order relation $\prec_{dec}$, and that is not possible.

So, N is not live under M_0 and there is no initial marking under which N is live. $\square$

Remark. In the context of net analysis, one can only consider minimal siphons of the net.

The previous theorem confirms why siphon-trap property is a necessary liveness condition for extended non self-controlling nets and for extended free-choice nets [1] otherwise minimal siphons are not controllable. Moreover, it explains why liveness monotonicity is true for these sub-classes.

Example : In this net from [7], consider the minimal siphon $S = \{b, c, d, e\}$. One can check that S contains no trap. The partition of S is indicated on figure, and we have $Dec(b) = \{t_1\}, Dec(c) = \{t_2\}, Dec(d) = \{t_7\}, Dec(e) = \{t_4\}$.

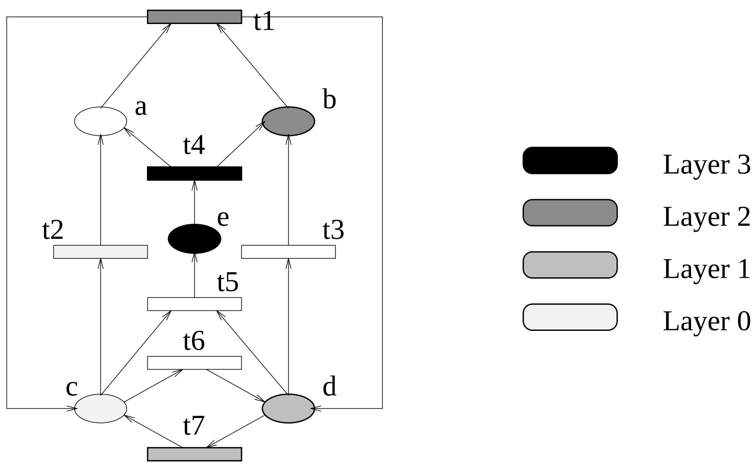

Fig. 5. A structurally non live net

We can also check that the "local free-choice" aspect is satisfied : ${}^\bullet t_2 \subseteq {}^\bullet t_5$, ${}^\bullet t_2 \subseteq {}^\bullet t_6$ for the place c, ${}^\bullet t_7 \subseteq {}^\bullet t_3$ ${}^\bullet t_5 \subseteq {}^\bullet t_3$ for the place d $Dec(b) = b^\bullet$ and $Dec(e) = e^\bullet$. Hence, $S = \{b, c, d, e\}$ is a non-controllable siphon.

So N is structurally non-live, thus N is not a well formed net although N is conservative, consistent, and satisfies the rank condition [6, 7].

The next section is devoted to investigate structural conditions ensuring the liveness once weak-liveness is established by means of the cs-property.

5 A Non-Liveness Structural Characterization

We consider in this section a non live marked Petri net (N, M_0) but for which all siphons are max-controlled ((N, M_0) is weakly-live). We suppose also that any shared place p ($Card(p^\bullet) \geq 2$) satisfies $min_{{}^\bullet p} \geq max_{p^\bullet}$. This constraint means that the valuation of the net is locally (around shared places) strongly non-blocking.

We are looking for graph-theoretical properties underlying to this pathological behavior (max-cs property but not live).

Proposition 20. *Let (N, M_0) be a marked Petri net satisfying the max-cs property ((N, M_0) is weakly-live) but not live. There exists a reachable marking M^* from M_0 and two subsets of transitions, T_L and T_D such that*

- *T_L and T_D is a partition of T ($T_L \neq \emptyset, T_D \neq \emptyset, T_L \cap T_D = \emptyset$, $T_L \cup T_D = T$)*
- *all transitions of T_L are live and all transitions of T_D are dead.*

Proof. The proof is trivial since the net is not deadlockable (weakly-live) but not live. □

Remark. This partition is not necessarily unique but there exists at least one. It is important to note that T_D is maximal in the sense that all transitions that do not belong to T_D, will never become dead.

For this particular marking M^*, we characterize now the existence of a particular dead transition, $t^* \in T_D{}^\bullet$.

Theorem 21. *Let (N, M_0) be a non live marked Petri net satisfying the max-cs property. There exists a transition $t^* \in T_D$ such that the set L_P defined by $L_P = \{p \in {}^\bullet(t^*) \mid {}^\bullet p \cap T_L \neq \emptyset$ and $p^\bullet \cap T_L \neq \emptyset\}$ contains at least two items. Furthermore, $\exists M \in Acc(N, M^*)$ such that $\forall p \in {}^\bullet(t^*) \setminus L_P,\ M(p) \geq V(p, t^*)$.*

Before proving this proposition we establish two lemmas.

Lemma 22. $\forall p \in P, {}^\bullet p \cap T_L = \emptyset \Longrightarrow p^\bullet \cap T_L = \emptyset$

Proof. Suppose that this lemma is not true. In this case there exists a place p with all its input transitions in T_D (${}^\bullet p \cap T_L = \emptyset$) and at least one output transition t_v in T_L ($p^\bullet \cap T_L \neq \emptyset$). Since t_v is live, after a finite number of firings place p becomes non min-marked because all its input transitions are dead. So, t_v is dead and denies the maximality of T_D. □

Lemma 23. *There exists $t^* \in T_D$ such that*

1. $\forall p \in {}^\bullet(t^*),\ {}^\bullet p \cap T_L = \emptyset \Longrightarrow \forall M \in Acc(N, M^*), M(p) \geq V(p, t^*)$
2. ${}^\bullet({}^\bullet t^*) \cap T_L \neq \emptyset$

Proof. Suppose that point 1 is false. Since $M^* \in Acc(N, M^*)$, $\forall t \in T_D, \exists p_t \in {}^\bullet t$ with ${}^\bullet p_t \cap T_L = \emptyset$ and $M^*(p_t) < V(p_t, t)$. Let $S = \{p_t\}_{t \in T_D}$. By construction, ${}^\bullet S \subseteq T_D$ and $T_D \subseteq S^\bullet$ (for all p_t in S, ${}^\bullet p_t \cap T_L = \emptyset$). So, S is a siphon. Since $\forall p_t \in S, M^*(p_t) < V(p,t)$, S is non max-marked for M^* and hence, the max-cs property is denied. If S is not minimal, then there exists a minimal siphon included in S which is not max-marked at M^*: that also denies hypothesis.

Using now the previous lemma (lemma 22), (if p has no live input then all outputs of p are dead), we can deduce that the marking of such places does not change for all markings reachable from M^*; that ends the proof of point 1.

If the second point of this lemma is false, then using the point 1 we obtain $\forall p \in {}^\bullet(t^*)$, $M^*(p) \geq V(p, t^*)$. So t^* is enabled at M^* that contradicts hypothesis. □

Proof of theorem 21. Let t^* be a transition fulfilling the conditions of the previous lemma. We first prove that L_p is not empty ($Card(L_p) > 0$).

Suppose that $L_P = \emptyset$: any input place of t^* having a live input transition (there is at least one because of point 2 of the previous lemma) has no live output transition. As the other input places of t^* are such that their pre-conditions on t^* are satisfied at M^* and remain satisfied (point 1 of the previous lemma), we can reach a marking M from M^* such that t^* would be enabled at M. So, $Card(L_P) \geq 1$.

Suppose now that $Card(L_P) = 1$ ($L_P = \{p_1\}$). The valuation is strongly non-blocking around p_1 (p_1 is a shared place), and p_1 has a live input transition. Since the other pre-conditions of t^* are satisfied for M^* (point 1 of lemma 23) then t^* would be enabled at M^*: That is not possible. So, $Card(L_P) \geq 2$.

The last point of the theorem is true for places having no live input transition (point 1 of lemma 23). The other places have a live input transition and are not in T_L. So, these places are not bounded. qed □

We now show how this non-liveness structural characterization allows us to state that the cs-property is a necessary and sufficient liveness condition for asymmetric choice nets with homogeneous valuation.

6 A Necessary and Sufficient Liveness Condition for A.C. Nets

We first establish that the cs-property is a necessary and sufficient liveness condition for simple nets having a modeling power more interesting than free choice nets.

Definition 24. A Petri net N is a simple net if and only if:

$$\forall t \in T, Card(\{p \in {}^\bullet t \mid Card(p^\bullet) > 1\}) \leq 1$$

Corollary 25 of theorem 21. *Let* (N, M_0) *be a simple marked net.*

1. *If* (N, M_0) *satisfies the max-cs property, then is live.*
2. *Furthermore, when* N *has an homogeneous valuation,* (N, M_0) *is live if and only if it satisfies the max-cs property.*

Proof. Suppose that the net is not live. As it is weakly-live since it satisfies the max-cs property, we know (th. 21) that there must exist a transition t^* having at least two input places p_1, p_2 with $Card({p_1}^\bullet \setminus \{t^*\}) \geq 1$ and $Card({p_2}^\bullet \setminus \{t^*\}) \geq 1$. So it comes $Card({p_1}^\bullet) \geq 2$, $Card({p_2}^\bullet) \geq 2$, $\{p_1, p_2\} \subseteq {}^\bullet(t^*)$ and then the net cannot be a simple net.

The proof of the second point of the corollary is obvious since the max-cs and min-cs properties coincide when the valuation is homogeneous. □

Example : Consider the following simple net of fig. 6 given in [7]. This net contains six minimal siphons: $D_1 = \{p_1, p_6, p8, p_9\}$, $D_2 = \{p_2, p_9\}$, $D_3 = \{p_6, p_7\}$, $D_4 = \{p_1, p_3, p_4, p_5, p_8\}$ and $D_5 = \{p_2, p_3, p_4, p_5\}$.

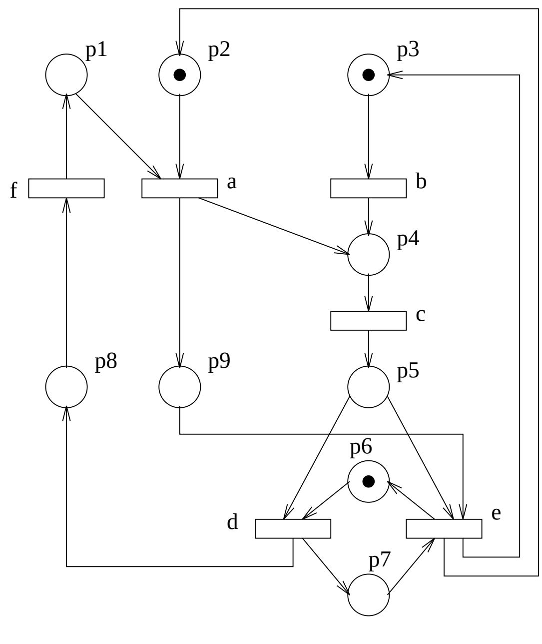

Fig. 6. A simple net

The siphons D_1, D_2, D_3, D_4 are traps and D_5 is controlled by the place invariant $f = p_2 + p_3 + p_4 + p_5 - p_6$.

So, the net is live if the initial marking M_0 satisfies :

1. $\forall i, M_0(D_i) > 0$
2. $M(p_2) + M(p_3) + M(p_4) + M(p_5) > M(p_6)$.

For the marking presented in the figure the net is live but if we e.g. increase by a token the marking of place p_6, the non-liveness holds.

Now we prove that an equivalent result of corollary 1 can be stated for asymmetric choice nets.

Definition 26. A Petri net N is an asymmetric choice net (AC net) if and only if:

$$\forall(p,q) \in P \times P, p^\bullet \cap\ q^\bullet \neq \emptyset \Longrightarrow p^\bullet \subseteq q^\bullet \text{ or } q^\bullet \subseteq p^\bullet$$

Corollary 27 of theorem 21. *An asymmetric choice net with homogeneous valuation is live if and only if it satisfies the max-cs property.*

Proof. Because min-cs and max-cs properties coincide when the valuation is homogeneous, a live AC net with homogeneous valuation fulfills the max-cs property.

Suppose now that (N, M_0) fulfills the max-cs property (it is weakly-live) but is not live. In this case, there exists (th. 21) a marking M^* and a transition t^* such that the set defined by $L_P = \{p \in {}^\bullet(t^*) \mid {}^\bullet p \cap T_L \neq \emptyset \text{ and } p^\bullet \cap T_L \neq \emptyset\}$ contains at least two items and such that there exists a reachable marking M satisfying $\forall p \in {}^\bullet(t^*) \setminus L_P,\ M(p) \geq V(p, t^*)$.

The AC net definition implies that the set of places $\{{}^\bullet(t^*)\}$ can be linearly ordered (with the relation $\subseteq$) and then one can derive an ordering on L_P.

Suppose that $L_P = \{p_1, p_2, \ldots, p_k\}$. Without loss of generality we may assume $p_1{}^\bullet \subseteq \ldots \subseteq p_k{}^\bullet$.

As $p_1 \in L_P$ $(p_1{}^\bullet \cap T_L \neq \emptyset)$, $\exists t \in p_1{}^\bullet$ such that t is enabled at a marking M' reachable from M. Since $\forall p \in Lp, p \in {}^\bullet t$ $(p_1{}^\bullet \subseteq \ldots \subseteq p_k{}^\bullet)$ and valuation is homogeneous, t^* would be enabled at M' that contradicts definition of t^*.

So the max cs-property implies the liveness. □

Example : This AC net (given in [14]) is not a simple net. It contains five minimal siphons :

- $D_1 = \{p_{11}, p_1, p_{12}, p_{22}, p_{13}, p_{23}, p_{14}, p_{24}\}$
- $D_2 = \{p_{21}, p_1 + p_{12}, p_{22}, p_{13}, p_{23}, p_{14}, p_{24}\}$
- $D_3 = \{robot, p_{12}, p_{22}, p_{14}, p_{24}\}$
- $D_4 = \{ressources, p_{13}, p_{23}\}$
- $D_5 = \{robot, ressources, p_{14}, p_{24}\}$

The four siphons D_1, D_2, D_3, D_4 are trap-controlled (they are all traps). The siphon D_5 is invariant-controlled. It is controlled either by the invariant $f_1 = robot + ressources - p_1 - p_{11}$ or by the invariant $f_2 = robot + ressources - p_1 - p_{21}$.

So, the net is live as soon as the initial marking M_0 satisfies :

1. $\forall i, M(D_i) > 0$,
2. $M_0(robot) + M_0(ressources) > M_0(p_1) + min(M_0(p_{11}) + M_0(p_{21}))$.

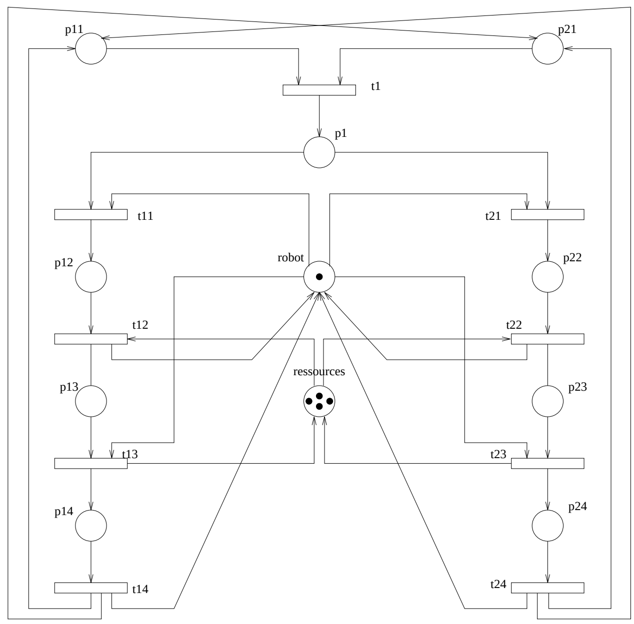

Fig. 7. An asymmetric choice Petri net

7 Conclusion

In this paper we have presented some new results related to structure theory of Petri nets [10, 4] which are more conclusive about liveness than Commoner's property and than algebraic conditions such as consistency, conservativeness and rank theorem. The technical results are based on the controlled-siphon property which uses in a refined manner both traps and invariants in order to "control" siphons. We have proved that this cs-property is a necessary and sufficient liveness condition for asymmetric choice nets. Also, we highlighted a new structural non-liveness condition by investigating the local structure of siphons containing no trap. The cs-property highlights the conditions markings since it provides an explanation of the non-monotonicity of liveness property. In particular, this property allows us to understand why a small modification of the initial marking can make a net non live even if it fulfills the good algebraic properties.

Among the open problems which seem to us relatively approachable in future, we mention the following: How to refine the proposed non-liveness characterization using both a graph theoretical characterization of minimal siphons [2] and the transition invariant notion [13] in order to define a new class of Petri nets having a strong description power and for which the controlled siphon property can be checked structurally in polynomial time.

References

1. K. Barkaoui, J.M Couvreur, and C. Duteilhet. On liveness in extended non self-controlling nets in application and theory of Petri nets. *LNCS*, 935, 1995.
2. K. Barkaoui and B. Lemaire. An effective characterization of minimal deadlocks and traps based on graph theory. 10^{th} *ICATPN*, 1989.
3. K. Barkaoui and M. Minoux. A polynomial time graph algorithm to decide liveness of some basic classes of bounded Petri nets. *LNCS*, No. 616:62–75, 1992.
4. E. Best. Structure theory of Petri nets : The free choice hiatus. In G.Rozenberg W.Brauer, W.Resig, editor, *LNCS*, volume No. 255. Springer-Verlag, 1986.
5. G.W. BRAMS. *Réseaux de Petri : Theorie et pratique*. Masson, 1983.
6. J. Desel. A proof of the rank theorem for extended free choice nets. *LNCS*, No. 616:134–153, 1992.
7. F. Dicesare, G. Harhalakis, J.M. Proth, M. Silva, and F.B. Vernadat. *Practice of Petri Nets in Manufacturing*. Chapman-Hall, 1995.
8. J. Esparza and M. Silva. A polynomial-time algorithm to decide liveness of bounded free-choice nets. *T.C.S*, N 102:185–205, 1992.
9. M.H.T. Hack. Analysis of production schemata by Petri nets. In *Cambridge, Mass.: MIT, MS Thesis*, 1974.
10. M. Jantzen and R. Valk. Formal properties of P/T nets. *LNCS*, No. 84, 1981.
11. P. Kemper and F. Bause. An efficient polynomial-time algorithm to decide liveness and boundedness of free-choice nets. *LNCS*, No. 616:263–278, 1992.
12. W. Reisig. *EATCS-An Introduction to Petri Nets*. Springer-Verlag, 1983.
13. H. Ridder and K. Lautenbach. Liveness in bounded Petri nets which are covered by t-invariants. *LNCS*, No. 815:358–375, 1994.
14. M. Zhou and F. DiCesare. *Petri nets Synthesis for Discrete Event Control of Manufacturing Systems*. Kluwer Academic, 1993.