Matrix interpretations revisited *

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1 Introduction

The property of termination, well-known to be undecidable, is fundamental by many aspects of computer sciences and logic. Many heuristics have been proposed to provide automation for termination proofs. Almost all of them require, possibly after several transformations of the initial termination problem, to search a well-founded orderings satisfying some properties. Among the different kinds of orderings, polynomial interpretations [10, 2, 4] and recursive path ordering [5] are the most used. Matrix interpretation is a recent powerful kind of well-founded ordering based on term interpretation into vectors, introduced by Endrullis, Waldmann and Zantema in [8]. This interpretation associates to each symbol a linear mapping with matrix coefficients. This paper presents a generalization of this method, which allows for more matrices and more orderings. In particular it allows for more systems to be proved to be terminating without increasing the bounds for coefficients or the size of matrices.

Due to the monotonicity requirement for interpretations, the original matrix interpretations are restricted to matrices with a strictly positive upper left coefficient, and the associated strict ordering only considers the upper coefficient on vectors. We propose in this paper a weaker limitation still preserving monotonicity. We require for each matrix to have a fixed sub-matrix with no null columns. The strict ordering will consider coefficients corresponding to this sub-matrix. Note that matrix interpretations of [8] is a particular case of our approach where the sub-matrix is reduced to the upper left coefficients. Note that similar ideas are used in [**?**] in the context of string rewriting.

Section 2 recalls preliminary notions on orderings by interpretation. Section 3 presents the extension we propose and the proof of its correctness. Sections 3.3 and 4 illustrate our method on an example and summarizes some experiments on the *termination problems database (TPDB)*. Finally we present future work and conclude in Section 5.

2 Preliminaries

We assume that the reader is familiar with basic concepts of term rewriting [6, 1, 7, 5] and termination. The typical approach of an (automated) termination prover is to transform recursively a well foundation problem to an equivalent set of simpler problems until we get problems that are proved directly using a well-founded ordering (pair). There are two kinds of desirable orderings: weakly monotonic (used for example to order dependency pairs) and strictly monotonic (rewriting rules) ones. We can use *interpretations* to produce such orderings. We recall notions of *ordering pairs* and *homomorphic interpretations*.

Ordering pairs. An ordering pair is a pair $(>, \geq)$ of relations over $T(\mathscr{F}, X)$ such that: 1) $\geq$ is a quasi-ordering, i.e. reflexive and transitive, 2) $>$ is a strict ordering, i.e. irreflexive and transitive, and 3) $\geq \cdot > \subseteq >$ (notice that $>$ is *not* the strict part of $\geq$). An ordering pair $(>, \geq)$ is *well-founded* if there is no infinite strictly decreasing sequence $t_1 > t_2 > \dots$. An ordering pair $(>, \geq)$ is *weakly monotonic* if for all terms t and u and any symbol f, $t \geq u \to f(\dots t \dots) \geq f(\dots u \dots)$. An ordering pair $(>, \geq)$ is *strictly monotonic* if for all terms t and u and any symbol f, $t > u \to f(\dots t \dots) > f(\dots u \dots)$.

Orderings by Interpretation. In the sequel, we suppose a non empty set D (domain), a quasi-ordering $\geq_D$ on D, and $>_D = \geq_D - \leq_D$. Therefore $(>, \geq)$ is an ordering pair.

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Definition 2.1. *A* homomorphic interpretation *is a function φ that takes a symbol f and returns a valuation function $[f]_\varphi : D^n \to D$, where n is the arity of f. We define the homomorphic interpretation $\varphi(t)$ of a (possibly non closed) term t as a function from valuation functions to D (i.e. $\varphi(t) : (X \to D) \to D$) by induction on t as follows: $\varphi(x)(v) = v(x)$ and $\varphi(f(t_1,\dots,t_n))(v) = [f]_\varphi(\varphi(t_1)(v),\dots,\varphi(t_n)(v))$*

Definition 2.2. *We define the ordering pair $(\succeq_\varphi, \succ_\varphi)$ on terms by: $s \succeq_\varphi t$ iff $\forall v \in (X \to D), \varphi(s)(v) \geq_D \varphi(t)(v)$ and $s \succ_\varphi t$ iff $\forall v \in (X \to D), \varphi(s)(v) >_D \varphi(t)(v)$.*

Theorem 2.3. *Let φ be a homomorphic interpretation on D, then*

- *$(\succeq_\varphi, \succ_\varphi)$ is stable by substitution. It is also well-founded if $(\geq_D, >_D)$ is.*
- *if the following property (strict monotonicity on each argument) holds for all symbol f: $\forall x,y \in D, x >_D y \Rightarrow [f]_\varphi(\dots x \dots) >_D [f]_\varphi(\dots y \dots)$, then $(\succeq_\varphi, \succ_\varphi)$ is strictly monotonic.*
- *if the following property (weak monotonicity on each argument) holds for all symbol f of arity n: $\forall x,y \in D, x \geq_D y \Rightarrow [f]_\varphi(\dots x \dots) \geq_D [f]_\varphi(\dots y \dots)$, then $(\succeq_\varphi, \succ_\varphi)$ is weakly monotonic.*

Matrix interpretation. The main idea of matrix interpretation in [8] is to define homomorphic interpretations using linear mappings represented by matrices. The ordering pair on $\mathbb{N}^d$, which we note $(\geq_{\mathbb{N}^d}, >_{\mathbb{N}^d})$ is defined as follows: $(u_i) \geq_{\mathbb{N}^d} (v_i)$ iff $\forall i, u_i \geq_{\mathbb{N}} v_i$ and $(u_i) >_{\mathbb{N}^d} (v_i)$ iff $\forall i, u_i \geq_{\mathbb{N}} v_i$ and $u_1 >_{\mathbb{N}} v_1$. As homomorphic interpretations defined by polynomials over matrices may not be monotonic, [8] proposes a restriction on the form of vectors and matrices to ensure strict monotonicity: the upper-left coefficient of vectors and matrices must be strictly positive.

3 Generalized matrix interpretation

We define a family of interpretations parametrized by the set E of column and line indexes used by the strict ordering. We use polynomials with *matrix* constants instead of vectors ($D = \mathbb{N}^{d\times d}$). This corresponds to the usual notion of polynomials over matrices. Results of [8] can be considered as consequences of what follows by forcing adequate columns to be null.

3.1 Ordering and interpretation

Definition 3.1. *Let $m, m' \in \mathbb{N}^{d\times d}$ and $E \subseteq \{1,...,d\}$, We define $\geq_{\mathbb{N}^{d\times d}}$ and $>^E_{\mathbb{N}^{d\times d}}$ on $\mathbb{N}^{d\times d}$ as: $m \geq_{\mathbb{N}^{d\times d}} m'$ iff $\forall i,j \in [1..d], m_{ij} \geq_{\mathbb{N}} m'_{ij}$, and $m >^E_{\mathbb{N}^{d\times d}} m'$ iff $\forall i,j \in [1..d], m_{ij} \geq_{\mathbb{N}} m'_{ij} \wedge \exists i,j \in E, m_{ij} >_{\mathbb{N}} m'_{ij}$.*

Lemma 3.2. *$(\geq_{\mathbb{N}^{d\times d}}, >^E_{\mathbb{N}^{d\times d}})$ is a well-founded ordering pair.*

Definition 3.3. *Let $E \subseteq \{1,...,d\}$, we call an E-position in a matrix $m \in \mathbb{N}^{d\times d}$ a position m_{ij} where $i \in E$ and $j \in E$. We also call E-columns the E-positions belonging to the same column. We say that m is E-*compatible *iff each E-column is non null, i.e. at least one E-position on each E-column is non null.*

For example the following matrix A is $\{1,3\}$-compatible whereas B is not: $A = \begin{pmatrix} \underline{\mathbf{0}} & 1 & \underline{\mathbf{0}} \\ 0 & 0 & 0 \\ \underline{\mathbf{2}} & 3 & \underline{\mathbf{1}} \end{pmatrix}, B = \begin{pmatrix} \underline{\mathbf{1}} & 1 & \underline{\mathbf{0}} \\ 0 & 0 & 0 \\ \underline{\mathbf{2}} & 3 & \underline{\mathbf{0}} \end{pmatrix}$.

Definition 3.4. *Given a signature Σ and a dimension $d \in \mathbb{N}$, a* matrix interpretation *φ is a homomorphic interpretation that takes a symbol f of arity n and returns a valuation function of the form: $[f]_\varphi(m_1,\dots,m_n) = F_1 m_1 + \cdots + F_n m_n + F_{n+1}$ where $F_i \in \mathbb{N}^{d\times d}$ and $m_1,\dots,m_n$ take their values in $\mathbb{N}^{d\times d}$. If the matrices F_i, **except** F_{n+1}, are E-compatible then the interpretation is said to be E-compatible and we call it an E-*interpretation*.*

Definition 3.5. $(\succeq_\varphi, \succ^E_\varphi)$ *is defined from* $(\geq_{\mathbb{N}^{d\times d}}, >^E_{\mathbb{N}^{d\times d}})$ *as in Definitions 2.2 (with* $D = \mathbb{N}^{d\times d}$*).*

Lemma 3.6. *Let* φ *be a matrix interpretation, then* $(\succeq_\varphi, \succ^E_\varphi)$ *is 1) stable wrt substitution, 2) well-founded, 3) weakly monotonic and 4) strictly monotonic if moreover* φ *is* E*-compatible.*

3.2 Proving termination

To prove termination we need to compare matrix interpretations of terms with $\succ_\varphi$ and $\succeq_\varphi$. Interpretations can be computed by developing and comparing (linear) polynomials, which is decidable as stated below:

Lemma 3.7. *Let* φ *be a matrix interpretation and* t *a term with free variables* $x_1 \dots x_n$*, then there exist* $n+1$ *matrices* $M_1 \dots M_{n+1}$ *such that* $\varphi(t)(v) = M_1 v(x_1) + \cdots + M_n v(x_n) + M_{n+1}$*. If moreover* φ *is* E*-compatible, then* $M_1 \dots M_n$ *are* E*-compatible.*

Lemma 3.8. *Let* t *and* u *be terms such that* $\varphi(t)(v) = L_1 v(x_1) + \cdots + L_k v(x_k) + L_{k+1}$ *and* $\varphi(u)(v) = R_1 v(x_1) + \cdots + R_k v(x_k) + R_{k+1}$*. If* $\forall 1 \leq i \leq k+1, L_i \geq_{\mathbb{N}^{d\times d}} R_i$ *then* $\varphi(t)(v) \geq_{\mathbb{N}^{d\times d}} \varphi(u)(v)$ *for any valuation* $\alpha : \mathbb{N}^k \to \mathbb{N}$*. If moreover* $L_{k+1} >^E_{\mathbb{N}^{d\times d}} R_{k+1}$*, then* $\varphi(t)(v) >^E_{\mathbb{N}^{d\times d}} \varphi(u)(v)$ *for any valuation* $\alpha : \mathbb{N}^k \to \mathbb{N}$*.*

3.3 Example

In this example matrix coefficients are forced to be 0 or 1. Within this bounds there exists a $\{1,2\}$-interpretation ($[.]_\varphi$ below) which allows to prove termination of the following system by Manna-Ness criterion, but no $\{1\}$-interpretations. It is worth noticing that this example can be solved by a $\{1\}$-interpretation with a higher bound, but at the price of a greater search space.

$$h(0) \to 0 \qquad h(s(s(x))) \to s(h(x)) \qquad l(s(0)) \to 0 \qquad l(s(s(x))) \to s(l(s(h(x))))$$

$$[h]_\varphi(x) = \begin{pmatrix} 1 0 \\ 0 1 \end{pmatrix} x + \begin{pmatrix} 0 1 \\ 0 0 \end{pmatrix}, [l]_\varphi(x) = \begin{pmatrix} 1 1 \\ 0 1 \end{pmatrix} x + \begin{pmatrix} 0 0 \\ 0 0 \end{pmatrix}, [s]_\varphi(x) = \begin{pmatrix} 1 0 \\ 0 1 \end{pmatrix} x + \begin{pmatrix} 0 0 \\ 1 1 \end{pmatrix}, [0]_\varphi = \begin{pmatrix} 0 0 \\ 0 0 \end{pmatrix}$$

4 Experiments

The proof search is an adaptation of the method of [8] for generating termination proofs. The main differences are the choice of an E, the treatment of E-compatibility and the ordering constraints using E. *Remark*: Given a set of constraints S, if there is a E-interpretation φ that solves S then any *simultaneous* permutation of lines and columns of all matrices of φ is a E'-interpretation for some E' which also solves S. Therefore it is enough to try only sets E of the form $\{1, \dots, n\}$ where $1 \leq n \leq d$.

The benchmarks were made on the Tpdb database with C*i*ME. The solving part of the criterion (MN=Manna-Ness, DP=Dependency pairs, LEX=Lexicographic combination, DPG=graph refinement of DP) were made by translation to the SAT solver `minisat2`. Figure 1 highlights differences depending on the size of E. Notice that, for the sake of comparison, inside one criterion *we only compare problems that did not timeout for any* $|E|$. Therefore this results do not compare the solving times.

We see that for strictly monotonic orderings the sets of problems solved are not included in each other as shown by line $|E| = 1$ or 2, therefore both sizes for E are worth trying. This is not the case for weakly monotonic orderings of DP criterion where the maximum size of E is the best. Finally we see that the choice of $|E|$ is not pertinent when considering graph refinement of dependency pairs.

5 Conclusion and Future work

As shown in previous section our approach generalizes the original matrix interpretations. It also allows for more generalization. First it should naturally extend to other refinement of matrix interpretations

Criterion	MN			LEX			DP			DPG		
coef. bound	1	2	3	1	2	3	1	2	3	1	2	3
$\|E\| = 1$	115	232	250	343	386	380	342	451	471	578	590	594
$\|E\| = 2$	167	236	252	296	369	377	469	467	480	578	590	594
$\|E\| = 1$ or 2	170	252	259	354	393	387	469	467	480	578	590	594

Figure 1: Experiments with 2×2 matrices

such as arctic interpretations (where the usual plus/times operations are generalized to an arbitrary semi-ring [9]). Second our approach using true polynomials over matrices, instead of mixing matrices and vectors, should allow for *non linear polynomials over matrices*.

Our matrix interpretations have been implemented in an early C*i*ME-3 prototype. Interpretations are found by solving linear constraints over matrix coefficients. We produce SAT constraints for each size of E, which forces us to call SAT solving once per size of E. It should be possible to build efficient SAT constraints corresponding to all possible sizes of E.

Another important point is that our implementation *produces proof traces*. We are currently adapting the translation of these traces into *proof certificates* [3] for verification by the Coq proof assistant.

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