Reconstruction of Convex Polyominoes from Orthogonal Projections of their Contours

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October 25, 2004
Abstract

The problem of reconstructing a convex polyominoes from its horizontal and vertical projections when the projections are defined as the number of cells of the polyomino in the different lines and columns was studied by Del Lungo and M. Nivat (cf [7]). In this paper we study the reconstruction of any convex polyomino when the orthogonal projections are defined as the contour length of the object intercepted by the ray. We prove the $NP$-hardness of this problem for several classes of polyominoes: general, $h$-convex, $v$-convex. For $hv$-convex polyominoes we give a polynomial time algorithm for the reconstruction problem.

**Key words**: Discrete Tomography, Polyomino, $NP$-complete, Graph, $2 - SAT$ problem
1 Introduction

In discrete tomography one wants to reconstruct a discrete object from its projections. This study is motivated by some interesting applications in image processing, data bases, crystallography, statistics, data compressing, scheduling, graph theory, ... Reconstruction of two dimensional objects from their two orthogonal projections have been studied by many authors. First, Ryser [15] showed how to reconstruct a binary matrix from its row and column sums. In [12] chapters deal with theoretical and practical aspects of discrete tomography. In [7] A. Del Lungo and M. Nivat study the reconstruction of convex polyominoes from their horizontal and vertical projections when the projections are defined as the number of cells of the polyomino in the considered ranks. Here we will study the reconstruction of convex polyominoes when the orthogonal projections represent the length of their contour lines. This problem is justified by the measurement of a hollow body with fine walls using a x-rays electroscope. To our knowledge this problem was not studied in the literature.

This article is organized as follows : the problem definition and some general properties are given in Section 2; Section 3 deals with the NP-hardness of reconstructing a polyomino, even in the cases of $h$-convexity or $v$-convexity; the main part of this paper is in Section 4 in which we design a polynomial time algorithm when the polyomino to reconstruct must be horizontally and vertically convex; we conclude and give some new perspectives in Section 5.

2 Preliminaries

Let $R$ be a discrete rectangle with $m$ rows and $n$ columns. We denote by $C(i,j)$ the cell of $R$ lying on the row $i$ and column $j$; the rows are indiced from top to bottom and the columns from left to right. The top edge of a cell $C(i,j)$ is denoted by $T(i,j)$, its right edge by $R(i,j)$, its bottom edge by $B(i,j)$, and its left edge by $L(i,j)$. We say that two cells are connected (or adjacent) if they share an edge. A polyomino (see Fig. 1) is a connected subset of cells of $R$. In the following we are interested in polyominoes without holes. Without loss of generality, we will consider polyominoes having at least one cell in every row and column. ($R$ is therefore the smallest rectangle containing the considered polyomino). Given a polyomino $P \subset R$, we can define $\Gamma(P)$ its contour line in the following way : $\Gamma(P) = \{T(i,j) : C(i,j) \in P, C(i-1,j) \notin P\} \cup \{R(i,j) : C(i,j) \in P, C(i,j+1) \notin P\} \cup \{B(i,j) : C(i,j) \in P, C(i+1,j) \notin P\} \cup \{L(i,j) : C(i,j) \in P, C(i,j-1) \notin P\}$ (see Fig. 1). Since $P$ is finite and without hole, $\Gamma(P)$ is a closed curve.

A polyomino is horizontally convex (in the following we write $h$-convex) (see Fig. 2) if for each row $i$, the cells $C(i,j) \in P$ form an interval, i.e. $\forall i, \exists (l,r) \in \mathbb{N}^2, l \leq r, \forall k \leq l, C(i,k) \notin P, \forall l \leq k \leq r, C(i,k) \in P, \forall k > r, C(i,k) \notin P$. Similarly a polyomino is vertically convex ($v$-convex) (see Fig. 2) if for each column $j$ the cells $C(i,j) \in P$ form an interval, i.e. $\forall j, \exists (t,b) \in \mathbb{N}^2, t \leq b, \forall k \leq t, C(k,j) \notin P, \forall t \leq k \leq b, C(k,j) \in P, \forall k > b, C(k,j) \notin P$. A polyomino is $hv$-convex iff it is $h$-convex and $v$-convex (see Fig. 2).

The two orthogonal projections of $\Gamma(P)$ are defined as following:
\( H(P) = (h_0, h_1, \ldots, h_m) \), the horizontal projection, is a \((m+1)\)-dimensional vector of nonnegative integers where

\[
  h_i = |\{T(i+1, j) : T(i+1, j) \in \Gamma(P)\}| + |\{B(i, j) : B(i, j) \in \Gamma(P)\}|
\]

\( V(P) = (v_0, v_1, \ldots, v_n) \), the vertical projection, is a \((n+1)\)-dimensional vector of nonnegative integers where

\[
  v_j = |\{R(i, j) : R(i, j) \in \Gamma(P)\}| + |\{L(i+1, j) : L(i+1, j) \in \Gamma(P)\}|
\]

(see Fig. 3).

We will now prove two basic properties linking the orthogonal projections with horizontal and vertical convexity.

**Property 1** A polyomino \( P \) is \( h \)-convex if and only if \( \sum_{j=0}^{n} v_j = 2m \)

**Proof:** If \( P \) is \( h \)-convex, then for each line \( i \) there is one edge \( L(i, j) \in \Gamma(P) \) and one edge \( R(i, j) \in \Gamma(P) \) and thus \( \sum_{j=0}^{n} v_j = 2m \). Reciprocally, let \( P \) be a polyomino : for each line \( i \) there is an edge \( L(i, j) \in \Gamma(P) \) and an edge \( R(i, k) \in \Gamma(P) \) and \( L(i, j) \neq R(i, k) \). If \( \sum_{j=0}^{n} v_j = 2m \) then for each \( i \) there is one edge \( L(i, j) \in \Gamma(P) \) and one edge \( R(i, j) \in \Gamma(P) \) and \( P \) is \( h \)-convex. \( \blacksquare \)

By symmetry we have the property hereafter.

**Property 2** A polyomino \( P \) is \( v \)-convex if and only if \( \sum_{i=0}^{m} h_i = 2n \)

Given two vectors \( H \) and \( V \), our goal is to reconstruct a polyomino \( P \) (possibly \( P \) may satisfy
some additional convexity constraints) such that $\Gamma(P)$ has projections $H$ and $V$. So our problem is formally defined as follows:

**INPUT**: two nonnegative integer vectors $H = (h_0, h_1, \ldots, h_m)$ and $V = (v_0, v_1, \ldots, v_n)$

**OUTPUT**: a polyomino $P$ such that $H(P) = H$ and $V(P) = V$ if such a $P$ exists

Before proving complexity results and giving algorithms for various problems, we will make a preliminary remark about these polyominoes having the same contour projections. In Figure 4 are shown two polyominoes (these polyominoes are both $hv$-convex) with the same contour projections. One can notice that the two polyominoes have not the same number of cells. Of course this fact is a major difference with the polyominoes reconstructing problems studied in [7].
3 Intractability results

The central purpose of this section is to show that the existence problems for the classes of $v$-convex polyominoes, $h$-convex polyominoes, and general polyominoes are $NP$-complete. We first prove the $NP$-completeness for $v$-convex polyominoes. Therefore the same result holds immediately for $h$-convex polyominoes. As corollary to the polynomial transformation used for Theorem 1 we will deduce the $NP$-completeness for polyominoes.

**Theorem 1** The existence problem for the class of $v$-convex polyominoes is $NP$-complete.

**Proof:** The transformation is from the problem **NUMERICAL MATCHING WITH TARGET SUMS (NMTS)** which is $NP$-complete in the strong sense [10]. NMTS is defined below:

**INSTANCE:** $\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_p\}, \{B_1, \ldots, B_p\}$ three sets of $p$ positive integers.

**QUESTION:** Is there a perfect matching between $a_i$’s and $b_j$’s such that for each target $B_k$ there is a pair $(a_i, b_j)$ of the matching such that $B_k = a_i + b_j$?

Let $a = \max\{a_i\}$ and $b = \max\{b_i\}$. Without loss of generality, one can consider the instances of NMTS such that $\min\{B_i\} > \max\{a, b\}$. For each instance of NMTS, we construct an instance of the $v$-convex polyominoes existence problem in the following way (see Figure 5 for an example): the rectangle $R$ containing the polyomino has $m$ rows and $n$ columns and we set $m = 1 + 2(a + b)$ and $n = 2p + 1$. Let $\alpha_i$ be the number of elements $a_k$ such that $i = a_k$; in the same way, we denote by $\beta_i$ the number of elements $b_k$ such that $i = b_k$. The horizontal projections are as follows:

$h_{2i} = \alpha_{a-i}, h_{2i+1} = 0, 0 \leq i \leq a - 1$

$h_{2a} = h_{2a+1} = p + 1$

$h_{2(a+i)+1} = \beta_i, h_{2(a+i)} = 0, 1 \leq i \leq b$

The vertical projections are:

$v_0 = v_{2p+1} = 1$

$v_{2i-1} = v_{2i} = 2B_i, 1 \leq i \leq p$

We prove that if NMTS has a positive answer then there exists a $v$-convex polyomino $P$ which satisfies the projections. The cells of $P$ are the following (see Figure 5): $C(2a + 1, j) \in P, 1 \leq j \leq m$; if for the $i$’st target we have $B_i = a_j + b_k$, then $C(l, 2i) \in P, 2(a - a_j) \leq l \leq 2a$ and $C(l, 2i) \in P, 2(a + 1) \leq l \leq 2(a + b_k) + 1$ (on column $2i$ there are a peak of $2a_j$ cells above the line $2a$ and a peak of $2b_k$ cells below the line $2a + 1$). Thus, $P$ is $v$-convex and the projections are satisfied.
Figure 5 : $a_1 = 3, a_2 = 2, a_3 = 2, b_1 = 4, b_2 = 1, b_3 = 3, B_1 = 4, B_2 = 6, B_3 = 5$

Now we prove that if a $v$-convex polyomino $P$ exists that satisfies the contour projections, there is a positive answer for $NMTS$. By the reduction we have $h_2a = h_{2a} + 1 = p + 1$ and $v_i > 2 \max \{a, b\}, i \neq 0, 2p + 1$, hence we cannot have $T(2a, j) \in \Gamma(P)$ and $T(2a, j + 1) \in \Gamma(P)$ or $B(2a, j) \in \Gamma(P)$ and $B(2a, j + 1) \in \Gamma(P)$. It follows that for every $j, 1 \leq j \leq p + 1$, we have $T(2a, 2j − 1) \in \Gamma(P), T(2a, 2j) \notin \Gamma(P)$ and $B(2a, 2j − 1) \in \Gamma(P), B(2a, 2j) \notin \Gamma(P)$. Therefore $C(2a + 1, 1) \in P$ and by the connectedness the cell $C(2a + 1, 2)$ belongs to $P$; condition $B_1 > \max \{a, b\}$ implies that there are cells of $P$ up and down the $(2a + 1)$-th row and, $P$ being vertically convex, they form a bar; by the reduction and the fact that $P$ satisfies the projections, there exists $a_k$ and $b_j$ such that $a_k + b_j = B_1$; and so on for $B_2, \ldots , B_p$. Thus we have a positive answer for $NMTS$.

Using symmetry we have the following:

**Corollary 1** The existence problem for $h$-convex polyominoes is NP-complete.

From the reduction used in the prove above we can deduce the following result:

**Corollary 2** The existence problem for general polyominoes is NP-complete.

**Proof**: In the reduction above one can observe that $\sum_{i=0}^{2a} h_i = \sum_{i=2a+1}^{2(a+b)+1} h_i = 2p + 1 = n$ and so in every column there are exactly two unit top or bottom contour of the object. Thus a set of unit lines consistent with such a projection corresponds to a simple closed curve, i.e. the contour of a polyomino.
4 \( hv \)-convex polyominoes

In this section we present a polynomial time algorithm that reconstructs a \( hv \)-convex polyomino from \( H \) and \( V \) if such a polyomino exists. The skeleton of our algorithm is the same as the one of the algorithm of Del Lungo and Nivat [7] used for the reconstruction of \( hv \)-convex polyominoes when the projections are the number of cells in each row. We will focus our attention to the specific points of our algorithm.

We introduce some useful notations: \( H^*_i = \sum_{k=0}^{i} h_k \), \( V^*_j = \sum_{k=0}^{j} v_k \).

4.1 Feet of a polyomino

We define the top foot of an \( hv \)-convex polyomino \( P \) as the edges of \( \Gamma(P) \) lying on the top segment of \( R \), that is the edges \( T(1, j) \in \Gamma(P) \). Since \( P \) is \( h \)-convex, this foot contains \( h_0 \) adjacent edges, where \( h_0 = |\{T(1, j) : T(1, j) \in \Gamma(P)\}| \). In the same way, the bottom foot of \( P \) consists of \( h_m \) adjacent edges \( B(m, j) \in \Gamma(P) \); the left foot is the \( v_0 \) adjacent edges \( L(i, 1) \in \Gamma(P) \); and the right foot is the \( v_n \) adjacent edges \( R(i, n) \in \Gamma(P) \).

In the following we suppose that the left most edge of the top foot is located to the left of the left most edge of the bottom foot. The converse case is obtained by symmetry.

**Property 3** If the left most edge of the top foot is \( T(1, k) \), then \( V^*_{j-1} \) equals the number of \( P \) cells in the \( j \)-th column.

![Figure 6: the top foot](image)

**Proof**: The Figure 6 illustrates the proof. Since the bottom foot is on the right of the top foot, the edges \( T(t_{j-1}, j - 1) \in \Gamma(P), T(t_j, j) \in \Gamma(P), 1 \leq j \leq k \) are such that \( t_{j-1} \leq t_j \) and the edges \( B(b_{j-1}, j - 1) \in \Gamma(P), B(b_j, j) \in \Gamma(P), 1 \leq j \leq k \) are such that \( b_{j-1} \leq b_j \) and we have \( v_{j-1} = (t_{j-1} - t_j) + (b_j - b_{j-1}) \). For the first column we have \(|\{C(i, j) : C(i, 1) \in P\}| = v_0\), so for a column \( j \) we obtain \(|\{C(i, j) : C(i, j) \in P\}| = |\{C(i, j - 1) : C(i, j - 1) \in P\}| + (t_{j-1} - t_j) + (b_j - b_{j-1}) = V^*_{j-2} + v_{j-1} = V^*_{j-1} \).
Property 4 If the left most edge of the top foot is $T(1,k)$ and the left most edge of the bottom foot is $B(m,j)$, then $B(V^*_{l-1}, l) \in \Gamma(P)$, $k \leq l \leq \min\{k + h_0, j\}$.

Proof: The Figure 7 illustrates the proof. Since the left most edge of the top foot is $T(1,k)$ then $T(1,l) \in \Gamma(P)$, $k \leq l \leq k + h_0$. From Property 3, we have that $B(V^*_{k-1}, k) \in \Gamma(P)$. Two consecutive bottom edges $B(i,l) \in \Gamma(P), B(i',l+1) \in \Gamma(P), k \leq l \leq j - 1$ must satisfy $i' - i = v_l$. So the property follows.

![Figure 7](image)

Property 5 If the right most edge of the top foot is $T(1,k)$ and the left most edge of the bottom foot is $B(m,j), k + 1 < j$ then for each column $l$ such that $k < l < j$, we have $C\left(\left\lfloor \frac{V^*_{l-1} + 1}{2} \right\rfloor, l\right) \in \Gamma(P)$ and the two edges $B(b,l) \in \Gamma(P)$ and $T(t,l) \in \Gamma(P)$ satisfy the relation: $b + t - 1 = V^*_{l-1}$.

Proof: The relation $b + t - 1 = V^*_{l-1}$ derives from the fact that $P$ is hv-convex and the column $l$ is situated between the two feet. Since $b \geq t$ we have $b \geq \left\lfloor \frac{V^*_{l-1} + 1}{2} \right\rfloor = t$ when $V^*_{l-1}$ is even, and $b \geq \left\lfloor \frac{V^*_{l-1} + 1}{2} \right\rfloor = \left\lceil \frac{V^*_{l-1}}{2} \right\rceil \geq t$ when $V^*_{l-1}$ is odd.

From the properties above, for a fixed position of the top foot and the bottom foot, we have the following information: by Property 3 we know the number of cells in each column on the left of the top foot; by Property 4 we can recognize the ‘bottom shape’ of the $P$ intersecting the strip of the top foot (and therefore the number of cells in the columns); by Property 5 we get a cell of $P$ in every column between the top and the bottom feet.

We define as the top base of $P$ the cells of $P$ that are in the columns containing an edge of the top foot; we denote the top base by $\beta^t = \{C(i,j) \in P, T(1,j) \in \Gamma(P)\}$, and we denote $\beta^b = \{C(i,j) \notin P, T(1,j) \in \Gamma(P)\}$. In the same way, we define the bottom base $\beta^b = \{C(i,j) \in P, B(m,j) \in \Gamma(P)\}$,
and \( \beta^b = \{ C(i, j) \notin P, B(m, j) \in \Gamma(P) \} \); the left base \( \beta^l \), the right base and \( \beta^r \) and the sets \( \beta^l, \beta^r \) are defined in the same way (see Figure 8).

We say that two bases \( \beta^1, \beta^2 \) are compatible if \( \beta^1 \cap \beta^2 = \emptyset \) and \( \beta^2 \cap \beta^1 = \emptyset \) (in Figure 8, \( \beta^b \) and \( \beta^r \) are not compatible). So we have that if \( P \) is a \( hv \)-convex polyomino, and \( \beta^1, \beta^2 \) are two bases of \( P \), then \( \beta^1 \) and \( \beta^2 \) are compatible.

We say that a base \( \beta^1 \) intersects a base \( \beta^2 \) if \( \beta^1 \cap \beta^2 \neq \emptyset \) (in Figure 8, \( \beta^b \) intersects \( \beta^r \)). We have that if \( P \) is a \( hv \)-convex polyomino with the left most edge of its top foot on the left of the left most edge of its bottom foot, then \( \beta^l \) intersects \( \beta^l \) and \( \beta^b \) intersects \( \beta^r \); and by symmetry, if \( P \) is a \( hv \)-convex polyomino with the left most edge of its bottom foot on the left of the left most edge of its top foot, then \( \beta^l \) intersects \( \beta^b \) and \( \beta^l \) intersects \( \beta^r \).

### 4.2 Filling and excluding operations

When the four bases are determined and their compatibility checked, our aim is to expand the set of cells that have to belong into \( P \) and the set of cells that have not to belong into \( P \). Recall that from the previous steps, for each row, we know at least one cell \( C(i, j) \) belonging to \( P \). The two operations we present hereafter are the same as in [7] (this reference gives a detailed description of these operations named the connecting and coherence operations). These operations take advantage of the convexity constraints. These operations are described for a column \( j \), but they can be easily adapted for a row \( i \). We denote by \( C(t, j) \) and \( C(b, j) \) the top most and the bottom most cells of column \( j \) known to be inside \( P \).

The filling operation consists to add to the set of cells already inside \( P \), every cell \( C(i, j) \) such that \( t < i < b \).

The excluding operation is as follows: if it exists a cell \( C(i, j) \notin P \) such that \( i < t \), then for each cell \( C(k, j) \) with \( 1 \leq k < i \), \( C(i, k) \notin P \); if there exists a cell \( C(i, j) \notin P \) such that \( b < i \), then for
Thus a filling-excluding algorithm can be designed as follows: perform filling or excluding operations for every line and column of $R$ until no more cell can be filled in $P$ or excluded of $P$ or until there is a contradiction (a cell $C(i, j)$ with $C(i, j) \in P$ and $C(i, j) \notin P$). The complexity of this algorithm is $O(mn)$.

After the filling-excluding algorithm the situation is the following: for each row we have a nonempty interval of cells that belong to $P$ and two intervals of cells that not belong to $P$ (see Figure 9). The other cells of the row form two intervals where the cells have an undetermined status (either inside $P$ or outside $P$). Hereafter we establish how to add new cells into $P$ and how to exclude new cells from $P$, depending upon the position of the row relatively to the feet. We describe these operations for the columns, so it is an easy task to adapt them to the lines.

In the case of an external column, from Property 3 we know that $|\{C(i, k) : C(i, k) \in P\}| = V_{k-1}^*$ (without loss of generality, $k$ is a column on the left of the top base), so we use the same operations as in [7]. Let $l \leq V_{k-1}^*$ be the number of cells inside $P$, these cells are $C(t+1, k), \ldots, C(t+l, k)$; the interval of cells outside $P$ are $C(1, k), \ldots, C(h, k), 1 \leq h \leq t$ and $C(b, k), \ldots, C(m, k), t+l < b \leq m$. We can remark that Property 3 implies $l \leq V_{k-1}^*$ and $V_{k-1}^* \leq b - h + 1$. The interval containing the $l$ cells inside $P$ must be extended by filling it with $V_{k-1}^* - l$ cells. We consider two cases in order to make this extension (see Figure 10): the first case is when $V_{k-1}^* - l < t - h$ or $t + V_{k-1}^* < b - 1$, the second case is when $V_{k-1}^* - l > t - h$ or $t + V_{k-1}^* > b - 1$. For the case $V_{k-1}^* - l < t - h$ (the
Now, one can remark that after these filling and excluding operations, for any row, the two intervals of undetermined cells contain the same number of cells.

Figure 11 : a cell is excluded
4.3 Relations between undetermined cells

After the previous filling and excluding operations if it remains no undetermined cell, we have reconstruct $P$ an $hv$-convex polyomino satisfying $H$ and $V$. When there are undetermined cells, we show how to link two undetermined cells of a same row. We have two kinds of relations according as the row containing the two cells is internal or external. Again we present these relations in the case of a column $j$.

First we treat the case where $j$ is an external column: let $C(i, j)$ be an undetermined cell situated above the interval of cells set inside $P$. From Property 3 we know $l$, the number of cells of a $hv$−convex polyomino, on the column $j$. Then we have the following property.

\textbf{Property 6} \hspace{1em} C(i, j) \in P \text{ if and only if } C(i + l, j) \notin P.

\textbf{Proof} : Since $P$ is $hv$-convex and satisfies $H$ and $V$, $C(i, j)$ and $C(i + l, j)$ cannot be inside $P$ together (otherwise the number of cells inside $P$ would be at least $l + 1$). After the filling operations, since $C(i, j)$ is above the cells set inside $P$, the cell $C(i + l, j)$ is below this set; thus $C(i, j)$ and $C(i + l, j)$ cannot be outside $P$ together (otherwise the number of cells inside $P$ would be at most $l - 1$).

Now we consider the case where $j$ is an internal column: let $C(i, j)$ be an undetermined cell such that $C(i, j)$ is above the interval of cells already set into $P$; we suppose that $V^*_{j-1} \leq m$. Recall that from Property 5 we have $i \leq \lceil V^*_j - 1/2 \rceil$. We establish the property below.

\textbf{Property 7} \hspace{1em} C(i, j) \in P \text{ if and only if } C(V^*_{j-1} - i - 1, j) \in P.

\textbf{Proof} : Figure 12 illustrates the proof. If $P$ is a $hv$−convex polyomino satisfying $H$ and $V$, there exist $k$ and $l (k \leq l)$, such that $T(k, j) \in \Gamma(P), B(l, j) \in \Gamma(P)$ and $(k - 1) + l = V^*_j$. If $C(i, j) \in P$, we have $i \geq k$ which implies $V^*_{j-1} - i - 1 \leq V^*_j - l$ and $C(V^*_{j-1} - i - 1, j) \in P$; if $C(i, j) \notin P$, $i < k$ and we obtain $V^*_{j-1} - i - 1 > V^*_j - l$, hence $C(V^*_{j-1} - i - 1, j) \notin P$.

Graph of undetermined cells

Now we construct a graph capturing the relations defined above. This graph is $G = (U, E)$ where the vertex set $U$ corresponds to the undetermined cells and the edge set $E$ is as follows:

If $C(i, j)$ is on an external row and $C(i, j)$ is on the left of the interval of cells that are inside $P$ then $\{C(i, j), C(i, j + l)\} \in E$, where $l$ is the number of cells of $P$ in the row $i$.

If $C(i, j)$ is on an external column and $C(i, j)$ is above the interval of cells that are inside $P$ then $\{C(i, j), C(i + l, j)\} \in E$, where $l$ is the number of cells of $P$ in the column $j$. 

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If \( C(i, j) \) is on an internal row and \( C(i, j) \) is on the left of the interval of cells that are inside \( P \), then \( \{C(i, j), C(i, H_{i-1}^* - j - 1)\} \in E \).

If \( C(i, j) \) is on an internal column and \( C(i, j) \) is above the interval of cells that are inside \( P \) then \( \{C(i, j), C(V_j^* - i - 1, j)\} \in E \).

Since for each row the two intervals of undetermined cells have the same size, each vertex of \( G \) has degree 2 and \( G \) is a collection of disjoint even cycles \( \{C_1, \ldots, C_k\} \).

In a same manner as in [7] we associate a boolean variable \( x_i \) with each vertex \( v_i \) of \( G \). If \( x_i = 1 \) then the corresponding cell is inside \( P \); if \( x_i = 0 \) the cell is outside \( P \). From Properties 6 and 7, we have that each cycle \( C_i \) corresponds to one unique variable : indeed for any edge \( \{v_k, v_l\} \in E \), when the two corresponding cells are in an external row we have \( x_k = \overline{x}_l \) (Property 6), and when they are in an internal row we have \( x_k = x_l \) (Property 7).

**How to satisfy the \( hv \)-convexity**

As in [7] we show that finding \( P \) a \( hv \)-convex polyomino consistent with \( H \) and \( V \), is equivalent to satisfy a boolean formula over the variables \( x_i \) associated with the cycles \( C_i \).

We will express the formula associated with a row \( i \) (the formula associated with a column can be easily derived). We study two cases, depending whether the row is external or internal.

When the row \( i \) is external we are in the same situation as in [7] : let \( h \) be the number of cells of \( P \) in row \( i \); we denote by \( x_1, \ldots, x_k \) the variables associated with the undetermined cells situated on the left of the cells that are inside \( P \) (from left to right, see Figure 13); thus the variables associated with the undetermined cells on the right are \( \overline{x}_1, \ldots, \overline{x}_k \) (from left to right, \( \{C(i, j), C(i, j+h)\} \in E \).
We have the next property (see [7] for its proof).

**Property 8** The boolean formula \( F_i = (\bar{x}_1 \lor x_2) \land (\bar{x}_2 \lor x_3) \land \ldots \land (\bar{x}_{k-1} \lor x_k) \) is satisfied if and only if there are exactly \( h_i \) successive cells in row \( i \) of \( P \).

Now we consider the case where the row \( i \) is internal : let \( H_{i-1} = h_1 + h_2 \) with \( 0 \leq h_1 \leq h_2 \); we denote by \( x_1, \ldots, x_k \) the variables associated with the undetermined cells situated on the left of the cells inside \( P \) (from left to right, see Figure 14); thus the variables associated with the undetermined cells situated on the right are \( x_1, \ldots, x_k \) (from the right to the left); thus if \( x_l \) corresponds to the cell \( C(i, h_1) \), \( x_l \) is also associated with the cell \( C(i, h_2) \) (see Property 7). The property below establishes the equivalence between the satisfaction of a boolean formula and the \( h \)-convexity and the horizontal projection constraints for the row \( i \).

**Property 9** \( F_i = (\bar{x}_1 \lor x_2) \land (\bar{x}_2 \lor x_3) \land \ldots \land (\bar{x}_{k-1} \lor x_k) \) is satisfied if and only if there is one unique interval of cells inside \( P \) in row \( i \) and the projection \( h_i \) is satisfied.

**Proof :** If the \( h \)-convexity and projection constraints are satisfied then \( F_i \) is true (using similar arguments used in the proof of Property 8).

If \( F_i \) is satisfied : let \( l \) be the smallest index such that \( x_l = 1 \), then we have \( x_p = 1 \) for \( l < p \leq k \). Let \( C(i, h_1) \) and \( C(i, h_2) \) be the two cells associated with \( x_l \), then cells \( C(i, h_1), C(i, h_1 + 1), \ldots, C(i, h_2) \) are inside \( P \) and the \( h \)-convexity holds. Since \( h_1 + h_2 = H_{i-1} \), we also satisfy the projection constraint.

Now, we are able to prove the main result.

**Theorem 2** For a fixed and coherent position of the four feet, \( P \) a \( hv \)-convex polyomino satisfying \( (H, V) \) exists if and only if the boolean formula \( F = \bigwedge_{i=1}^m F_i \land \bigwedge_{j=1}^n F_j \) is satisfied.
**Proof**: If \( P \) is a \( hv \)-convex polyomino with projections \((H, V)\) the status of each cell is determined, and so the value of the variable associated with each cycle \( C_i \) of \( G \). Thus the boolean formula \( F_i \) associated with each row \( i \) has value true and \( F \) is satisfied.

If \( F \) is satisfied, each \( F_i \) is true. So from Properties 3, 8 and 9, \( P \) is a \( hv \)-convex polyomino with contour projections \( H \) and \( V \).

### 4.4 Reconstruction algorithm

Here we give an algorithm that builds \( P \) an \( hv \)-convex polyomino that satisfies \((H, V)\) is such a \( P \) exists. The algorithm is as follows:

1. **compute** \( H^*_i = \sum_{j=0}^{i} h_j, 1 \leq i \leq m \), and \( V^*_i = \sum_{j=0}^{i} v_j, 1 \leq i \leq n \), and check that \( H^*_m = 2n, V^*_n = 2m \) (Properties 1 and 2)

2. **repeat**

   2.1 **choose** a coherent position for the four feet;

   2.2 **compute** the bases \( \beta^t, \beta^b, \beta^l, \beta^r \) and **check** their compatibility

   2.2 **perform** filling and excluding operations and **check** their consistency

   2.3 **construct** \( G \) and match a boolean variable \( x_i \) with each cycle \( C_i \) of \( G \)

   2.4 **check** whether the boolean formula \( F \) associated with \( G \) is satisfied

   **until** there exists \( P \) an \( hv \)-convex polyomino that satisfies \((H, V)\) or all the feet positions have been examined

**Theorem 3** The reconstruction problem of an \( hv \)-convex polyomino \( P \) satisfying the projections \( H \) and \( V \) can be solved in polynomial time.

**Proof**: from Theorem 2 one can check that our algorithm returns a \( hv \)-convex polyomino satisfying \( H \) and \( V \) if such a polyomino exists.

The number of feet positions is \( O(m^2n^2) \). The determination of the four bases can be performed in time \( O(m + n) \) from the partial sums \( H^*_i \) and \( V^*_i \); their compatibility can be checked in time \( O(mn) \). Filling and excluding operations can be performed in time \( O(m^2n^2) \). The construction of \( G \) and the determination of its cycles can be done in time \( O(mn) \). Build the boolean formula \( F \) takes \( O(mn) \), and since \( F \) is a 2−SAT formula it can be solved in time \( O(mn) \) (see [1]). Thus the time complexity of the algorithm is \( O(m^4n^4) \).
5 Conclusion

We have studied problems arising in the reconstruction of convex polyominoes when the orthogonal projections are defined as the length of their contour lines. We have proved the $NP$-completeness of the related existence problems in the case of $h$-convex or $v$-convex polyominoes. For the class of $hv$-convex polyominoes we gave a polynomial time reconstruction algorithm.

From a practical point of view, the problem of unicity of a solution is often a crucial aspect in discrete tomography (see [14]). This problem is not studied in this paper but should be the subject of a future work. Other future researches arise with the following optimization problems: Figure 4 shown two $hv$-convex polyominoes with the same projections but with two different areas; let $\mathcal{P}$ be the set of $hv$-convex polyominoes consistent with the projections $(H,V)$, one can be interested in finding a thin polyomino that is $p \in \mathcal{P}$ such that $|\{(i,j) : C(i,j) \in p\}| = \min_{p' \in \mathcal{P}} \{|\{(i,j) : C(i,j) \in p'\}|\}$ or, on the opposite, in finding a fat polyomino that is $p \in \mathcal{P}$ such that $|\{(i,j) : C(i,j) \in p\}| = \max_{p' \in \mathcal{P}} \{|\{(i,j) : C(i,j) \in p'\}|\}$. A further work is trying to adapt our polynomial algorithm to these optimization problems.

Acknowledgments:

The author express its gratitude to an anonymous referee for helpful comments that substantially contributed to improve the presentation of the paper.

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